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# CONGRUENCES FOR COEFFICIENTS OF DRINFELD MODULAR FORMS

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Abstract. We find congruences for the $t$-expansion coefficients of a Drinfeld modular form for $\Gamma_0^+(Q)$, where $Q$ is a monic irreducible polynomial. As an application we obtain new congruence relations for the $t$-expansion coefficients of Drinfeld modular forms for $GL_2(\mathbb{F}_q[T])$.

## 1. Introduction

By using the action of the Hecke operators Gekeler [4] and López [7] proved the existence of congruences for the coefficients of two distinguished Drinfeld modular forms, the Poincaré series $P_{q+1,1}$ and the discriminant function $\Delta$, respectively. Gallardo and López [2] showed that there exist congruences for the $s$-expansion coefficients of the Eisenstein series of weight $q^k-1$ for any positive integer $k$. By using the residue theorem, we [1] found divisibility properties for $t$-expansion coefficients of Drinfeld modular forms for $GL_2(\mathbb{F}_q[T])$ and as a consequence we obtained further congruence relations.

Let $Q$ be a monic irreducible polynomial in $\mathbb{F}_q[T]$. Let $\Gamma_0(Q)$ be the subgroup of $GL_2(\mathbb{F}_q[T])$ of matrices that are congruent to $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ modulo $Q$. Let $\Gamma_0^+(Q)$ be the subgroup of $GL_2(\mathbb{F}_q(T))$ generated by $\Gamma_0(Q)$ and the Atkin-Lehner involution $W_Q := \left(\begin{smallmatrix} 0 & -1 \\ Q & 0 \end{smallmatrix}\right)$. Let $X_{\Gamma_0^+(Q)}$ be the compactification of $\Gamma_0^+(Q)\backslash\Omega$, where $\Omega$ is the Drinfeld upper half-plane.

In this paper we define Drinfeld modular forms for $\Gamma_0^+(Q)$ of weight $k$ and type $m$. Let $M_k^m(\Gamma_0^+(Q))$ be the $C$-vector space of such Drinfeld modular forms. Then $M_k^m(\Gamma_0^+(Q))$ is a subspace of the space of Drinfeld modular forms for $\Gamma_0(Q)$ of weight $k$ and type $m$. We generalize results in [1] to these subspaces.

By using the residue theorem on the curve $X_{\Gamma_0^+(Q)}$ we find congruences for the $t$-expansion coefficients of Drinfeld modular forms for $\Gamma_0^+(Q)$ when $q$ is odd (Theorem 3.3). As an application, we obtain new congruence relations for the coefficients of Drinfeld modular forms for the modular group $GL_2(\mathbb{F}_q[T])$ (Theorem 3.6).

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## 2. Preliminary

Let $A$ be the ring $\mathbb{F}_q[T]$ of polynomials over a finite field $\mathbb{F}_q$ and $K$ the rational function field $\mathbb{F}_q(T)$ over $\mathbb{F}_q$. Let $C$ be the completion of the algebraic closure of the completion, $K_\infty$, of $K$ at $1/T$ and $\Omega := C - K_\infty$ the Drinfeld upper half plane. Then the group $GL_2(K)$ acts on $\Omega$ in the following way: if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(K)$ and $z \in \Omega$, then

$$\gamma z = \frac{az+b}{cz+d}.$$

Let $L = \tilde{\pi}A$ be the rank 1 $A$-lattice in $C$ associated to the Carlitz module $\rho$,

$$\rho_T(X) = TX + X^q.$$

Let $e_L$ be the exponential function associated to $L$, i.e.,

$$e_L : C \to C, \qquad e_L(z) = z \prod_{\lambda \in L - \{0\}} \left(1 - \frac{z}{\lambda}\right).$$

We let $t = t(z) := 1/(\tilde{\pi} e_A(z))$ and $s = s(z) := t(z)^{q-1}$. For any nonzero $a \in A$ we define $t_a = t_a(z) := t(az)$ and the $a$-th inverse cyclotomic polynomial $f_a(X) \in A[X]$ by

$$f_a(X) = \rho_a(X^{-1}) X^{q^{\deg a}}.$$

Then we have $t_a = t^{q^{\deg a}}/f_a(t)$.

Hereafter $Q$ is a monic irreducible polynomial in $A$. Let $\Gamma_0^+(Q)$ be the subgroup of $GL_2(K)$ generated by $\Gamma_0(Q)$ and the Atkin-Lehner involution $W_Q := \left(\begin{smallmatrix} 0 & -1 \\ Q & 0 \end{smallmatrix}\right)$. Let $X_{\Gamma_0^+(Q)}$ be the compactification of $\Gamma_0^+(Q)\backslash\Omega$ and $X_{\Gamma_0^+(Q)} - \Gamma_0^+(Q)\backslash\Omega := \{\infty\}$.

