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NOTES ON THE SUPERSTABILITY OF D'ALEMBERT TYPE FUNCTIONAL EQUATIONS

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Abstract. In this paper we will investigate the superstability of the generalized d'Alembert type functional equations $\sum_{i=1}^{m} f(x+\sigma^i(y)) = kg(x)f(y)$ and $\sum_{i=1}^{m} f(x+\sigma^i(y)) = kf(x)g(y)$.

1. Introduction

D'Alembert functional equation

$$f(x+y)+f(x-y)=2f(x)f(y), \tag{1.1}$$

also called the cosine functional equation, has a long history going back to J. D'Alembert [4]. The equation (1.1) plays an important role in determining the sum of two vectors in various Euclidean and non-Euclidean geometries.

The superstability of the d'Alembert functional equation (1.1) originated from J. A. Baker [2] under the condition $|f(x+y)+f(x-y)-2f(x)f(y)| \leq \varepsilon$. R. Badora and R. Ger [1] generalized this result, by replacing ε by $\varepsilon(x)$ or $\varepsilon(y)$. Recently, G. H. Kim investigated the superstability of the generalized d'Alembert type functional equations as follows:

$$f(x+y)+f(x+\sigma y)=2f(x)f(y), \tag{1.2}$$

$$f(x+y)+f(x+\sigma y)=2g(x)f(y), \tag{1.3}$$

$$f(x+y)+f(x+\sigma y)=2f(x)g(y). \tag{1.4}$$

Some special cases of the aforementioned functional equations were investigated (see, e.g., [3, 5, 6, 7, 8, 10, 11]).

The main purpose of this paper is to study the superstability of the more general d'Alembert type functional equations

$$\sum_{i=1}^{m} f(x+\sigma^i(y)) = kg(x)f(y), \tag{1.5}$$

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$$\sum_{i=1}^{m} f(x+\sigma^i(y)) = kf(x)g(y), \tag{1.6}$$

where m is a positive integer and k is an arbitrary nonzero complex number.

As an application, we obtain the superstability of functional equation

$$\sum_{i=1}^{m} f(x+\sigma^i(y)) = kf(x)f(y). \tag{1.7}$$

In this paper, let ($\mathbf{G}$,+) be an Abelian group, denote by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, as usual, the set of positive integer, the real and complex number field, respectively. Moreover, let σ be an endomorphism of $\mathbf{G}$ with $\sigma^m(x) = x$. We may assume that f and g are nonzero functions, and $\varphi : \mathbf{G} \to \mathbb{R}$.

2. Superstability of the equation (1.5)

Theorem 1. *Suppose that* $f, g : \mathbf{G} \to \mathbb{C}$ *satisfy the inequality*

$$\Big|\sum_{i=1}^{m} f(x+\sigma^i(y)) - kg(x)f(y)\Big| \le \begin{cases} \text{(i) } \varphi(x) \\ \text{(ii) } \varphi(y) \textit{ and } \varphi(x) \end{cases} \tag{2.1}$$

for all $x, y \in \mathbf{G}$. *Then*

(i) *either* f *is bounded or* g *satisfies* (1.7),

(ii) *either* g (*or* f) *is bounded or* g *satisfies* (1.7),

also f *and* g *satisfy* (1.5) *and* (1.6).

Proof. For the case (i), let f be unbounded. Then we can choose a sequence $(y_n)_{n>0}$ in $\mathbf{G}$ such that $|f(y_n)| \neq 0$ and

$$|f(y_n)| \to \infty \quad \text{as} \quad n \to \infty. \tag{2.2}$$

We will show that g satisfies (1.7). Taking $y = y_n$ in (i) of (2.1) we obtain

$$\left|\frac{\sum_{i=1}^{m} f(x+\sigma^i(y_n))}{kf(y_n)} - g(x)\right| \le \frac{\varphi(x)}{|kf(y_n)|}.$$

Taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{m} f(x+\sigma^i(y_n))}{kf(y_n)} = g(x) \tag{2.3}$$

for all $x \in \mathbf{G}$. Using (i) of (2.1) we have

$$\left|\sum_{j=1}^{m}\sum_{i=1}^{m} f\Big(x+\sigma^i\big(y+\sigma^j(y_n)\big)\Big) - kg(x)\sum_{j=1}^{m} f\big(y+\sigma^j(y_n)\big)\right| \le m\varphi(x) \tag{2.4}$$

for all $x, y \in \mathbf{G}$ and every $n \in \mathbb{N}$. Consider the condition $\sigma^m(x) = x$, we obtain

$$\sum_{i=1}^{m} f(x+\sigma^i(y)) \equiv \sum_{i=1}^{m} f(x+\sigma^{s+i}(y)) \tag{2.5}$$

for all $x, y \in \mathbf{G}$ and every $s \in \mathbb{N}$. Then by (2.5) and (2.4), we have

$$\begin{aligned} & \left|\sum_{i=1}^{m} \frac{\sum_{j=1}^{m} f\Big(\big(x+\sigma^i(y)\big)+\sigma^j(y_n)\Big)}{kf(y_n)} - kg(x)\frac{\sum_{j=1}^{m} f\big(y+\sigma^j(y_n)\big)}{kf(y_n)}\right| \\ = & \left|\sum_{i=1}^{m} \frac{\sum_{j=1}^{m} f\Big(\big(x+\sigma^i(y)\big)+\sigma^{i+j}(y_n)\Big)}{kf(y_n)} - kg(x)\frac{\sum_{j=1}^{m} f\big(y+\sigma^j(y_n)\big)}{kf(y_n)}\right| \\ = & \frac{\left|\sum_{j=1}^{m}\sum_{i=1}^{m} f\Big(x+\sigma^i\big(y+\sigma^j(y_n)\big)\Big) - kg(x)\sum_{j=1}^{m} f\big(y+\sigma^j(y_n)\big)\right|}{|kf(y_n)|} \\ \leq & \frac{m\varphi(x)}{|kf(y_n)|} \end{aligned}$$

for all $x, y \in \mathbf{G}$. By virtue of (2.3), we have

$$\left|\sum_{i=1}^{m} g(x+\sigma^i(y)) - kg(x)g(y)\right| = 0$$

for all $x, y \in \mathbf{G}$. Therefore g satisfies (1.7).

For the proof of the case (ii), first we show that f (or g) is unbounded if and only if g (or f) is also unbounded. Putting $y = 0$ in (ii) of (2.1) we obtain

$$|mf(x) - kg(x)f(0)| \leq \varphi(0) \tag{2.6}$$

for all $x \in \mathbf{G}$. If g is bounded, then by (2.6), we have

$$|f(x)| \leq \frac{1}{m}|kg(x)f(0)| + \frac{1}{m}\varphi(0),$$

which shows that f is also bounded. On the other hand, if f is bounded, we choose $y_0 \in \mathbf{G}$ such that $f(y_0) \neq 0$, and then by (2.1) we obtain

$$\left|\frac{\sum_{i=1}^{m} f\big(x+\sigma^i(y_0)\big)}{kf(y_0)} - g(x)\right| \leq \frac{\varphi(y_0)}{|kf(y_0)|}$$

and it follows that g is also bounded on $\mathbf{G}$.

Namely, if f (or g) is unbounded, then so is g (or f).

Let g be unbounded, then f is also unbounded. Then we can choose sequences $(x_n)_{n>0}$ and $(y_n)_{n>0}$ in $\mathbf{G}$ such that $|g(x_n)| \neq 0$ and $|g(x_n)| \to \infty$, $|f(y_n)| \neq 0$ and $|f(y_n)| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ in (ii) of (2.1) we deduce

