

THE ATOMIC DECOMPOSITION OF HARMONIC BERGMAN FUNCTIONS, DUALITIES AND TOEPLITZ OPERATORS

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ABSTRACT. On the setting of the unit ball of \mathbb{R}^n , we consider a Banach space of harmonic functions motivated by the atomic decomposition in the sense of Coifman and Rochberg [5]. First we identify its dual (resp. predual) space with certain harmonic function space of (resp. vanishing) logarithmic growth. Then we describe these spaces in terms of boundedness and compactness of certain Toeplitz operators.

1. Introduction

Let B be the unit ball of \mathbb{R}^n ($n \geq 2$) and V denote the Lebesgue volume measure on B . Given $1 \leq p < \infty$, the harmonic Bergman space b^p is the space of all complex-valued harmonic functions f on B such that

$$\|f\|_p = \left\{ \int_B |f|^p dV \right\}^{1/p} < \infty.$$

As is well-known, the harmonic Bergman space b^p is a closed subspace of the Lebesgue space $L^p = L^p(B, V)$. Since each point evaluation is a bounded linear functional on b^2 for each $x \in B$, there exists a unique function $R(x, \cdot) \in b^2$ which has the following reproducing property:

$$(1) \quad f(x) = \int_B f(y) \overline{R(x, y)} dV(y)$$

for all $f \in b^2$. The kernel $R(x, y)$ is the well-known harmonic Bergman kernel whose explicit formula is given by

$$R(x, y) = \frac{(n-4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)(1 - 2x \cdot y + |x|^2|y|^2)^{1+\frac{n}{2}}}, \quad y \in B,$$

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where $x \cdot y$ denotes the usual inner product for points $x, y \in \mathbb{R}^n$. Thus, the kernel $R(x, y)$ is real and the complex conjugate in the integral of (1) can be removed. It is known that the reproducing property (1) still remains valid for every functions in b^1 . See Chapter 8 of [1] for details and related facts.

In [5], Coifman and Rochberg proved that every function in b^1 admits an atomic decomposition based on the weighted harmonic Bergman kernels. To be more precise, let us introduce some notation. Given $\alpha \geq -1$, we let $R_\alpha(x, y)$ be the reproducing kernel for the weighted harmonic Bergman space with respect to the weight $(1 - |x|)^\alpha$. So, $R_0 = R$ is the usual harmonic Bergman kernel mentioned above. For the explicit formula of R_α and related facts, see [14] or Chapter 3 of [5].

The result [5, Theorem 3] of Coifman and Rochberg implies that every function in b^1 admits the following atomic decomposition: Given an integer $m > 0$, there exist a sequence $\{a_j\}$ in B and a constant C such that every $f \in b^1$ can be represented as

$$(2) \quad f(x) = \sum_{j=1}^{\infty} \lambda_j R_m(x, a_j) (1 - |a_j|)^m, \quad x \in B$$

for some sequence $\{\lambda_j\} \in \ell^1$ with

$$\sum_{j=1}^{\infty} |\lambda_j| \leq C \|f\|_1.$$

Here and in what follows, the notation ℓ^p denotes the usual p -summable sequence space. On the other hand, using the similar argument as in Proposition 8 of [11], together with Lemma 3.2 of [5], we see that for each $m > 0$, there exist constants $C_1, C_2 > 0$ such that

$$\frac{C_1}{(1 - |x|)^m} \leq \|R_m(\cdot, x)\|_1 \leq \frac{C_2}{(1 - |x|)^m}$$

for all $x \in B$. Hence the decomposition (2) yields

$$f(x) = \sum_{j=1}^{\infty} \lambda_j \frac{R_m(x, a_j)}{\|R_m(\cdot, a_j)\|_1}, \quad x \in B$$

for some sequence $\{\lambda_j\} \in \ell^1$.

Recalling $R_0 = R$, one may naturally consider functions in b^1 which admit the following atomic decomposition as the limiting case of $m = 0$:

$$(3) \quad f(x) = \sum_{j=1}^{\infty} \lambda_j \frac{R(x, a_j)}{\|R(\cdot, a_j)\|_1}, \quad x \in B$$

for some $\{a_j\}$ in B and $\{\lambda_j\} \in \ell^1$.

Motivated by such atomic decomposition, we consider the space of harmonic Bergman integrals of certain complex Borel measures. To be more precise, we

put $R_x(y) = R(x, y)$ for convenience and let \mathfrak{M} denote the space of all complex Borel measures μ on B for which

$$\int_B \|R_x\|_1 d|\mu|(x) < \infty,$$

where $|\mu|$ is the total variation of μ . We let Y be the space of all harmonic functions f on B of the form

$$f(x) = \int_B R(x, y) d\mu(y)$$

for some $\mu \in \mathfrak{M}$ and a norm $\|f\|_*$ of $f \in Y$ is defined by the quotient norm

$$\|f\|_* = \inf \int_B \|R_x\|_1 d|\mu|(x),$$

where the infimum is taken over all $\mu \in \mathfrak{M}$ which represent f . Then, it is easy to check that Y equipped with the norm $\|\cdot\|_*$ is a Banach space. Also, one can see that $\|f\|_1 \leq \|f\|_*$ for $f \in Y$ by an application of Fubini's theorem. Hence $Y \subset b^1$. Moreover, since each R_x is represented by the unit point mass δ_x at x , we see $\|R_x\|_* \leq \|R_x\|_1$. It follows that $\|R_x\|_1 = \|R_x\|_*$ for all $x \in B$.