For a 1-form $\omega$ on $X_{\Gamma_0^+(Q)}$ we have the residue theorem ([6, Theorem 7.14.2]) as follows

$$\sum_{p \in X_{\Gamma_0^+(Q)}} \mathrm{Res}_p \omega = 0.$$

**Lemma 2.1.** *$s$ is a uniformizing parameter at $\infty$ for $X_{\Gamma_0^+(Q)}$ .*

*Proof.* This is the same proof as (2.7) in [5]. □

## 3. Congruences for coefficients of Drinfeld modular forms for $\Gamma_0^+(Q)$

We can define a Drinfeld modular form for $\Gamma_0^+(Q)$ of weight $k$ and type $m$ as follows: The usual slash operator $f|_k^m\gamma$ is defined as

$$(f|_k^m\gamma)(z) := (\det\gamma)^m (cz+d)^{-k} f(\gamma z),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(K)$.

A (respectively meromorphic) Drinfeld modular form for $\Gamma_0^+(Q)$ of weight $k$ and type $m$ (where $k \geq 0$ and $m$ are integers) is a holomorphic (respectively meromorphic) function $f : \Omega \to C$ that satisfies:

(i) $(f|_k^m\gamma)(z) = f(z)$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^+(Q)$,
(ii) $f$ is holomorphic (respectively meromorphic) at the cusp $\infty$.

The condition (ii) says that $f$ has a power series (respectively Laurent series) expansion at the cusp. If $f$ is a meromorphic Drinfeld modular form for $\Gamma_0^+(Q)$ of weight $k$ and type $m$, then the $t$-expansion of $f$ is of the form

$$f = \sum_i a_f((q-1)i+m)t^{(q-1)i+m}.$$

Moreover, it becomes a meromorphic Drinfeld modular form for $\Gamma_0(Q)$ of weight $k$ and type $m$. Let $M_k^m(\Gamma_0^+(Q))$ be the $C$-vector space of Drinfeld modular forms for $\Gamma_0^+(Q)$ of weight $k$ and type $m$. We assume that $k = 2m$ is an even nonnegative integer. If not, the space $M_k^m(\Gamma_0^+(Q))$ is the zero space. Indeed, since $\begin{pmatrix} -Q & 0 \\ 0 & -Q \end{pmatrix} \in \Gamma_0^+(Q)$, $f$ is invariant under the action of $\begin{pmatrix} -Q & 0 \\ 0 & -Q \end{pmatrix}$ by the slash operator, that is, $f(z) = Q^{-2m}(-Q)^k f(z)$, which implies that $k = 2m$ if $f(z) \neq 0$.

**Example 3.1.** If $k = 2m$ and $f$ is a meromorphic Drinfeld modular form for $GL_2(A)$ of weight $k$ and type $m$, then $f(z) + Q^m f(Qz)$ is a meromorphic Drinfeld modular form for $\Gamma_0^+(Q)$ of weight $k$ and type $m$.

For any meromorphic Drinfeld modular form $G(z)$ for $\Gamma_0^+(Q)$ of weight 2 and type 1, $\omega := G(z)dz$ can be considered as a 1-from on $X_{\Gamma_0^+(Q)}$. Let

$$G(z) = \sum_{i=n_0}^{\infty} a((q-1)i+1)t^{(q-1)i+1}$$

be the $t$-expansion of $G(z)$ at $\infty$ and

$$\pi : \Omega \to \Gamma_0^+(Q)\backslash\Omega$$

be the quotient map. For any point $\tau \in \Omega$, let

$$e_\tau = |\Gamma_0^+(Q)_\tau/(\Gamma_0^+(Q)_\tau \cap Z(K))|,$$

where $\Gamma_0^+(Q)_\tau$ is the stabilizer of $\tau$ in $\Gamma_0^+(Q)$ and $Z(K)$ is the center of scalar matrices.

In what follows we suppose that $q = p^l$ is odd. Then $e_\tau$ is prime to $p$ because $e_\tau$ equals $n_\tau$ or $2n_\tau$, where $n_\tau = |\Gamma_0(Q)_\tau/(\Gamma_0(Q)_\tau \cap Z(K))|$ has the value 1 or $q+1$. Moreover we have the following.

**Lemma 3.2.**

(i) $\mathrm{Res}_\infty\omega = -a(1)/\widetilde{\pi}$.
(ii) $\mathrm{Res}_{\pi(\tau)}\omega = e_\tau^{-1}\mathrm{Res}_\tau G(z)$ *for each* $\tau \in \Omega$.