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{m} f\big(x_n+\sigma^i(y)\big)}{kg(x_n)} = f(y) \tag{2.7}$$

for all $y \in \mathbf{G}$. Using (ii) of (2.1) we have

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{m} f\Big(\big(x_n+\sigma^i(x)\big)+\sigma^j(y)\Big) - k\sum_{i=1}^{m} g\big(x_n+\sigma^i(x)\big)f(y)\right| \leq m\varphi(y) \tag{2.8}$$

for all $x, y \in \mathbf{G}$ and every $n \in \mathbb{N}$. Then by (2.5) and (2.8), we obtain

$$\begin{aligned}
&\left|\sum_{j=1}^{m} \frac{\sum_{i=1}^{m} f\Big(x_n + \sigma^i\big(x + \sigma^j(y)\big)\Big)}{kg(x_n)} - k\frac{\sum_{i=1}^{m} g\big(x_n + \sigma^i(x)\big)}{kg(x_n)} f(y)\right| \\
= &\left|\frac{\sum_{i=1}^{m}\sum_{j=1}^{m} f\Big(\big(x_n + \sigma^i(x)\big) + \sigma^{i+j}(y)\Big)}{kg(x_n)} - k\frac{\sum_{i=1}^{m} g\big(x_n + \sigma^i(x)\big)}{kg(x_n)} f(y)\right| \\
= &\frac{\left|\sum_{i=1}^{m}\sum_{j=1}^{m} f\Big(\big(x_n + \sigma^i(x)\big) + \sigma^{j}(y)\Big) - k\sum_{i=1}^{m} g\big(x_n + \sigma^i(x)\big) f(y)\right|}{|kg(x_n)|} \\
\leq &\frac{m\varphi(y)}{|kg(x_n)|}
\end{aligned}$$

for all $x, y \in \mathbf{G}$ and every $n \in \mathbb{N}$. Passing here to the limit as $n \to \infty$ with the use of $|g(x_n)| \to \infty$ and (2.7). Since g satisfies (1.7) by (i), we have $\big|\sum_{j=1}^{m} f\big(x + \sigma^j(y)\big) - kg(x)f(y)\big| = 0$, that is to say f and g are solutions of (1.5).

Applying (ii) of (2.1) again, we get

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{m} f\Big(\big(x_n + \sigma^i(y)\big) + \sigma^j(x)\Big) - k\sum_{i=1}^{m} g\big(x_n + \sigma^i(y)\big) f(x)\right| \leq m\varphi(x)$$

and using (2.5), we have

$$\begin{aligned}
&\left|\sum_{i=1}^{m} \frac{\sum_{j=1}^{m} f\Big(x_n + \sigma^j\big(x + \sigma^i(y)\big)\Big)}{kg(x_n)} - kf(x)\frac{\sum_{i=1}^{m} g\big(x_n + \sigma^i(y)\big)}{kg(x_n)}\right| \\
= &\left|\sum_{j=1}^{m} \frac{\sum_{i=1}^{m} f\Big(\big(x_n + \sigma^j(x)\big) + \sigma^{j+i}(y)\Big)}{kg(x_n)} - kf(x)\frac{\sum_{i=1}^{m} g\big(x_n + \sigma^i(y)\big)}{kg(x_n)}\right| \\
= &\left|\sum_{j=1}^{m} \frac{\sum_{i=1}^{m} f\Big(\big(x_n + \sigma^j(x)\big) + \sigma^{i}(y)\Big)}{kg(x_n)} - kf(x)\frac{\sum_{i=1}^{m} g\big(x_n + \sigma^i(y)\big)}{kg(x_n)}\right| \\
= &\frac{\left|\sum_{i=1}^{m}\sum_{j=1}^{m} f\Big(\big(x_n + \sigma^i(y)\big) + \sigma^{j}(x)\Big) - kf(x)\sum_{i=1}^{m} g\big(x_n + \sigma^i(y)\big)\right|}{|kg(x_n)|} \\
\leq &\frac{m\varphi(x)}{|kg(x_n)|}
\end{aligned}$$

for all $x, y \in \mathbf{G}$ and every $n \in \mathbb{N}$. Using (2.7) and the fact that g satisfies (1.7) by (i), we have $\big|\sum_{i=1}^{m} f\big(x + \sigma^i(y)\big) - kf(x)g(y)\big| = 0$, that is to say f and g are solutions of (1.6). $\square$

In the case of $m = k = 2$ in Theorem 1, we can obtain the following corollary, which was investigated by G. H. Kim [9].