In this paper, we first identify the dual space and pre-dual space of Y with some natural function spaces. Then we describe those dual and pre-dual spaces in terms of the boundedness and compactness of certain Toeplitz operators acting on the harmonic Bergman space b^2 . To introduce two function spaces, we note $\|R_x\|_1 > 0$ for all $x \in B$; see Section 2. Put $\eta(x) = \|R_x\|_1^{-1}$ for notational simplicity. Let X denote the space of all harmonic functions f on B such that

$$\|f\|_{**} = \sup_{x \in B} |f(x)|\eta(x) < \infty.$$

In addition, if $f \in X$ satisfies the following boundary vanishing condition

$$\lim_{|x| \rightarrow 1} |f(x)|\eta(x) = 0,$$

then we say $f \in X_0$. By a standard argument, one can verify that the space X equipped with the norm $\|\cdot\|_{**}$ is a Banach space and X_0 is a closed subspace of X .

In this paper, we first establish the dualities $X_0^* = Y$ and $Y^* = X$ using the standard integral pairing. These results resemble known dualities between the harmonic Bergman space b^1 , the harmonic Bloch space and the harmonic little Bloch space; see [5] or [10] for details. As a consequence of these dualities, we show that Y is in fact the space of all harmonic functions which admit the atomic decomposition such as (3); see Proposition 8. Next, we characterize the spaces X and X_0 in terms of the boundedness and compactness of certain Toeplitz operators on b^2 . Also, we prove the corresponding result for Schatten p -class Toeplitz operators. See Section 4.

The holomorphic versions of these characterizations have been obtained in [7], [8] on the unit disk and in [3] on the unit ball of \mathbb{C}^n . Our results show

that the known characterizations on the holomorphic Bergman spaces continue to hold on the harmonic cases. While main scheme of our proofs are adapted from [3], we need to establish corresponding theories for the harmonic Bergman spaces.

2. The space Y

In this section, we investigate some useful properties of the space Y . We first have the following growth estimate of $\|R_x\|_1$. By Lemma 3.1 of [2], we see that there exists a positive constant C , depending only on n , such that

$$(4) \quad C^{-1} \leq \|R_x\|_1 \left\{ 1 + \log \frac{1}{1-|x|} \right\}^{-1} \leq C$$

for all $x \in B$. It follows that

$$(5) \quad \int_B \|R_x\|_1 dV(x) < \infty.$$

Let h^∞ be the space of all bounded harmonic functions on B . Also, let \mathcal{A} denote the space of all functions harmonic on B and continuous on \bar{B} . Given $f \in h^\infty$, we have by the reproducing property (1)

$$f(x) = \int_B R(x, y) d\mu(y), \quad x \in B,$$

where $\mu = f dV$. On the other hand, by (5), we see

$$\int_B \|R_x\|_1 d|\mu|(x) \leq \|f\|_\infty \int_B \|R_x\|_1 dV(x) < \infty.$$

Thus we have $h^\infty \subset Y$. Moreover, we will show that \mathcal{A} is actually dense in Y . To prove this, we recall that the harmonic Bergman kernel can be expressed in terms of the so-called zonal harmonics. To be more precise, let $S = \partial B$. A spherical harmonic of degree m is the restriction to S of a harmonic homogeneous polynomial on \mathbb{R}^n of degree m . We write $\mathcal{H}_m(S)$ for the space of all spherical harmonics of degree m . It is known that each $\mathcal{H}_m(S)$ is a finite dimensional Hilbert space with respect to the usual inner product in $L^2(S, d\sigma)$, where σ is the normalized surface-area measure on S . For each $\zeta \in S$, the linear functional $p \mapsto p(\zeta)$ on $\mathcal{H}_m(S)$ is uniquely represented by a harmonic m -homogeneous polynomial $Z_m(\cdot, \zeta)$ called the zonal harmonic of degree m at pole ζ . Extending Z_m to a function on $\mathbb{R}^n \times \mathbb{R}^n$ by setting

$$Z_m(x, y) = |x|^m |y|^m Z_m\left(\frac{x}{|x|}, \frac{y}{|y|}\right).$$

It is known that $Z_m(\cdot, y)$ is a harmonic m -homogeneous polynomial on \mathbb{R}^n for each $y \in B$. Also, letting h_m be the dimension of $\mathcal{H}_m(S)$, we have $Z_m(\zeta, \eta) = h_m$ for all $\zeta, \eta \in S$ and

$$(6) \quad \frac{h_m}{m^{n-2}} \rightarrow \frac{2}{(n-2)!} \quad \text{as} \quad m \rightarrow \infty.$$

See Chapter 5 of [1] for details and related facts. It turns out that the harmonic Bergman kernel can be expressed in terms of the zonal harmonics as follows:

$$(7) \quad R(x, y) = \frac{1}{nV(B)} \sum_{m=0}^{\infty} (n+2m) Z_m(x, y), \quad x, y \in B.$$

Moreover, the series converges absolutely and uniformly on $K \times B$ for every compact subsets $K \subset B$. See Theorem 8.9 of [1].

The following proposition shows that \mathcal{A} is a dense subset of Y .