*Proof.* (i) follows from the simple fact that $dt = -\widetilde{\pi}t^2dz$. Let $\tau \in \Omega$. We choose uniformizers $x$ and $y$ on $\Omega$ and $\Gamma_0^+(Q)\backslash\Omega$, respectively, with $y = x^{e_\tau}$. Then $dy = e_\tau x^{e_\tau-1}dx$, which gives the assertion (ii). □

To state the main theorem we let

$$\rho_Q(t) = \sum_{0 \le j \le \deg Q} l_j t^{q^j}.$$

Then for a nonnegative integer $a$,

$$1/t(Qz)^{p^a} = \left(\frac{f_Q(t)}{t^{q^{\deg Q}}}\right)^{p^a} = \rho_Q(t^{-1})^{p^a} = \sum_{0 \le j \le \deg Q} l_j^{p^a} t^{-p^a q^j}.$$

**Theorem 3.3.** *Suppose that $k = 2m$ and $q$ is odd. Let $\overline{m}$ be the nonnegative integer such that $m \equiv \overline{m} \mod (q-1)$ and $0 \le \overline{m} < q-1$ and let*

$$r := \dim_C M_k^m(\Gamma_0(1)) = \left[\frac{k-(q+1)\overline{m}}{q^2-1}\right] + 1 \qquad (\textit{see } [1]).$$

*Let $f \in M_k^m(\Gamma_0^+(Q))$ have the $t$-expansion as follows*

$$f = \sum_{i \ge -m/(q-1)} a_f((q-1)i+m) t^{(q-1)i+m} \in O_L[[t]],$$

*where $L$ is a finite algebraic extension field of $K$ with ring $O_L$ of integers.*

*For any integer $a$ such that $p^a + 1 \ge \overline{m} + r(q-1)$ and $\overline{m} \equiv p^a + 1 \mod (q-1)$, we have that*

$$Q^{m-1} a_f\left((q-1)\frac{p^a+1-m}{q-1} + m\right) + \sum_{0 \le j \le \deg Q} l_j^{p^a} a_f\left((q-1)\frac{p^a q^j + 1 - m}{q-1} + m\right) \equiv 0 \mod (T^q - T).$$

*Proof.* Let

$$\alpha := \frac{k - 2\overline{m}}{q-1} + (1-r)(q+1) - \overline{m} \quad \text{and} \quad l_a := \frac{p^a + 1 - \overline{m}}{q-1} - r.$$

Let $h$ be the Pincaré series $P_{q+1,1}$ and $g$ the normalized Eisenstein series of weight $q-1$. Noticing that we can consider $h^{(q-1)(r+l_a)+\overline{m}-1}$ as a Drinfeld modular form for $GL_2(A)$ of type $m-1$ and $g^{(q+1)l_a-\alpha+q}$ as a Drinfeld modular form for $GL_2(A)$ of type 0, we choose

$$B(z) := \frac{g^{(q+1)l_a-\alpha+q}}{h^{(q-1)(r+l_a)+\overline{m}-1}} \qquad \text{and} \qquad C(z) := B(Qz),$$

which are of weight $2-k$ and

$$F(z) := Q^{m-1}B(z)f(z) + C(z)f(z).$$

Then clearly we have $((Q^{m-1}B+C)|_{2-k}^{1-m}\gamma)(z) = Q^{m-1}B(z)+B(Qz)$ for any $\gamma \in \Gamma_0(Q)$. Moreover,

$$\begin{aligned}
& \left((Q^{m-1}B+C)|_{2-k}^{1-m}\left(\begin{smallmatrix} 0 & -1 \\ Q & 0 \end{smallmatrix}\right)\right)(z) \\
= & \left(Q^{m-1}B|_{2-k}^{1-m}\left(\begin{smallmatrix} 0 & -1 \\ Q & 0 \end{smallmatrix}\right)\right)(z) + \left(C|_{2-k}^{1-m}\left(\begin{smallmatrix} 0 & -1 \\ Q & 0 \end{smallmatrix}\right)\right)(z) \\
= & Q^{1-m}Q^{m-1}(Qz)^{k-2}B\left(\frac{-1}{Qz}\right) + Q^{1-m}(Qz)^{k-2}B\left(\frac{-1}{z}\right) \\
= & B(Qz)+Q^{m-1}B(z).
\end{aligned}$$

Hence $B(Qz)+Q^{m-1}B(z)$ is invariant under the action of $\Gamma_0^+(Q)$ by the slash operator, which implies that $F(z)$ is a meromorphic Drinfeld modular form of weight 2 and type 1 for $\Gamma_0^+(Q)$. Since $F(z)$ is holomorphic on $\Omega$, the residue theorem on $X_{\Gamma_0^+(Q)}$ implies that

$$\sum_{p\in X_{\Gamma_0^+(Q)}} \mathrm{Res}_p F(z)dz = \mathrm{Res}_\infty F(z)dz = 0.$$