Corollary 1 ([9]). *Suppose that* $f, g : \mathbf{G} \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) + f\big(x+\sigma(y)\big) - 2g(x)f(y)| \le \begin{cases} \text{(i) } \varphi(x) \\ \text{(ii) } \varphi(y) \text{ and } \varphi(x) \end{cases}$$

for all $x, y \in \mathbf{G}$. *Then*

(i) *either* f *is bounded or* g *satisfies* (1.2),
(ii) *either* g (*or* f) *is bounded or* g *satisfies* (1.2),

also f *and* g *satisfy* (1.3) *and* (1.4).

If $g = f$ in Theorem 1, then the stability problem of the functional equation (1.7) is proved as a corollary.

Corollary 2. *Suppose that* $f : \mathbf{G} \to \mathbb{C}$ *satisfies the inequality*

$$\left|\sum_{i=1}^{m} f\big(x+\sigma^i(y)\big) - kf(x)f(y)\right| \le \begin{cases} \text{(i) } \varphi(x) \\ \text{(ii) } \varphi(y) \text{ and } \varphi(x) \end{cases}$$

for all $x, y \in \mathbf{G}$. *Then, in all cases* (i) *and* (ii), *either* f *is bounded or* f *satisfies* (1.7).

3. Superstability of the equation (1.6)

We will prove the stability of (1.6) using a strategy similar to that of Theorem 1.

Theorem 2. *Suppose that* $f, g : \mathbf{G} \to \mathbb{C}$ *satisfy the inequality*

$$\left|\sum_{i=1}^{m} f\big(x+\sigma^i(y)\big) - kf(x)g(y)\right| \le \begin{cases} \text{(i) } \varphi(y) \\ \text{(ii) } \varphi(x) \text{ and } \varphi(y) \end{cases} \tag{3.1}$$

for all $x, y \in \mathbf{G}$. *Then*

(i) *either* f *is bounded or* g *satisfies* (1.7),
(ii) *either* g (*or* f) *with* $f(\sigma x) = f(x)$ *is bounded or* g *satisfies* (1.7),

also f *and* g *satisfy* (1.5) *and* (1.6).

Proof. For the case (i), let f be unbounded. Then we can choose a sequence $(x_n)_{n>0}$ in $\mathbf{G}$ such that $|f(x_n)| \ne 0$ and

$$|f(x_n)| \to \infty \quad \text{as} \quad n \to \infty. \tag{3.2}$$

We will show that g satisfies (1.7). Taking $x = x_n$ in (3.1) we obtain

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{m} f(x_n + \sigma^i(y))}{kf(x_n)} = g(y) \tag{3.3}$$

for all $y \in \mathbf{G}$. Using (i) of (3.1) we have

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{m} f\Big(\big(x_n+\sigma^i(x)\big)+\sigma^j(y)\Big) - k\sum_{i=1}^{m} f\big(x_n+\sigma^i(x)\big)g(y)\right| \le m\varphi(y) \tag{3.4}$$

for all $x, y \in \mathbf{G}$ and every $n \in \mathbb{N}$. Then by (2.5) and (3.4), we obtain

$$\begin{aligned}
&\left|\sum_{j=1}^{m} \frac{\sum_{i=1}^{m} f\Big(x_n+\sigma^i\big(x+\sigma^j(y)\big)\Big)}{kf(x_n)} - k\frac{\sum_{i=1}^{m} f\big(x_n+\sigma^i(x)\big)}{kf(x_n)}g(y)\right| \\
= &\left|\sum_{i=1}^{m} \frac{\sum_{j=1}^{m} f\Big(\big(x_n+\sigma^i(x)\big)+\sigma^{i+j}(y)\Big)}{kf(x_n)} - k\frac{\sum_{i=1}^{m} f\big(x_n+\sigma^i(x)\big)}{kf(x_n)}g(y)\right| \\
= &\frac{\left|\sum_{i=1}^{m}\sum_{j=1}^{m} f\Big(\big(x_n+\sigma^i(x)\big)+\sigma^j(y)\Big) - k\sum_{i=1}^{m} f\big(x_n+\sigma^i(x)\big)g(y)\right|}{|kf(x_n)|} \\
\le &\frac{m\varphi(y)}{|kf(x_n)|}
\end{aligned}$$

for all $x, y \in \mathbf{G}$. By virtue of (3.2) and (3.3), we have

$$\left|\sum_{j=1}^{m} g\big(x+\sigma^j(y)\big) - kg(x)g(y)\right| = 0$$

for all $x, y \in \mathbf{G}$. Therefore g satisfies (1.7).