Proposition 1. *The space \mathcal{A} is dense in Y .*

Proof. Let $f \in Y$ and assume

$$f(x) = \int_B R(x, y) d\mu(y), \quad x \in B$$

for some $\mu \in \mathfrak{M}$. For $0 < r < 1$, let ψ_r be the function represented by the restriction of μ to $r\bar{B}$. Thus

$$\psi_r(x) = \int_{r\bar{B}} R(x, y) d\mu(y), \quad x \in B.$$

We first show that each ψ_r belongs to \mathcal{A} . Fix $r \in (0, 1)$. Note that

$$|Z_m(x, y)| \leq |x|^m |y|^m Z_m\left(\frac{x}{|x|}, \frac{y}{|y|}\right) \leq r^m h_m$$

for all $|x| \leq 1$ and $|y| \leq r$. Hence, for each $N \geq 1$, we have

$$(8) \quad \begin{aligned} & \left| \psi_r(x) - \frac{1}{nV(B)} \sum_{m=0}^N (n+2m) \int_{r\bar{B}} Z_m(x, y) d\mu(y) \right| \\ & \leq \frac{1}{nV(B)} \sum_{m=N+1}^{\infty} (n+2m) \int_{r\bar{B}} |Z_m(x, y)| d|\mu|(y) \\ & \leq \frac{|\mu|(B)}{nV(B)} \sum_{m=N+1}^{\infty} (n+2m) r^m h_m, \quad x \in \bar{B}. \end{aligned}$$

On the other hand, using (6), we see that the series $\sum_{m=1}^{\infty} (n+2m) r^m h_m$ converges. It follows from (8) that ψ_r is a uniform limit of harmonic polynomials on \bar{B} and hence $\psi_r \in \mathcal{A}$. We now show that ψ_r converges to f in Y as $r \rightarrow 1$. For each r , we note

$$f(x) - \psi_r(x) = \int_{r < |y| < 1} R(x, y) d\mu(y), \quad x \in B$$

and therefore

$$\|f - \psi_r\|_* \leq \int_{r < |x| < 1} \|R_x\|_1 d|\mu|(x) \rightarrow 0$$

as $r \rightarrow 1$. This completes the proof. \square

Next, we show that functions in \mathcal{A} can be approximated in Y by discrete sums based on the harmonic Bergman kernels. Before proceeding, we need a couple of two lemmas. The first one shows that the unit ball can be decomposed into smaller balls in a canonical way. Since the proof is a trivial modification of that of Corollary 1 of [7], we omit the details.

In the following, we write the notation $B(a, r)$ for the usual Euclidean ball centered at $a \in B$ with radius $r > 0$. Also, $\text{dist}(E, F)$ denotes the Euclidean distance between two sets E and F .

Lemma 2. *Given $\beta > 0$, there exists a family of balls $\{B_k\} = \{B_k(a_k, r_k)\}$ in B such that*

- (a) $2r_k \leq \min\{\beta, \text{dist}(B_k, S)\}$, $k = 1, 2, \dots$
- (b) $V(B \setminus \cup_{k=1}^{\infty} B_k) = 0$.
- (c) $B_j \cap B_k = \emptyset$ whenever $j \neq k$.

We also need the following characterization of harmonic functions on B in terms of the volume version of the mean value property.

Lemma 3. *Let $f \in L^1$. Then f is harmonic on B if and only if*

$$f(a) = \frac{1}{V(B(a, r))} \int_{B(a, r)} f dV$$

whenever $\overline{B(a, r)} \subset B$.

Proof. See (1.3) and Theorem 1.21 of [1]. □

We are now ready to prove that functions in \mathcal{A} can be approximated in Y by the discrete sums involving L^1 -normalized harmonic Bergman kernels. The following shows the space Y can be viewed as the “completion” of the set of functions in b^1 which admit the atomic decomposition as in (3). The term “completion” will be justified in Proposition 8.

Theorem 4. *For $f \in \mathcal{A}$ and $\epsilon > 0$, there exist sequences $\{\lambda_j\}$ in \mathbb{C} and $\{a_j\}$ in B such that*

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty \quad \text{and} \quad \left\| f - \sum_{j=1}^{\infty} \lambda_j \frac{R_{a_j}}{\|R_{a_j}\|_1} \right\|_* < \epsilon.$$

Proof. By (5), we note

$$N := \int_B \|R_x\|_1 dV(x) < \infty.$$

By the uniform continuity of f , there exists $\delta > 0$ such that $x_1, x_2 \in B$ and $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \epsilon/N$. With this δ in place of β in Lemma 2, choose a family $\{B_j\} = \{B_j(a_j, r_j)\}$ of Euclidean balls satisfying

three conditions in Lemma 2. Using the reproducing property (1) and conditions (b), (c) in Lemma 2, one obtains

$$f(x) = \sum_{j=1}^{\infty} \int_{B_j} f(y) R_x(y) dV(y), \quad x \in B.$$

Note $R_x(y) = R_y(x)$ for every $x, y \in B$. It follows from Lemma 3 that

$$\begin{aligned} (9) \quad f(x) &= \sum_{j=1}^{\infty} \int_{B_j} [f(y) - f(a_j)] R_x(y) dV(y) + \sum_{j=1}^{\infty} f(a_j) \int_{B_j} R_x(y) dV(y) \\ &= \sum_{j=1}^{\infty} \int_{B_j} [f(y) - f(a_j)] R_x(y) dV(y) + \sum_{j=1}^{\infty} \lambda_j \frac{R_{a_j}(x)}{\|R_{a_j}\|_1}, \end{aligned}$$

where $\lambda_j = f(a_j) \|R_{a_j}\|_1 V(r_j B)$. On the other hand, by condition (a) in Lemma 2, one can check that

