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THE NEARLY ADDITIVE MAPS

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Abstract. This note is a verification on the relations between almost linear and nearly additive maps; and the continuity of almost multiplicative nearly additive maps. Also we consider the stability of nearly additive and almost linear maps.

1. Introduction

In 1952, Michael asked his famous question about the continuity of multiplicative linear functionals on complete metrizable locally convex topological algebras. From that time mathematicians have studied this interesting problem in various directions. A part of studies is on complete metrizable topological algebras which are not necessarily locally convex. It is proved in [2] that every multiplicative linear functional is automatically continuous on complete metrizable FLM algebras, where by an FLM algebra we mean a fundamental topological algebra A for which there exists a neighborhood U_0 of zero such that, for every neighborhood V of zero, the sufficiently large power of U_0 lie in V (See also [1]). The other group of mathematicians have replaced multiplicative linear functionals with almost multiplicative one, or even with almost linear operators.

In 2002, Šemrl [19] introduces the concept of almost linear maps, and Jarosz [11] discusses the continuity of almost multiplicative linear functionals on Banach algebras. In [3] it is proved that an almost multiplicative linear map from a Banach algebra to a semi-simple Banach algebra is continuous. We will show here that (Theorem 3.3) every homogeneous nearly additive almost multiplicative functional on a Banach algebra is continuous.

The stability of functional equations is an interesting area of research for mathematicians, but it can be also of importance to persons who work outside of the realm of pure mathematics. For example, physicists are interested in the stability of the mathematical formulae which they use to model physical

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processes. More precisely, physicists and other scientists are interested in determining when a small change in an equation used to the model of a phenomenon, gives a large changes in the results predicted by the equation.

At first, in 1940, Ulam [20] studied the stability of functional equations: Given a group G_1, a metric group G_2 and $\epsilon > 0$, find $\delta > 0$ such that, if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x, y \in G_1$, then there exists a homomorphism $g : G_1 \to G_2$ such that $d(f(x), g(x))) \leq \epsilon$ for all $x \in G_1$.

In 1941, Hyers [8] showed that if $\delta > 0$ and if $f : E_1 \to E_2$ is a mapping between Banach spaces E_1 and E_2 with $||f(x + y) - f(x) - f(y)|| \leq \delta$ for all $x, y \in E_1$, then there exists a unique $T : E_1 \to E_2$ such that $T(x + y) = T(x) + T(y)$ with $||f(x) - T(x)|| \leq \delta$ for all $x, y \in E_1$.

In 1978, a generalized solution to Ulam's problem for approximately linear mappings was given by Th. M. Rassias [15]. He considered E_1 and E_2 to be two Banach spaces and assumed the existence of $\delta \geq 0$ and $p \in [0, 1)$ such that $||f(x + y) - f(x) - f(y)|| \leq \delta(||x||^p + ||y||^p)$ for every $x, y \in E_1$. Then he proved that there exists a unique linear mapping $T : E_1 \to E_2$ such that $||f(x) - T(x)|| \leq 2\delta/(2 - 2^p)||x||^p$ for every $x \in E_1$. It is easy to see that the proof of Th. M. Rassias [15] also is valid for $p < 0$.

In 1991, Gajda [5] gave a solution to this question for $p > 1$. For the case $p = 1$, Th. M. Rassias and Šemrl [16] show that there exists a continuous real-valued function $f : \mathbb{R} \to \mathbb{R}$ such that f can not be approximated with an additive map. In other words, an analogue of Rassias's result for $p = 1$ can not be obtained.

Finally in 1992, Gavruta [6] generalized the result of Th. M. Rassias for the admissible control functions.

The approximated mappings have been studied extensively in several papers. (See for example [9, 10, 13]).

In this note at first, we have gathered a collection of definitions and related results. In Section 3, we introduce nearly additive maps and have a discussion on the properties of these maps which are different from almost linear maps. In Section 4, we have a more verification on nearly additive maps and the stability of these maps.

2. Preliminaries

In this section we recall some definitions and related theorems on perturbations of linear maps and in the next section we introduce a new concept about this matter.

Definition. Let A be a normed algebra, B a complete normed algebra, and $\varphi : A \longrightarrow B$ a linear map. We say that φ is almost multiplicative if there exists an $\epsilon > 0$ such that, for all $x, y \in A$, $||\varphi(xy) - \varphi(x)\varphi(y)|| \leq \epsilon||x||||y||$.

Definition. Let A be a normed algebra and B a complete normed algebra. A mapping $\varphi : A \longrightarrow B$ is said to be an almost linear map if there exists a $\delta > 0$ such that, for every $x, y \in A$, $||\varphi(x + y) - \varphi(x) - \varphi(y)|| \leq \delta(||x|| + ||y||)$.