For the case (ii), we can see that, similar to Theorem 1, f (or g) is unbounded if and only if g (or f) is also unbounded. Namely, if f is bounded, choose $x_0 \in \mathbf{G}$ such that $f(x_0) \neq 0$ and use (ii) of (3.1) to get

$$|g(y)| - \frac{\left|\sum_{i=1}^{m} f\big(x_0+\sigma^i(y)\big)\right|}{|kf(x_0)|} \le \left|\frac{\sum_{i=1}^{m} f\big(x_0+\sigma^i(y)\big)}{kf(x_0)} - g(y)\right| \le \frac{\varphi(x_0)}{|kf(x_0)|},$$

which shows that g is also bounded.

Suppose f is unbounded. Putting $x = 0$ in (ii) of (3.1), we have

$$\Big|\sum_{i=1}^{m} f(\sigma^i y) - kf(0)g(y)\Big| \le \varphi(0),$$

that is, $|mf(y) - kf(0)g(y)| \le \varphi(0)$, since $f(\sigma x) = f(x)$ for all $x \in \mathbf{G}$. This implies that g is also unbounded.

Let g be unbounded, then f is also unbounded. Then we can choose sequences $(x_n)_{n>0}$ and $(y_n)_{n>0}$ in $\mathbf{G}$ such that $|f(x_n)| \neq 0$ and $|f(x_n)| \to \infty$, $|g(y_n)| \neq 0$ and $|g(y_n)| \to \infty$ as $n \to \infty$.

Taking $y = y_n$ in (ii) of (3.1) we deduce

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{m} f\big(x+\sigma^i(y_n)\big)}{kg(y_n)} = f(x) \tag{3.5}$$

for all $x \in \mathbf{G}$. Using (ii) of (3.1) we have

$$\left|\sum_{j=1}^{m}\sum_{i=1}^{m} f\Big(x+\sigma^i\big(y+\sigma^j(y_n)\big)\Big) - kf(x)\sum_{j=1}^{m} g\big(y+\sigma^j(y_n)\big)\right| \le m\varphi(x) \tag{3.6}$$

for all $x, y \in \mathbf{G}$ and every $n \in \mathbb{N}$. Then by (2.5) and (3.6), we obtain

$$\begin{aligned}
&\left|\sum_{i=1}^{m} \frac{\sum_{j=1}^{m} f\Big(\big(x+\sigma^i(y)\big)+\sigma^j(y_n)\Big)}{kg(y_n)} - kf(x)\frac{\sum_{j=1}^{m} g\big(y+\sigma^j(y_n)\big)}{kg(y_n)}\right| \\
=& \left|\sum_{i=1}^{m} \frac{\sum_{j=1}^{m} f\Big(\big(x+\sigma^i(y)\big)+\sigma^{i+j}(y_n)\Big)}{kg(y_n)} - kf(x)\frac{\sum_{j=1}^{m} g\big(y+\sigma^j(y_n)\big)}{kg(y_n)}\right| \\
=& \frac{\left|\sum_{j=1}^{m}\sum_{i=1}^{m} f\Big(x+\sigma^i\big(y+\sigma^j(y_n)\big)\Big) - kf(x)\sum_{j=1}^{m} g\big(y+\sigma^j(y_n)\big)\right|}{|kg(y_n)|} \\
\le& \frac{m\varphi(x)}{|kg(y_n)|}
\end{aligned}$$

for all $x, y \in \mathbf{G}$. Since g satisfies (1.7), it follows from (3.5) that

$$\left|\sum_{i=1}^{m} f\big(x+\sigma^i(y)\big) - kf(x)g(y)\right| = 0$$

for all $x, y \in \mathbf{G}$. Hence f and g are solutions of (1.6).