$$(10) \quad \frac{1}{1 - |a_j|} \leq \frac{4}{3(1 - |y|)}$$

for all $y \in B_j$ and $j = 1, 2, \dots$. Also, by (4), we see

$$\|R_{a_j}\|_1 \leq C \left(1 + \log \frac{1}{1 - |a_j|} \right), \quad j = 1, 2, \dots$$

for some constant C independent on j . It follows from (10) that

$$\begin{aligned} \sum_{j=1}^{\infty} |\lambda_j| &\leq C \|f\|_{\infty} \sum_{j=1}^{\infty} V(r_j B) \left(1 + \log \frac{1}{1 - |a_j|} \right) \\ &\leq C \|f\|_{\infty} \sum_{j=1}^{\infty} \int_{B_j} \left(1 + \log \frac{4}{3} + \log \frac{1}{1 - |y|} \right) dV(y) \\ &= C \|f\|_{\infty} \int_B \left(1 + \log \frac{4}{3} + \log \frac{1}{1 - |y|} \right) dV(y) \\ &< \infty, \end{aligned}$$

which gives the first part of the results. To prove the second part, we put $d\mu = \sum_{j=1}^{\infty} [f - f(a_j)] \chi_{B_j} dV$, where χ_K is the characteristic function of $K \subset B$. Then, we have by (9)

$$f(x) - \sum_{j=1}^{\infty} \lambda_j \frac{R_{a_j}(x)}{\|R_{a_j}\|_1} = \int_B R(x, y) d\mu(y), \quad x \in B.$$

By condition (a) in Lemma 2, we note that $|y - a_j| < \delta$ for $y \in B_j$. It follows that

$$\left\| f - \sum_{j=1}^{\infty} \lambda_j \frac{R_{a_j}}{\|R_{a_j}\|_1} \right\|_* \leq \sum_{j=1}^{\infty} \int_{B_j} \|R_y\|_1 |f(y) - f(a_j)| dV(y)$$

$$\begin{aligned}
&\leq \frac{\epsilon}{N} \int_B \|R_y\|_1 dV(y) \\
&= \epsilon,
\end{aligned}$$

so we have the second part of the results. The proof is complete. \square

Combining Proposition 1 with Theorem 4, we obtain the following corollary which will be used in the proof of the duality $Y^* = X$.

Corollary 5. *The set of finite sums of the form $\sum_{j=1}^N \lambda_j R_{a_j}$ is dense in Y .*

3. Dualities

In this section we prove that the dual of Y is X and its predual is X_0 under the standard integral pairing. The basic idea comes from [7] or [3] where the holomorphic version of these dualities have been established.

For $r \in (0, 1)$ and a function f on B , we let f_r denote the r -dilated function defined by $f_r(x) = f(rx)$ for $x \in B$. Note that each dilation of a harmonic function is also harmonic on B . Moreover, we can see that for $f \in X$, $f \in X_0$ if and only if $\|f - f_r\|_{**} \rightarrow 0$ as $r \rightarrow 1$ using the same argument as in the proof of Proposition 4 of [6, Section 2.6] where the similar characterization is proved for the well known holomorphic little Bloch functions of the unit disk. Since $\eta(x) \rightarrow 0$ as $|x| \rightarrow 1$ by (4), we note $\mathcal{A} \subset X_0$. In particular, \mathcal{A} is densely contained in X_0 . This fact will be used in the proof of dualities below.

Theorem 6. *Every $f \in Y$ induces $\Lambda_f \in X_0^*$ defined by*

$$\Lambda_f h = \int_B h f dV, \quad h \in \mathcal{A}$$

and, conversely, to each $\Lambda \in X_0^$ there corresponds a unique function $f \in Y$ such that $\Lambda = \Lambda_f$. Moreover, the operator norm of Λ_f is equal to $\|f\|_*$.*

Proof. Let $f \in Y$ and suppose f is represented by $\mu \in \mathfrak{M}$. Recall that $R_x(y) = R_y(x)$ for all $x, y \in B$. Applying Fubini's theorem and using (1), one obtains

$$\Lambda_f(h) = \int_B \int_B h(x) R_y(x) dV(x) d\mu(y) = \int_B h(y) d\mu(y)$$

and hence

$$|\Lambda_f(h)| \leq \int_B |h(y)| d|\mu|(y) \leq \|h\|_{**} \int_B \|R_y\|_1 d|\mu|(y)$$

for every $h \in \mathcal{A}$. This is true for all $\mu \in \mathfrak{M}$ which represent f . It follows that $|\Lambda_f(h)| \leq \|h\|_{**} \|f\|_*$ for all $h \in \mathcal{A}$. Since \mathcal{A} is dense in X_0 , each $f \in Y$ induces a bounded linear functional $\Lambda_f \in X_0^*$ with $\|\Lambda_f\| \leq \|f\|_*$.

