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ON RICCI CURVATURES OF LEFT INVARIANT METRICS ON $SU(2)$

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Abstract. In this paper, we shall prove several results concerning Ricci curvature of a Riemannian manifold $(M,g) := (SU(2),g)$ with an arbitrary given left invariant metric g.

First of all, we obtain the maximum (resp. minimum) of $\{r(X) := Ric(X,X) \mid ||X||_g = 1, X \in \mathfrak{X}(M)\}$, where Ric is the Ricci tensor field on (M,g), and then get a necessary and sufficient condition for the Levi-Civita connection ∇ on the manifold (M,g) to be projectively flat. Furthermore, we obtain a necessary and sufficient condition for the Ricci curvature $r(X)$ to be always positive (resp. negative), independently of the choice of unit vector field X.

1. Introduction

One of the present authors ([5]) completely classified harmonic inner automorphisms of $(SU(2),g)$ with arbitrary given left invariant metric g onto $(SU(2),g)$.

In general, for the Ricci tensor field Ric on a Riemannian manifold (M,g),

$$r(X) = \frac{Ric(X,X)}{||X||_g^2}, \quad (X \neq 0)$$

is said to be the *Ricci curvature* of (M,g) with respect to $X \in T_pM$, $(p \in M)$.

In this paper, we completely estimate the Ricci curvatures on $(SU(2),g)$ with an arbitrary given left invariant Riemannian metric g (Theorem 2.3). And then, we obtain a necessary and sufficient condition for the Ricci curvature $r(X)$ to be always positive (resp. negative), independently of the choice of nonzero vector X (Theorem 2.5). In general, the Levi-Civita connection for the metric g of a constant curvature space (M,g) is projectively flat, but the converse is not true. We get a necessary and sufficient condition for the Levi-Civita connection ∇ for g on $(SU(2),g)$ with a left invariant Riemannian metric g to

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be projectively flat (Proposition 2.6). Using these results, we get the following (Theorem 2.8):

On $(SU(2), g)$ with a left invariant metric g, the following are equivalent;

(a) The Levi-Civita connection for the metric g is projectively flat.
(b) $(SU(2), g)$ is a space of constant curvature.
(c) The metric g is the bi-invariant metric which is induced by the Killing form on the simple Lie algebra $\mathfrak{su}(2)$.
(d) For an arbitrary given nonzero vector $X \in T_x SU(2)$ $(x \in SU(2))$, the Ricci curvature $r(X)$ with respect to X is $\frac{1}{4}$.

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2. Ricci curvatures of left invariant metrics on $SU(2)$

Let M denote the Lie group $SU(2)$ and let $\mathfrak{su}(2)$ be the Lie algebra of all left invariant vector fields on $SU(2)$. The Killing form B of the simple Lie algebra $\mathfrak{su}(2)$ satisfies

$$B(X, Y) = 4\ \mathrm{Trace}(XY), \quad (X, Y \in \mathfrak{su}(2)).$$

We define an inner product $\langle\ ,\ \rangle_0$ on $\mathfrak{su}(2)$ by

$$\langle X, Y\rangle_0 \ := -B(X, Y), \quad (X, Y \in \mathfrak{su}(2)). \tag{2.1}$$

Then the inner product $\langle\ ,\ \rangle_0$ determines a left invariant metric g_0 on M. The following lemma is known ([8, Lemma 1.1, p. 154]).

Lemma 2.1. *Let g be an arbitrary left invariant Riemannian metric on M and let $\langle\ ,\ \rangle$ be an inner product on $\mathfrak{su}(2)$ defined by*

$$\langle X, Y\rangle \ := g_e\langle X_e, Y_e\rangle, \quad (X, Y \in \mathfrak{su}(2)),$$

where e is the identity matrix of M. Then there exists an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $\langle\ ,\ \rangle$ such that

$$\begin{cases} [X_1, X_2] = \frac{1}{\sqrt{2}}X_3, & [X_2, X_3] = \frac{1}{\sqrt{2}}X_1, \\ [X_3, X_1] = \frac{1}{\sqrt{2}}X_2, & \langle X_i, X_j\rangle = \delta_{ij}a_i, \end{cases} \tag{2.2}$$

where a_i $(i = 1, 2, 3)$ are positive constant real numbers determined by the given left invariant Riemannian metric g on M.

We fix an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to g_0 with the property (2.2) in Lemma 2.1 and denote by $g_{(a_1,a_2,a_3)}$ the left invariant Riemannian metric on M which is determined by positive real numbers a_1, a_2, a_3 in Lemma 2.1. Moreover, we normalize left invariant Riemannian metrics on M by putting $a_3 = 1$. We denote by $g_{(a_1,a_2,1)}$, or simply by $g_{(a_1,a_2)}$, the left invariant Riemannian metric which is determined by positive real numbers $a_1, a_2, a_3 = 1$.

