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ON GENERALIZED UPPER SETS IN BE-ALGEBRAS

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Abstract. In this paper, we develop the idea of a generalized upper set in a BE-algebra. Furthermore, these sets are considered in the context of transitive and self distributive BE-algebras and their ideals, providing characterizations of one type, the generalized upper sets, in terms of the other type, ideals.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3, 4], Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([10]) introduced the notion of d-algebras which is another generalization of BCK-algebras. S. S. Ahn and Y. H. Kim ([1]) gave some constructions of implicative/commutative d-algebras which are not BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim ([7]) introduced the notion of BH-*algebra*, which is a generalization of $BCH/BCI/BCK$-algebras. In [8], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of generalization of a BCK-algebra. Using the notion of upper sets they provided an equivalent condition describing filters in BE-algebras. Using the notion of upper sets they gave an equivalent condition for a subset to be a filter in BE-algebras. In [2], we introduced the notion of ideals in BE-algebras, and then stated and proved several characterizations of such ideals.

In this paper, we generalize the notion of upper sets in BE-algebras, and discuss properties of the characterizations of generalized upper sets $A_n(u,v)$ while relating them to the structure of ideals in transitive and self distributive BE-algebras.

2. Preliminaries

We recall some definitions and results (See [2, 8]).

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Definition 2.1. An algebra $(X; *, 1)$ of type (2,0) is called a *BE-algebra* ([8]) if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(BE3) $1 * x = x$ for all $x \in X$;
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$. (*exchange*)

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$. Note that if $(X; *, 1)$ is a BE-algebra, then $x * (y * x) = 1$ for any $x, y \in X$.

Example 2.2 ([8])**.** Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra.

Definition 2.3. A BE-algebra $(X, *, 1)$ is said to be *self distributive* ([8]) if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Example 2.4 ([8])**.** Let $X := \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to see that X is a BE-algebra satisfying self distributivity.

Note that the BE-algebra in Example 2.2 is not self distributive, since $d * (a * 0) = d * d = 1$, while $(d * a) * (d * 0) = 1 * a = a$.

Definition 2.5 ([2])**.** A non-empty subset I of X is called an *ideal* of X if

(I1) $\forall x \in X$ and $\forall a \in I$ imply $x * a \in I$, i.e., $X * I \subseteq I$;
(I2) $\forall x \in X$, $\forall a, b \in I$ imply $(a * (b * x)) * x \in I$.

In Example 2.2, $\{1, a, b\}$ is an ideal of X, but $\{1, a\}$ is not an ideal of X, since $(a * (a * b)) * b = (a * a) * b = 1 * b = b \notin \{1, a\}$.

It was proved that every ideal I of a BE-algebra X contains 1, and if $a \in I$ and $x \in X$, then $(a * x) * x \in I$. Moreover, if I is an ideal of X and if $a \in I$ and $a \leq x$, then $x \in I$ (see [2]).

Lemma 2.6 ([2])**.** *Let I be a subset of X such that*

(I3) $1 \in I$;
(I4) $x * (y * z) \in I$ *and* $y \in I$ *imply* $x * z \in I$ *for all* $x, y, z \in X$.
If $a \in I$ *and* $a \leq x$, *then* $x \in I$.

Definition 2.7. A BE-algebra $(X; *, 1)$ is said to be *transitive* ([2]) if for any $x, y, z \in X$,

$$y * z \leq (x * y) * (x * z).$$

Example 2.8 ([2])**.** Let $X := \{1, a, b, c\}$ be a set with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is a transitive BE-algebra.

Proposition 2.9 ([2])**.** *If* X *is a self distributive* BE*-algebra, then it is transitive.*

The converse of Proposition 2.9 need not be true in general. In Example 2.8, X is a transitive BE-algebra, but $a * (a * b) = a * a = 1$, while $(a * a) * (a * b) = 1 * a = a$, showing that X is not self distributive.

Theorem 2.10 ([2])**.** *Let* X *be a transitive* BE*-algebra. A subset* I $(\neq \emptyset)$ *of* X *is an ideal of* X *if and only if it satisfies conditions* (I3) *and* (I4).