Next assume that $\Lambda \in X_0^*$. It suffices to show that there exists a unique function $f \in Y$ such that $\Lambda = \Lambda_f$ on \mathcal{A} . To do so, we first define a linear operator $\Pi : X_0 \rightarrow C_0(B)$ by $\Pi h = h\eta$. Here $C_0(B)$ denotes the space of all functions continuous on B and vanishing on S . Then Π is an isometry of X_0 into $C_0(B)$. By the Hahn-Banach extension theorem, $\Lambda \circ \Pi^{-1}$ extends to

a bounded linear functional on $C_0(B)$. By the Riesz representation theorem, there exists a complex Borel measure μ on B such that $|\mu|(B) = \|\Lambda \circ \Pi^{-1}\|$ and

$$\Lambda \circ \Pi^{-1}(g) = \int_B g d\mu, \quad g \in C_0(B).$$

In particular, if $h \in \mathcal{A}$, using (1) and Fubini's theorem, we see

$$\Lambda(h) = \Lambda \circ \Pi^{-1} \circ \Pi(h) = \int_B h \eta d\mu = \int_B h f dV,$$

where

$$f(y) = \int_B R_x(y) \eta(x) d\mu(x), \quad y \in B.$$

Now, since Π is an isometry, we obtain

$$\int_B \|R_x\|_1 \eta(x) d|\mu|(x) = |\mu|(B) = \|\Lambda \circ \Pi^{-1}\| = \|\Lambda\|,$$

so that $f \in Y$ and $\Lambda = \Lambda_f$ on \mathcal{A} . In addition, we have $\|f\|_* \leq \|\Lambda\| = \|\Lambda_f\|$. To prove the uniqueness, suppose $f \in Y$ induces a zero functional. Write

$$f(x) = \sum_{m=0}^{\infty} p_m(x), \quad x \in B,$$

where each p_m is a harmonic homogeneous polynomial of degree m (see, for example, [1, Theorem 1.27]). Since two harmonic homogeneous polynomials with different degree are orthogonal in L^2 , by applying the integral pairing to each p_m , we see that $p_m = 0$ for all m and hence $f = 0$. The proof is complete. \square

Theorem 7. *Every $h \in X$ induces $\Phi_h \in Y^*$ defined by*

$$\Phi_h(f) = \int_B f h dV, \quad f \in \mathcal{A}$$

and, conversely, to each $\Phi \in Y^$ there corresponds a unique function $h \in X$ such that $\Phi = \Phi_h$. Moreover, the operator norm of Φ_h is equal to $\|h\|_{**}$.*

Proof. Let $h \in X$ and consider functions of the form $f = \sum_{j=1}^N \lambda_j R_{a_j}$. Note $X \subset b^1$ by (4). Since $f \in Y$ is represented by $d\mu = \sum_{j=1}^N \lambda_j \delta_{a_j}$, we have by (1)

$$\Phi_h(f) = \sum_{j=1}^N \lambda_j \int_B h R_{a_j} dV = \int_B h d\mu.$$

It follows that

$$|\Phi_h(f)| \leq \int_B |h(w)| d|\mu|(w) \leq \|h\|_{**} \int_B \|R_w\|_1 d|\mu|(w)$$

and hence $|\Phi_h(f)| \leq \|h\|_{**} \|f\|_*$. Since such functions f form a dense subset of Y by Corollary 5, the above observation shows that every $h \in X$ induces a bounded linear functional $\Phi_h \in Y^*$ with $\|\Phi_h\| \leq \|h\|_{**}$.

Now assume $\Phi \in Y^*$ and define

$$h(x) = \Phi(R_x), \quad x \in B.$$

First, we will show $h \in X$. For $x, y \in B$, we note

$$|h(x) - h(y)| = |\Phi(R_x - R_y)| \leq \|\Phi\| \|R_x - R_y\|_*.$$

On the other hand, by Lemma 3, we have

$$\begin{aligned} & R_x(a) - R_y(a) \\ &= \frac{1}{V(B(0, r))} \left(\int_{B(x, r)} R_a dV - \int_{B(y, r)} R_a dV \right) \\ &= \frac{1}{V(B(0, r))} \left(\int_{B(x, r) \setminus B(y, r)} R_a dV - \int_{B(y, r) \setminus B(x, r)} R_a dV \right) \end{aligned}$$

for $a \in B$ and for some r sufficiently small. It follows that

$$(11) \quad \|R_x - R_y\|_* \leq \frac{1}{V(B(0, r))} \sup \|R_a\|_1 V[B(x, r) \triangle B(y, r)],$$

where the supremum is taken over all $a \in B(x, r) \cup B(y, r)$ and \triangle denotes the symmetric set difference. From this, we can see that h is continuous on B . Note that (11) implies the map $x \mapsto R_x$ is continuous from B into Y . Also, note from (5) that

$$\begin{aligned} \int_B |h| dV &\leq \|\Phi\| \int_B \|R_x\|_* dV(x) \\ &= \|\Phi\| \int_B \|R_x\|_1 dV(x) \\ &< \infty, \end{aligned}$$

where we use the fact that $\|R_x\|_* = \|R_x\|_1$ for all $x \in B$. Hence $h \in L^1$. It follows from Lemma 3 that

$$\begin{aligned} \frac{1}{V(B(a, r))} \int_{B(a, r)} h(x) dV(x) &= \Phi \left(\frac{1}{V(B(a, r))} \int_{B(a, r)} R_x(y) dV(x) \right) \\ &= \Phi(R_a) \\ &= h(a) \end{aligned}$$

whenever $\bar{B}(a, r) \subset B$. Therefore, by Lemma 3, we conclude that h is harmonic on B . Recall that $\|R_x\|_* = \|R_x\|_1 = 1/\eta(x)$ for $x \in B$. Hence

$$(12) \quad |h(x)|\eta(x) = |\Phi(R_x)|\eta(x) \leq \|\Phi\| \|R_x\|_*\eta(x) = \|\Phi\| < \infty$$

for every $x \in B$, so $h \in X$. Finally we show that $\Phi = \Phi_h$ on Y . To do this, we only prove $\Phi f = \Phi_h f$ for functions f of the form $f = \sum_{j=1}^N \lambda_j R_{a_j}$ by

Corollary 5. By (1), one obtains

$$\Phi(f) = \sum_{j=1}^N \lambda_j h(a_j) = \sum_{j=1}^N \lambda_j \int_B R_{a_j} h dV = \int_B f h dV = \Phi_h(f).$$

Hence $\Phi = \Phi_h$ and $\|h\|_* \leq \|\Phi\| = \|\Phi_h\|$ by (12). The uniqueness is easily seen as in the proof of Theorem 6. The proof is complete. \square

As a consequence of the duality $Y^* = X$, we now show that Y is in fact the space of all harmonic functions which admit the atomic decomposition as in (3). To show this, let us write Y' for the class of all such functions of the form (3) and define a norm on Y' by

$$\|f\|_{*'} = \inf \sum_{j=1}^{\infty} |\lambda_j|,$$

where the infimum is taken over all possible decompositions (3) of f .