In general, the Riemannian connection ∇ for the Riemannian metric g on a Riemannian manifold (M, g) is given by:

$$\begin{aligned} 2g(\nabla_X Y, Z) = &\, Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]) \end{aligned} \tag{2.3}$$

for $X, Y, Z \in \mathfrak{X}M)$, and the curvature tensor field R is:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad (X, Y \in \mathfrak{X}(M)). \tag{2.4}$$

For the orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $\langle\ ,\ \rangle_0 = -B$ in Lemma 2.1, if we put

$$Y_1 := \frac{1}{\sqrt{a_1}} X_1, \quad Y_2 := \frac{1}{\sqrt{a_2}} X_2, \quad Y_3 := X_3,$$

then $\{Y_1, Y_2, Y_3\}$ is an orthonormal frame basis of $(M, g_{\langle\ ,\ \rangle} := g_{(a_1,a_2)})$. From (2.2) we have

$$[Y_1, Y_2] = \frac{1}{\sqrt{2a_1a_2}} Y_3, \quad [Y_2, Y_3] = \frac{\sqrt{a_1}}{\sqrt{2a_2}} Y_1, \quad [Y_3, Y_1] = \frac{\sqrt{a_2}}{\sqrt{2a_1}} Y_2. \tag{2.5}$$

By virtue of (2.3) and (2.5), we get

$$\begin{cases} \nabla_{Y_1} Y_2 = c^{-1}(-a_1 + a_2 + 1)Y_3, & \nabla_{Y_2} Y_1 = c^{-1}(-a_1 + a_2 - 1)Y_3, \\ \nabla_{Y_2} Y_3 = c^{-1}(a_1 - a_2 + 1)Y_1, & \nabla_{Y_3} Y_2 = c^{-1}(-a_1 - a_2 + 1)Y_1, \\ \nabla_{Y_3} Y_1 = c^{-1}(a_1 + a_2 - 1)Y_2, & \nabla_{Y_1} Y_3 = c^{-1}(a_1 - a_2 - 1)Y_2, \\ \nabla_{Y_1} Y_1 = \nabla_{Y_2} Y_2 = \nabla_{Y_3} Y_3 = 0, & \end{cases} \tag{2.6}$$

where $c := \sqrt{8a_1a_2}$. Furthermore, from (2.4) and (2.6), we obtain

$$\begin{cases} R(Y_1, Y_2)Y_1 = c^{-2}\{(3 - 2(a_1 + a_2) - (a_1 - a_2)^2\}Y_2, \\ R(Y_1, Y_2)Y_2 = c^{-2}\{-3 + 2(a_1 + a_2) + (a_1 - a_2)^2\}Y_1, \\ R(Y_1, Y_3)Y_1 = c^{-2}\{3{a_2}^2 - 2(1 + a_1)a_2 - (1 - a_1)^2\}Y_3, \\ R(Y_1, Y_3)Y_3 = c^{-2}\{-3{a_2}^2 + 2(1 + a_1)a_2 + (1 - a_1)^2\}Y_1, \\ R(Y_2, Y_3)Y_2 = c^{-2}\{3{a_1}^2 - 2(a_2 + 1)a_1 - (a_2 - 1)^2\}Y_3, \\ R(Y_2, Y_3)Y_3 = c^{-2}\{-3{a_1}^2 + 2(a_2 + 1)a_1 + (a_2 - 1)^2\}Y_2, \\ R(Y_1, Y_2)Y_3 = R(Y_2, Y_3)Y_1 = R(Y_3, Y_1)Y_2 = 0. \end{cases} \tag{2.7}$$

The Ricci tensor field Ric, of type (0,2), is defined by

$$Ric(Y, Z) = \text{trace}\{X \mapsto R(X, Y)Z\}. \tag{2.8}$$

By help of (2.7) and (2.8), we obtain

$$
(2.9)\qquad \begin{cases} Ric(Y_1,Y_1) = \frac{1}{4a_1a_2}\{{a_1}^2 - (a_2-1)^2\}, \\ Ric(Y_2,Y_2) = \frac{1}{4a_1a_2}\{{a_2}^2 - (a_1-1)^2\}, \\ Ric(Y_3,Y_3) = \frac{1}{4a_1a_2}\{1 - (a_1-a_2)^2\}, \\ Ric(Y_i,Y_j) = 0 \ \text{ if } \ i \neq j. \end{cases}
$$

To simplify notation, we put $Ric(Y_i,Y_j) = R_{ij}$. Then, from (2.9) we get:

Lemma 2.2. *On* $(M, g_{\langle\ ,\ \rangle} := g_{(a_1,a_2)})$, *we have the following equations*;

$$
\begin{cases} R_{22} - R_{11} = \frac{1}{2a_1a_2}\ (a_2-a_1)(a_2+a_1-1), \\ R_{33} - R_{22} = \frac{1}{2a_1a_2}\ (1-a_2)(a_2-a_1+1), \\ R_{11} - R_{33} = \frac{1}{2a_1a_2}\ (1-a_1)(a_2-a_1-1). \end{cases}
$$

For the Ricci curvature tensor Ric of (0,2)-type in a Riemannian manifold (M,g) and a nonzero vector $v_p \in T_pM$,

$$
r(v_p) := \frac{Ric(v_p,v_p)}{||v_p||_g^2}
$$

is said to be the *Ricci curvature* of (M,g) with respect to v_p. Now, we have from (2.9) and Lemma 2.2.

Theorem 2.3. *Let X be an arbitrary nonzero vector field which is left invariant on $SU(2)$. Then the Ricci curvatures $r(X)$ is completely estimated as follows*:

(a) $r(Y_1) \leq r(X) \leq r(Y_3)$ (*resp.* $r(Y_1) \leq r(X) \leq r(Y_2)$) *if and only if*

$$
(a_2-a_1)(a_2+a_1-1) \geq 0 \ \ \textit{and} \ \ (a_2-1)(a_2-a_1+1) \leq 0
$$
$$
(\textit{resp.}\ (a_1-1)(a_2-a_1-1) \geq 0 \ \ \textit{and} \ \ (a_2-1)(a_2-a_1+1) \geq 0),
$$

(b) $r(Y_2) \leq r(X) \leq r(Y_3)$ (*resp.* $r(Y_2) \leq r(X) \leq r(Y_1)$) *if and only if*

$$
(a_2-a_1)(a_2+a_1-1) \leq 0 \ \ \textit{and} \ \ (a_1-1)(a_2-a_1-1) \geq 0
$$
$$
(\textit{resp.}\ (a_2-1)(a_2-a_1+1) \leq 0 \ \ \textit{and} \ \ (a_1-1)(a_2-a_1-1) \leq 0),
$$

(c) $r(Y_3) \leq r(X) \leq r(Y_2)$ (*resp.* $r(Y_3) \leq r(X) \leq r(Y_1)$) *if and only if*

$$
(a_1-1)(a_2-a_1-1) \leq 0 \ \ \textit{and} \ \ (a_2-a_1)(a_2+a_1-1) \geq 0
$$
$$
(\textit{resp.}\ (a_2-1)(a_2-a_1+1) \geq 0 \ \ \textit{and} \ \ (a_2-a_1)(a_2+a_1-1) \leq 0).
$$

By help of (2.9), the scalar curvature $S = \sum_{i=1}^{3} R_{ii}$ on $(SU(2), g_{(a_1,a_2)})$ is given by

$$
(2.10)\qquad S = -\frac{1}{4a_1a_2}(a_2 - a_1 - 2\sqrt{a_1} - 1)(a_2 - a_1 + 2\sqrt{a_1} - 1).
$$

From (2.10), we get:

Proposition 2.4. *For the scalar curvature S on $(SU(2), g_{(a_1,a_2)})$,*
(a) $S > 0$ *if and only if* $(a_2 - a_1 - 1)^2 < 4a_1$,
(b) $S = 0$ *if and only if* $(a_2 - a_1 - 1)^2 = 4a_1$,
(c) $S < 0$ *if and only if* $(a_2 - a_1 - 1)^2 < 4a_1$.

By virtue of (2.9) we obtain:

Theorem 2.5. *For any nonzero tangent vector X of $SU(2)$, the Ricci curvature $r(X)$ on $(SU(2), g_{(a_1,a_2)})$ is always positive (resp. negative) if and only if*

$$a_2 + a_1 - 1 > 0 \ (\textit{resp. } a_2 + a_1 - 1 < 0), \quad a_2 - a_1 - 1 < 0, \quad a_2 - a_1 + 1 > 0.$$

Moreover, for any given left invariant Riemannian metric g on $SU(2)$, there exists a nonzero left invariant vector field Y such that $r(Y) \neq 0$.

Now, we introduce the notion of projectively flat connection in the tangent bundle TM of an n-dimensional manifold M. We say that two affine connections D and $\tilde{D}$ are *projectively equivalent* if there exists a 1-form τ on M such that

$$\tilde{D}_X Y = D_X Y + \tau(X)Y + \tau(Y)X, \quad (X, Y \in \mathfrak{X}(M)). \tag{2.11}$$

And, we say that an affine connection D is *projectively flat* if D is projectively equivalent to a flat affine connection in a neighborhood of an arbitrary point of M.