3. Main results

In what follows let X denote a BE-algebra unless otherwise specified. For any elements u and v of X and $n \in \mathbb{N}$, we use the notation $u^n * v$ instead of $u * (\cdots * (u * v)) \cdots)$ in which u occurs n times. Let X be a BE-algebra and let $u, v \in X$. Define

$$A(u, v) := \{z \in X \mid u * (v * z) = 1\}$$

We call $A(u, v)$ an *upper set* ([8]) of u and v. It is easy to see that $1, u, v \in A(u, v)$ for any $u, v \in X$. We generalize the notion of the upper set $A(u, v)$ using the concept of $u^n * v$ as follows.

For any $u, v \in X$, consider a set

$$A_n(u, v) := \{z \in X | u^n * (v * z) = 1\}.$$

We call $A_n(u, v)$ an *generalized upper set* of u and v in a BE-algebra X. In Example 2.2, the set $A_n(1, a) = \{1, a\}$ is not an ideal of X. Hence we know that $A_n(u, v)$ may not be an ideal of X in general.

Theorem 3.1. *If* X *is a self distributive* BE*-algebra, then* $A_n(u, v)$ *is an ideal of* X, $\forall u, v \in X$, *where* $n \in \mathbb{N}$.

Proof. Let $a \in A_n(u,v)$ and $x \in X$. Then $u^n * (v * a) = 1$. It follows from the self distributivity law that

$$\begin{aligned}
&u^n * (v * (x * a)) \\
&= u^{n-1} * [u * (v * (x * a))] \\
&= u^{n-1} * [u * ((v * x) * (v * a))] && \text{[self distributive]} \\
&= u^{n-1} * ([u * (v * x)] * [u * (v * a)]) && \text{[self distributive]} \\
&= (u^{n-1} * [u * (v * x)]) * (u^{n-1} * [u * (v * a)]) && \text{[self distributive]} \\
&= (u^{n-1} * [u * (v * x)]) * (u^n * (v * a)) \\
&= (u^{n-1} * (u * (v * x))) * 1, && [a \in A_n(u,v)] \\
&= 1 && \text{[(BE2)]}
\end{aligned}$$

whence $x * a \in A_n(u,v)$. Thus, (I1) holds.

Let $a, b \in A_n(u,v)$ and $x \in X$. Then $u^n * (v * a) = 1$ and $u^n * (v * b) = 1$. It follows from the self distributivity law that

$$\begin{aligned}
&u^n * (v * ((a * (b * x)) * x)) \\
&= u^{n-1} * (u * [v * ((a * (b * x)) * x)]) \\
&= u^{n-1} * (u * [(v * (a * (b * x))) * (v * x)]) \\
&= u^{n-1} * ([u * (v * (a * (b * x)))] * [u * (v * x)]) \\
&= u^{n-1} * ([(u * (v * a)) * (u * (v * (b * x)))] * [u * (v * x)]) \\
&= (u^{n-1} * [(u * (v * a)) * (u * (v * (b * x)))]) * (u^{n-1} * [u * (v * x)]) \\
&= [(u^{n-1} * (u * (v * a))) * (u^{n-1} * (u * (v * (b * x))))] * (u^{n-1} * [u * (v * x)]) \\
&= [(u^n * (v * a)) * (u^{n-1} * (u * (v * (b * x))))] * (u^{n-1} * [u * (v * x)]) \\
&= [1 * (u^{n-1} * (u * (v * (b * x))))] * (u^{n-1} * [u * (v * x)]) \\
&= [u^{n-1} * (u * (v * (b * x)))] * (u^{n-1} * [u * (v * x)]) \\
&= [u^{n-1} * (u * ((v * b) * (v * x)))] * (u^{n-1} * [u * (v * x)]) \\
&= [(u^{n-1} * (u * (v * b))) * (u^{n-1} * (v * x))] * (u^{n-1} * [u * (v * x)]) \\
&= [(u^n * (v * b)) * (u^{n-1} * (v * x))] * (u^{n-1} * [u * (v * x)]) \\
&= [1 * (u^{n-1} * (v * x))] * (u^{n-1} * [u * (v * x)]) \\
&= [u^{n-1} * (v * x)] * [u^{n-1} * (u * (v * x))] \\
&= u^{n-1} * [(v * x) * (u * (v * x))] \\
&= u^{n-1} * [u * ((v * x) * (v * x))] \\
&= u^{n-1} * (u * 1) \\
&= u^{n-1} * 1 = 1
\end{aligned}$$

whence $(a*(b*x))*x \in A_n(u,v)$. Thus, (I2) holds. This proves that $A_n(u,v)$ is an ideal of X. □

Lemma 3.2. *Let X be a BE-algebra. If $y \in X$ satisfies $y*z = 1$ for all $x \in X$, then*

$$A_n(x,y) = X = A_n(y,x)$$

for all $x \in X$, where $n \in \mathbb{N}$.