Proposition 8. $Y' = Y$ and two norms are equivalent.

Proof. Clearly, we have $Y' \subset Y$ and $\|f\|_* \leq \|f\|_{*'}$ for $f \in Y'$. Now, let $f \in Y$. Consider the set $F \subset Y$ of all functions of the form $\sum_{j=1}^N \lambda_j R_{a_j}$. Let E denote the convex set of all functions $g \in F$ with $\|g\|_{*' } \leq \|f\|_*$. We claim $f \in \bar{E}$. Suppose not. Then, by the Hahn-Banach theorem, f and \bar{E} are separated by some bounded linear functional on Y . In particular, by Theorem 7, there is some $h \in X$ such that

$$\sup_{a \in B} \frac{|\Phi_h(R_a)|}{\|R_a\|_1} \|f\|_* < |\Phi_h(f)| \leq \|\Phi_h\| \|f\|_* = \|h\|_{**} \|f\|_*.$$

On the other hand, by (1), we see that the leftmost side of the above is precisely the same as $\|h\|_{**} \|f\|_*$. This is a contradiction and we have $f \in \bar{E}$. Now, using the same argument as in the proof of Proposition 7 of [3], we have $\|f\|_{*' } \leq \|f\|_*$. The proof is complete. \square

4. Toeplitz operators

In this section, we characterize the spaces X and X_0 in terms of boundedness and compactness of certain Toeplitz operators on b^2 . To begin with, we let P be the Hilbert space orthogonal projection from L^2 onto b^2 . Using the reproducing formula (1), we see that the projection P has the following integral representation:

$$Pf(x) = \int_B R(x, y) f(y) dV(y), \quad x \in B$$

for functions $f \in L^2$. Also, it turns out that the projection P extends to an integral operator via the above integral representation taking L^1 into the space of harmonic functions.

For a function $u \in L^1$, the Toeplitz operator T_u with symbol u is defined by

$$T_u f = P(uf)$$

whenever $uf \in L^1$. If $u \in L^\infty$, then Toeplitz operator T_u is bounded on b^2 and

$$(13) \quad \|T_u\| \leq \|u\|_\infty.$$

Here, the notation $\|T\|$ denotes the operator norm of a bounded operator T on b^2 . For positive symbols, there are several characterizations of symbols for associated Toeplitz operators to be bounded. See [12] on the ball and [4] on general bounded smooth domains. One of them is in terms of the averaging function of symbol. Given $x \in B$ and $0 < r < 1$, we let

$$E_r(x) = \{y \in B : |x - y| < r(1 - |x|)\}$$

denote the Euclidean ball centered at x with radius $r(1 - |x|)$. It is easy to see that

$$(14) \quad (1 - r)(1 - |x|) \leq (1 - |y|) \leq (1 + r)(1 - |x|)$$

for $y \in E_r(x)$. For $f \in L^1$ and $0 < r < 1$, the averaging function \widehat{f}_r of f over the ball $E_r(x)$ is defined on B by

$$\widehat{f}_r(x) = \frac{1}{V(E_r(x))} \int_{E_r(x)} f dV, \quad x \in B.$$

It turns out that the boundedness of Toeplitz operators with positive symbol can be characterized by the boundedness of the averaging function of its symbol as shown in the following which will be used in the description of the space X .

Lemma 9. *Let $u \in L^1$ be positive. Then T_u is bounded on b^2 if and only if \widehat{u}_r is bounded on B for every (or some) $0 < r < 1$. Moreover, for each r , $\|T_u\|$ is equivalent to the sup-norm of \widehat{u}_r .*

Proof. See Theorem 6 of [12] or Theorem 3.9 of [4]. □

To describe the space X_0 , we need the corresponding characterization of compact Toeplitz operators, together with the vanishing Berezin transform characterization. For $f \in L^1$, the Berezin transform \widetilde{f} of f is the function on B defined by

$$\widetilde{f}(x) = \frac{1}{\|R_x\|_2^2} \int_B u(y) |R(x, y)|^2 dV(y), \quad x \in B.$$

Lemma 10. *Let $u \in L^1$ be positive. Then the following statements are equivalent.*

- (a) T_u is compact on b^2 .
- (b) $\widehat{u}_r(x) \rightarrow 0$ as $|x| \rightarrow 1$ for every (or some) $0 < r < 1$.
- (c) $\widetilde{u}(x) \rightarrow 0$ as $|x| \rightarrow 1$.

Proof. See Theorem 7 of [12] or Theorem 3.12 of [4]. □

Since $E_r(x)$ is the Euclidean ball centered at x with radius $r(1-|x|)$, we have the following submean value type inequality whose the proof can be founded in Lemma 5 of [13] for example.