Moreover note that $g \equiv 1 \bmod (T^q - T)$ and $h = -t - t^{(q-1)^2-1} + \cdots$ ([3, Cor. 10.4]). Therefore, the coefficient of $t$ in

$$\begin{aligned}
& Q^{m-1}\frac{g^{(q+1)l_a-\alpha+q}f}{h^{(q-1)(r+l_a)+\overline{m}-1}} + \frac{g(Qz)^{(q+1)l_a-\alpha+q}f}{h(Qz)^{(q-1)(r+l_a)+\overline{m}-1}} \quad (\in M_2^1(\Gamma_0^+(Q))) \\
\equiv\ & Q^{m-1}\frac{f}{h^{p^a}} + \frac{f}{h(Qz)^{p^a}} \qquad \bmod\ (T^q-T) \\
\equiv\ & Q^{m-1}\left(\sum_{i\geq -m/(q-1)} a_f((q-1)i+m)t^{(q-1)i+m}\right)\left(\frac{-1}{t^{p^a}} + t^{((q-1)^2-1)p^a}+\cdots\right) \\
& + \left(\sum_{i\geq -m/(q-1)} a_f((q-1)i+m)t^{((q-1)i+m}\right)\left(\frac{-1}{t(Qz)^{p^a}} + t(Qz)^{(q-1)^2-1)p^a}+\cdots\right) \\
& \bmod\ (T^q-T)
\end{aligned}$$

is zero. This gives the assertions. ∎

*Remark* 3.4. Note that for $f$, $m$ is necessarily larger than 1.

We obtain the following result.

**Corollary 3.5.** *With the notations and assumptions of Theorem 3.3 we have that for any monic irreducible polynomial $Q$ of degree 1 in $A$,*

$$a_f\left((q-1)\frac{p^a q+1-m}{q-1}+m\right) \equiv 0 \bmod Q.$$

Now by using theorem 3.3 we obtain new congruence relations for the $t$-expansion coefficients of Drinfeld modular forms for $GL_2(\mathbb{F}_q[T])$ as follows.

**Theorem 3.6.** *Let $k$, $m$, $q$, $\overline{m}$ and $r$ be as in Theorem 3.3 and suppose that $f \in M_k^m(\Gamma_0(1))$ has the $t$-expansion as follows*

$$f = \sum_{i \geq -m/(q-1)} a_f((q-1)i+m)t^{(q-1)i+m} \in O_L[[t]],$$

*where $L$ is a finite algebraic extension field of $K$ with ring $O_L$ of integers.*

*For any integer $a$ such that $p^a + 1 \geq \overline{m} + r(q-1)$ and $\overline{m} \equiv p^a + 1 \bmod (q-1)$, we have that for any monic irreducible polynomial $Q$ of degree 1 in $A$,*

$$a_f\left((q-1)\frac{p^a q + 1 - m}{q-1} + m\right) \equiv 0 \bmod Q.$$

*Proof.* For $f \in M_k^m(\Gamma_0(1))$, we obtain that $f(z) + Q^m f(Qz) \in M_k^m(\Gamma_0^+(Q))$. Since $f(z) + Q^m f(Qz) \equiv f(z) \bmod Q$, by Theorem 3.3 and Remark 3.4, we have that

$$\sum_{0 \leq j \leq 1} l_j^{p^a} a_f\left((q-1)\frac{p^a q^j + 1 - m}{q-1} + m\right)$$
$$\equiv a_f\left((q-1)\frac{p^a q + 1 - m}{q-1} + m\right) \equiv 0 \bmod Q.$$ □

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## References

[1] S. Choi, *Linear relations and congruences for the coefficients of Drinfeld modular forms*, Israel J. Math. **165** (2008), 93–101.

[2] J. Gallardo and B. López, *"Weak" congruences for coefficients of the Eisenstein series for $\mathbb{F}_q[T]$ of weight $q^k - 1$*, J. Number Theory **102** (2003), no. 1, 107–117.

[3] E. U. Gekeler, *On the coefficients of Drinfel?d modular forms*, Invent. Math. **93** (1988), no. 3, 667–700.

[4] ———, *Growth order and congruences of coefficients of the Drinfeld discriminant function*, J. Number Theory **77** (1999), no. 2, 314–325.

[5] E. U. Gekeler and M. Reversat, *Jacobians of Drinfeld modular curves*, J. Reine Angew. Math. **476** (1996), 27–93.

[6] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[7] B. López, *A congruence for the coefficients of the Drinfeld discriminant function*, C. R. Acad. Sci. Paris Ser. I Math. **330** (2000), no. 12, 1053–1058.

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