Proof. The proof is straightforward. □

Example 3.3. Let $X := \{1,a,b,c,d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then X is a self distributive BE-algebra. By Lemma 3.2, we have $A_n(x,d) = A_n(d,x) = X$ for all $x \in X$. Furthermore, we have that $A_n(1,1) = 1$, $A_n(1,a) = A_n(a,1) = A_n(a,a) = A_n(a,b) = \{1,a\}$, $A_n(1,b) = A_n(b,1) = A_n(b,b) = \{1,b\}$, $A_n(1,c) = A_n(a,c) = A_n(c,1) = A_n(c,a) = A_n(c,c) = \{1,a,c\}$, $A_n(b,a) = \{1,a,b\}$, and $A_n(c,b) = X$ are ideals of X, where $n \in \mathbb{N}$.

Using the notion of upper set $A(u,v)$, we given an equivalent condition for a non-empty subset to be an ideal in BE-algebras.

Theorem 3.4. *Let X be a transitive BE-algebra. A subset I $(\neq \emptyset)$ of X is an ideal of X if and only $A_n(u,v) \subseteq I$, $\forall u,v \in I$, where $n \in \mathbb{N}$.*

Proof. Assume that I is an ideal of X. If $z \in A_n(u,v)$, then $u^n*(v*z) = 1$ and so $z = 1*z = (u^n*(v*z))*z \in I$ by (I2). Hence $A_n(u,v) \subseteq I$.

Conversely, suppose that $A_n(u,v) \subseteq I$ for all $u,v \in I$. Note that $1 \in A_n(u,v) \subseteq I$. Hence (I3) holds. Let $x,y,z \in X$ with $x*(y*z), y \in I$. Since

$$\begin{aligned}(x*(y*z))^n*(y*(x*z)) &= (x*(y*z))^{n-1}*[(x*(y*z))*(y*(x*z))]\\ &= (x*(y*z))^{n-1}*[(x*(y*z))*(x*(y*z))]\\ &= (x*(y*z))^{n-1}*1 = 1,\end{aligned}$$

we have $x*z \in A_n(x*(y*z),y) \subseteq I$. Hence (I4) holds. By Theorem 2.10, I is an ideal of X. □

Corollary 3.5. *Let X be a self distributive BE-algebra. A subset I $(\neq \emptyset)$ of X is an ideal of X if and only $A_n(u,v) \subseteq I$, $\forall u,v \in I$, where $n \in \mathbb{N}$.*

Proof. The proof follows from Proposition 2.9 and Theorem 2.10. □

Theorem 3.6. *Let X be a transitive BE-algebra. If I is an ideal of X, then*

$$I = \bigcup_{u,v \in I} A_n(u,v),$$

where $n \in \mathbb{N}$.

Proof. Let I be an ideal of X and let $x \in I$. Obviously, $x \in A_n(u,1)$ and so

$$I \subseteq \bigcup_{x \in I} A_n(x,1) \subseteq \bigcup_{u,v \in I} A_n(u,v).$$

Now, let $y \in \cup_{u,v \in I} A(u,v)$. Then there exist $a, b \in I$ such that $y \in A_n(a,b) \subseteq I$ by Theorem 3.4. Hence $y \in I$. Therefore $\cup_{u,v \in I} A_n(u,v) \subseteq I$. This completes the proof. □

Corollary 3.7. *Let X be a self distributive BE-algebra. If I is an ideal of X, then*

$$I = \bigcup_{u,v \in I} A_n(u,v),$$

where $n \in \mathbb{N}$.

Proof. The proof follows from Proposition 2.9 and Theorem 3.6. □

Corollary 3.8. *Let X be a transitive BE-algebra. If I is an ideal of X, then*

$$I = \bigcup_{w \in I} A_n(w,1),$$

where $n \in \mathbb{N}$.

Corollary 3.9. *Let X be a self distributive BE-algebra. If I is an ideal of X, then*

$$I = \bigcup_{w \in I} A_n(w,1),$$

where $n \in \mathbb{N}$.

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