Suppose that D is torsion free and the Ricci tensor Ric^D is symmetric. We define the *projective curvature tensor* W_p by

$$W_p(X,Y)Z = R^D(X,Y)Z - \frac{1}{n-1}\{Ric^D(Y,Z)X - Ric^D(X,Z)Y\}. \tag{2.12}$$

If affine connections D and $\tilde{D}$ are projectively equivalent, then their projective curvature tensors coincide. For $n \geq 3$, an affine connection D is projectively flat if and only if its projective curvature tensor vanishes ([4, Theorem 3.3, p. 18]).

From this point of view, we obtain:

Proposition 2.6. *The Levi-Civita connection ∇ on $(SU(2), g_{(a_1,a_2)})$ is projectively flat if and only if $a_1 = a_2 = 1$.*

Proof. For the orthonormal frame $\{Y_1, Y_2, Y_3\}$ on $(SU(2), g_{(a_1,a_2)})$, from (2.12), we prove that

$$R^\nabla(Y_i, Y_j)Y_k = \frac{1}{2}\{Ric^\nabla(Y_j, Y_k)Y_i - Ric^\nabla(Y_i, Y_k)Y_j\} \tag{2.13}$$

if and only if $a_1 = a_2 = 1$ for any $i, j, k = 1, 2, 3$.

Suppose that the connection ∇ is projectively flat. Then, for indices (i, j, k) which appear in (2.13), we get the following from (2.7), (2.9) and $W_p = 0$ in (2.12);

(a) $(a_2-a_1+1)(a_2-1)=0$ in the cases $(i,j,k)=(1,2,1),\ (1,3,1),\ (2,1,1)$ and $(3,1,1)$,
(b) $(a_1-a_2+1)(a_1-1)=0$ in the cases $(i,j,k)=(2,1,2),\ (2,3,2),\ (1,2,2)$ and $(3,2,2)$,
(c) $(a_1+a_2-1)(a_1-a_2)=0$ in the cases $(i,j,k)=(3,1,3),\ (3,2,3),\ (1,3,3)$ and $(2,3,3)$.

Furthermore, from the properties of curvature tensor field, (2.7) and (2.9), we get

$$R^{\nabla}(Y_i,Y_j)Y_k=\frac{1}{2}\{Ric^{\nabla}(Y_j,Y_k)Y_i-Ric^{\nabla}(Y_i,Y_k)Y_j\}=0$$

except for the above 3-cases. From these facts, we obtain the fact that $a_1=a_2=1$.

Conversely, suppose that $a_1=a_2=1$ in $(SU(2),g_{(a_1,a_2)})$. Then, by virtue of (2.7), (2.9) and (2.12), we get $W_p=0$ on M. So, the Levi-Civita connection ∇ in $(SU(2),g_{(1,1)})$ is projectively flat.

Thus, the proof of this proposition is completed. □

From (2.9), we obtain:

Lemma 2.7. *A necessary and sufficient condition for the metric $g_{(a_1,a_2)}$ on $(SU(2),g_{(a_1,a_2)})$ to be Einstein is $a_1=a_2=1$.*

Remark. It is well known that *three dimensional Einstein manifold is a space of constant curvature* ([2, Proposition 2, p. 293]). In general, *a Riemannian manifold (M,g) is a space of constant curvature if and only if the Levi-Civita connection for the Riemannian metric g is projectively flat.* Wolf ([9]) showed the fact that *three dimensional nilpotent Lie group with an invariant metric is not constant curvature space.* As a result on the existence of Einstein metric on a Lie group, the following theorem ([3, Theorem 2.4, p. 301]) is well known:

Let G be a nilpotent Lie group. Then there does not exist any left invariant Einstein metric on G.

From the above remark, (2.9) and Proposition 2.6, we obtain:

Theorem 2.8. *On $(SU(2),g_{(a_1,a_2)})$, the following statements are equivalent;*
(a) *The Levi-Civita connection ∇ for the left invariant metric $g_{(a_1,a_2)}$ on $(SU(2),g_{(a_1,a_2)})$ is projectively flat.*
(b) *The Riemannian manifold $(SU(2),g_{(a_1,a_2)})$ is a space of constant curvature.*
(c) *In the Riemannian manifold $(SU(2),g_{(a_1,a_2)})$, $a_1=a_2=1$.*
(d) *For any nonzero tangent vector X of $SU(2)$, the Ricci curvature $r(X)$ with respect to X is $\frac{1}{4}$.*

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