Lemma 11. *For $0 < p < \infty$, there exists a constant C , depending only on n and p , such that*

$$|f(x)|^p \leq \frac{C}{V(E_r(x))} \int_{E_r(x)} |f|^p dV, \quad x \in B$$

for every $r \in (0, 1)$ and f harmonic on B .

Now, we are ready to characterize the spaces X and X_0 in terms of boundedness and compactness of Toeplitz operators, respectively.

Theorem 12. *Let $f \in b^1$. Then $T_{|f|\eta}$ is bounded on b^2 if and only if $f \in X$. Moreover, $\|T_{|f|\eta}\|$ is equivalent to $\|f\|_{**}$.*

Proof. Assume $f \in X$. Then $\|T_{|f|\eta}\| \leq \|f\eta\|_\infty = \|f\|_{**}$ by definition of X and (13). Now assume that $T_{|f|\eta}$ is bounded and show $f \in X$. By (4), there exist $\ell_0 \in (0, 1)$ and a constant $C > 0$, depending only on n , such that

$$(15) \quad \frac{1}{C} \log \frac{1}{1-|a|} \leq \|R_a\|_1 \leq C \log \frac{1}{1-|a|}$$

whenever $|a| \geq \ell_0$. Fix $0 < r < 1$ and put $\rho = \max\{\frac{\ell_0+r}{1+r}, r\}$. Note $\rho > \ell_0$. Using (14), we can see that if $|x| \geq \rho$ and $y \in E_r(x)$, then $|y| \geq \ell_0$. Also, note that $\log \frac{1}{1-r} \leq \log \frac{1}{1-|x|}$ for $|x| \geq \rho$. It follows from (14) and (15) that

$$\begin{aligned} \|R_y\|_1 &\leq C \log \frac{1}{1-|y|} \\ &\leq C \left\{ \log \frac{1}{1-r} + \log \frac{1}{1-|x|} \right\} \\ &\leq 2C \log \frac{1}{1-|x|} \\ &\leq 2C^2 \|R_x\|_1 \end{aligned}$$

for every $|x| \geq \rho$ and $y \in E_r(x)$. Since $\eta(x) = \|R_x\|_1^{-1}$, the above implies

$$\eta(x) \leq 2C^2 \inf\{\eta(y) : y \in E_r(x)\}$$

for all $|x| \geq \rho$. It follows from Lemma 11 (with $p = 1$) that there exists another constant $C > 0$, depending only on n and r , such that

$$(16) \quad |f(x)|\eta(x) \leq \frac{C}{V(E(x, r))} \int_{E(x, r)} |f|\eta dV = C(\widehat{|f|\eta})_r(x)$$

for $|x| \geq \rho$. On the other hand, by Lemma 9, the averaging function of $|f|\eta$ is bounded and its sup-norm is equivalent to the operator norm of $T_{|f|\eta}$. Hence

we have

$$(17) \quad \sup_{\rho \leq |x| < 1} |f(x)|\eta(x) \leq C \|T_{|f|\eta}\|$$

for some constant C depending only on n and r . On the other hand, by (4), we have

$$\sup_{x \in B} \eta(x) < \infty \quad \text{and} \quad \sup_{|x|=\rho} \|R_x\|_1 \leq C$$

for some constant C depending only on n and r . It follows from the maximum principle for harmonic functions and (17) that

$$\begin{aligned} \sup_{|x| \leq \rho} |f(x)|\eta(x) &\leq C \sup_{|x|=\rho} |f(x)| \\ &= C \sup_{|x|=\rho} |f(x)|\eta(x) \|R_x\|_1 \\ &\leq C \sup_{|x|=\rho} |f(x)|\eta(x) \\ &\leq C \|T_{|f|\eta}\| \end{aligned}$$

for some constant C depending only on n and r . Combining the above with (17), we see that

$$\|f\|_{**} = \sup_{x \in B} |f(x)|\eta(x) \leq C \|T_{|f|\eta}\|$$

for some constant C depending only on n and r . Hence $f \in X$ and $\|T_{|f|\eta}\|$ is equivalent to $\|f\|_{**}$. The proof is complete. \square

Next, we prove a corresponding result for the compactness. In the proof, we use the following growth estimates for $x \in B$:

$$(18) \quad \frac{1}{C_1} \leq \|R_x\|_2^2 (1 - |x|)^n \leq C_1$$

and

$$(19) \quad |R_x(y)| \leq \frac{C_2}{(1 - |x||y|)^n}, \quad y \in B$$

for some positive constants C_1, C_2 depending only on n . See Proposition 4 of [11] for example.

Theorem 13. *Let $f \in b^1$. Then $T_{|f|\eta}$ is compact on b^2 if and only if $f \in X_0$.*

Proof. Let $u = |f|\eta$ for simplicity. First assume $f \in X_0$. Since $u(x) \rightarrow 0$ as $|x| \rightarrow 1$, given $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that $u(x) < \epsilon$ whenever $\delta < |x| < 1$. By (18) and (19), we have for $x \in B$

$$\begin{aligned} \tilde{u}(x) &= \frac{1}{\|R_x\|_2^2} \int_B u |R_x|^2 dV \\ &\leq C(1 - |x|)^n \int_{\delta B} u |R_x|^2 dV + \frac{1}{\|R_x\|_2^2} \int_{B \setminus \delta B} u |R_x|^2 dV \end{aligned}$$

$$\begin{aligned} &\leq C \frac{\|u\|_\infty V(\delta B)}{(1-\delta)^{2n}} (1-|x|)^n + \frac{\epsilon}{\|R_x\|_2^2} \int_B |R_x|^2 dV \\ &= C(1-|x|)^n + \epsilon \end{aligned}$$

for some constants C independent of x . Since ϵ is arbitrary, this implies $\tilde{u}(x) \rightarrow 0$ as $|x| \rightarrow 1$. Hence T_u is compact on b^2 by Lemma 10.