Applying (ii) of (3.1) again, we get

$$\left|\sum_{j=1}^{m}\sum_{i=1}^{m} f\Big(y+\sigma^i\big(x+\sigma^j(y_n)\big)\Big) - kf(y)\sum_{j=1}^{m} g\big(x+\sigma^j(y_n)\big)\right| \le m\varphi(y) \tag{3.7}$$

for all $x, y \in \mathbf{G}$. Since $f(\sigma x) = f(x)$ for all $x \in \mathbf{G}$ and (3.7), we have

$$\begin{aligned}
&\left|\sum_{i=1}^{m} \frac{\sum_{j=1}^{m} f\Big(\big(x+\sigma^{m-i}(y)\big)+\sigma^j(y_n)\Big)}{kg(y_n)} - k\frac{\sum_{j=1}^{m} g\big(x+\sigma^j(y_n)\big)}{kg(y_n)}f(y)\right| \\
=& \left|\frac{\sum_{i=1}^{m}\sum_{j=1}^{m} f\Big(\sigma^i\big(x+\sigma^{m-i}(y)+\sigma^j(y_n)\big)\Big)}{kg(y_n)} - k\frac{\sum_{j=1}^{m} g\big(x+\sigma^j(y_n)\big)}{kg(y_n)}f(y)\right| \\
=& \frac{\left|\sum_{j=1}^{m}\sum_{i=1}^{m} f\Big(y+\sigma^i\big(x+\sigma^j(y_n)\big)\Big) - k\sum_{j=1}^{m} g\big(x+\sigma^j(y_n)\big)f(y)\right|}{|kg(y_n)|} \\
\le& \frac{m\varphi(y)}{|kg(y_n)|}
\end{aligned}$$

for all $x, y \in \mathbf{G}$. Since g satisfies (1.7), using (3.5), we have

$$\left|\sum_{i=1}^{m} f\big(x+\sigma^i(y)\big) - kg(x)f(y)\right| = \left|\sum_{i=1}^{m} f\big(x+\sigma^{m-i}(y)\big) - kg(x)f(y)\right| = 0$$

for all $x, y \in \mathbf{G}$. Therefore f and g are solutions of (1.5). □

In the case of $m = k = 2$ in Theorem 2, we can obtain the following corollary, which was investigated by G. H. Kim [9].

Corollary 3 ([9]). *Suppose that $f, g : \mathbf{G} \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f\big(x+\sigma(y)\big) - 2f(x)g(y)| \le \begin{cases} \text{(i) } \varphi(y) \\ \text{(ii) } \varphi(x) \text{ and } \varphi(y) \end{cases}$$

for all $x, y \in \mathbf{G}$. Then

(i) *either f is bounded or g satisfies* (1.2),

(ii) *either g (or f) with $f(\sigma x) = f(x)$ is bounded or g satisfies* (1.2),

also f and g satisfy (1.3) *and* (1.4).

If we apply the case $g = f$ to Theorem 2, then the stability problem of the functional equation (1.7) is proved as a corollary.

Corollary 4. *Suppose that $f : \mathbf{G} \to \mathbb{C}$ satisfies the inequality*

$$\left|\sum_{i=1}^{m} f\big(x+\sigma^i(y)\big) - kf(x)f(y)\right| \le \begin{cases} \text{(i) } \varphi(y) \\ \text{(ii) } \varphi(x) \text{ and } \varphi(y) \end{cases}$$

for all $x, y \in \mathbf{G}$. Then, in all cases (i) *and* (ii)*, either f is bounded or f satisfies* (1.7).

Example. Let $\sigma(x) = ax$, where a is a constant in $\mathbb{C}$ satisfying $a^m = 1$. Then Theorem 1 and Theorem 2 hold for the functional equations

$$\sum_{i=1}^{m} f(x + a^i y) = kg(x)f(y),$$
$$\sum_{i=1}^{m} f(x + a^i y) = kf(x)g(y).$$

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