Conversely, assume T_u is compact on b^2 . By (16), one can see that for a fixed $r \in (0, 1)$ and $x \in B$ near the boundary, $u(x) \leq C(\widehat{|f|\eta})_r(x)$ for some constant C independent of x . Combining this with Lemma 10 again, we have $u(x) \rightarrow 0$ as $|x| \rightarrow 1$. Hence $f \in X_0$ and the proof is complete. \square

Remark. The proofs of Theorems 12 and 13 show that in fact a little more hold: For $0 < p < \infty$ and $f \in b^1$, $T_{|f|\eta^p}$ is bounded (resp. compact) on b^2 if and only if $f \in X$ (resp. X_0).

Finally, we characterize functions $f \in b^1$ for which the Toeplitz operators $T_{|f|\eta}$ belong to Schatten p -class on b^2 . Before proceeding, we recall the notion of Schatten class operators. For a compact operator T on a separable Hilbert space H , it is known that there exist orthonormal sets $\{e_m\}$ and $\{\sigma_m\}$ in H such that

$$Tx = \sum_m \lambda_m \langle x, e_m \rangle \sigma_m, \quad x \in H,$$

where $\{\lambda_m\}$ is the n th singular value of T and $\langle \cdot, \cdot \rangle$ denotes the inner product on H . For $1 \leq p < \infty$, we define the Schatten p -class of H , denoted $S_p(H)$, to be the space of all compact operators T with singular value sequence $\{\lambda_m\}$ belonging to ℓ^p . Of course, we will take $H = b^2$ in our applications below and, in that case, we put $S_p = S_p(b^2)$.

We also need a characterization for Schatten p -class Toeplitz operators with positive symbol. For the proof, see [12, Theorem 11] or [4, Theorem 3.13]. In the following, the measure $d\lambda$ is defined on B by $d\lambda(x) = (1-|x|)^{-n} dV(x)$.

Lemma 14. *Let $1 \leq p < \infty$ and $u \in L^1$ be positive. Then $T_u \in S_p$ if and only if $\hat{u}_r \in L^p(B, d\lambda)$ for every (or some) $0 < r < 1$.*

As a final result, we prove that there is no nontrivial Toeplitz operator $T_{|f|\eta}$ in the Schatten p -class of b^2 .

Theorem 15. *Let $1 \leq p < \infty$ and $f \in b^1$. Then $T_{|f|\eta} \in S_p$ if and only if $f = 0$ on B .*

Proof. Suppose $T_{|f|\eta} \in S_p$. By (4), we recall that

$$\left\{ 1 + \log \frac{1}{1-|x|} \right\}^{-1} \leq C\eta(x), \quad x \in B$$

for some constant $C > 0$ depending on n . Using (16) and Lemma 14, we can see

$$N := \int_B |f(x)|^p \left(1 + \log \frac{1}{1-|x|} \right)^{-p} d\lambda(x) < \infty.$$

Recall that the integral $\int_{\partial B} |f(r\zeta)|^p d\sigma(\zeta)$ is increasing with r ; see Chapter 2 of [9] for example. Now, using the integrating in polar coordinates, we obtain

$$\begin{aligned} N &\geq nV(B) \int_{1-t}^{1-\frac{t}{2}} \frac{r^{n-1}}{(1-r)^n \left(1 + \log \frac{1}{1-r}\right)^p} \int_S |f(r\zeta)|^p d\sigma(\zeta) dr \\ &\geq \frac{nV(B)(1-t)^{n-1}}{2t^{n-1} \left(1 + \log \frac{2}{t}\right)^p} \int_S |f((1-t)\zeta)|^p d\sigma(\zeta) \end{aligned}$$

and hence

$$(20) \quad \int_S |f((1-t)\zeta)|^p d\sigma(\zeta) \leq \frac{2Nt^{n-1} \left(1 + \log \frac{2}{t}\right)^p}{nV(B)(1-t)^{n-1}}$$

for any $0 < t < 1$. Let $|x| < 1/2 < 1-t < 1$. Since the $(1-t)$ -dilated function f_{1-t} is continuous on \bar{B} and harmonic on B , we have

$$f(x) = \int_S \frac{1 - \left|\frac{x}{1-t}\right|^2}{\left|\frac{x}{1-t} - \zeta\right|^n} f((1-t)\zeta) d\sigma(\zeta);$$

see Chapter 1 of [1]. It follows from Jensen's inequality and (20) that

$$\begin{aligned} |f(x)|^p &\leq \left(\frac{2-2t}{1-2t}\right)^p \int_S |f((1-t)\zeta)|^p d\sigma(\zeta) \\ &\leq \left(\frac{2-2t}{1-2t}\right)^p \frac{2Nt^{n-1} \left(1 + \log \frac{2}{t}\right)^p}{nV(B)(1-t)^{n-1}} \end{aligned}$$

whenever $|x| < \frac{1}{2} < 1-t < 1$. Since the last term goes to 0 as $t \rightarrow 0$, we have $f(x) = 0$ for all $|x| < \frac{1}{2}$. Hence $f = 0$ on B , as desired. The converse implication is clear. The proof is complete. \square

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