Definition. Let A be a normed algebra and B a complete normed algebra. A mapping $\varphi : A \longrightarrow B$ is said to be an (ϵ, δ)-homomorphism if there exist $\epsilon > 0$ and $\delta > 0$ such that, for every $x, y \in A$, $||\varphi(xy) - \varphi(x)\varphi(y)|| \leq \epsilon||x||||y||$ and $||\varphi(x + y) - \varphi(x) - \varphi(y)|| \leq \delta(||x|| + ||y||)$.

Theorem 2.1. *Let A be a normed algebra. If $\varphi : A \to \mathbb{C}$ is an (ϵ, δ)-homomorphism which is not additive, then φ is bounded.*

Proof. See the proof of [18, Theorem 2.1]. □

Definition. Suppose that X and Y are topological vector spaces and Γ is a collection of linear mappings from X into Y. We say that Γ is equicontinuous if for every neighborhood W of zero in Y, there corresponds a neighborhood V of zero in X such that $\Lambda(V) \subseteq W$ for all $\Lambda \in \Gamma$.

Definition. Suppose that $(G, +)$ is an abelian group. We call a function $\varphi : G \times G \to [0, \infty)$ to be an admissible control function if, for all $x, y \in G$:

$$\widehat{\varphi}(x, y) = 1/2 \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty.$$

Theorem 2.2. *Suppose that $(G, +)$ is an abelian group, $(X, || \cdot ||)$ a complete normed algebra, and φ an admissible control function. If $f : G \to X$ is a mapping with $||f(x + y) - f(x) - f(y)|| \leq \varphi(x, y)$ for all $x, y \in G$, then, there exists a unique additive mapping $T : G \to X$ such that $||f(x) - T(x)|| \leq \varphi(x, x)$ for all $x \in G$.*

Proof. See [6]. □

3. The nearly additive maps

In this section we define a new concept about perturbations of linear maps, that is, *nearly additive maps.*

Definition. Let A be a normed space and B a complete normed space. A mapping $\varphi : A \longrightarrow B$ is a nearly additive map if there exists a $\delta > 0$ such that $||\varphi(x + y) - \varphi(x) - \varphi(y)|| \leq \delta||x + y||$ for all $x, y \in A$.

If $\varphi : A \longrightarrow B$ is an almost linear map, then the continuity at the origin of A can be the only point of continuity (see for example [18]). But we have the following proposition:

Proposition 3.1. *Let A be a normed space, B a complete normed space and $\varphi : A \longrightarrow B$ a nearly additive map which is continuous at zero. Then φ is continuous.*

Proof. It is completely straightforward. □

Now, we state and prove some well known theorems for nearly additive maps. At first, we have a proposition on the boundedness of these maps.

Proposition 3.2. *Let A be a normed space, B a complete normed space and $\varphi : A \to B$ a homogeneous nearly additive map. Then φ is continuous if and only if it is bounded.*

Proof. For a proof, see [17, Theorem 1.32]. $\square$

Theorem 3.3. *Every homogeneous nearly additive ϵ-multiplicative functional φ on a Banach algebra X is continuous. Moreover $||\varphi|| \leq 1 + \epsilon$.*

Proof. When φ is an additive functional, the proof is given in [11]. In the other case, by Theorem 2.1 and Proposition 3.2 the proof is complete. $\square$

The Theorems 3.4 and 3.7 are the extensions of *Banach-Steinhaus Theorem* and *Closed graph Theorem* [17].

Theorem 3.4. *Let X be a normed space, Y a complete normed space, Γ a collection of homogeneous nearly additive continuous mappings from X into Y, and B the set of all $x \in X$ whose orbits $\Gamma(x) = \{\Lambda(x) : \Lambda \in \Gamma\}$ are bounded in Y. If B is of the second category in X, then $B = X$ and Γ is equicontinuous.*

Proof. Pick $U = B(0, \delta_2)$ a neighborhood in Y. Put $E = \bigcap_{\Lambda \in \Gamma} \Lambda^{-1}(\overline{U})$. It is clear that E is a closed subset of X. If $x \in B$, then $\Gamma(x) \subseteq nU$ for some positive integer n, so that $\Lambda x \in \Gamma(x) \subseteq nU \subseteq n\overline{U}$ for every $\Lambda \in \Gamma$. Therefore $x \in nE$. Consequently, $B \subseteq \bigcup_{n=1}^{\infty} nE$. At least one nE is of the second category in X. Since the mapping $x \mapsto nx$ is a homeomorphism of X onto X, E is itself of the second category in X. But E is closed and therefore $\mathrm{int}(E) \neq \emptyset$. So there exists an $x_0 \in \mathrm{int}(E)$ such that $B(x_0, r_0) \subseteq E$ for some r_0. Suppose that there exists δ_1 such that $x \in B(0, \delta_1)$, so $r_0 x \in B(0, r_0\delta_1)$. Set $z = r_0 x + x_0$. It follows that $z \in B(x_0, r_0\delta_1) \subseteq E$. Hence $||\Lambda x|| = ||\Lambda((z - x_0)/r_0)|| = 1/r_0(||\Lambda(z - x_0)||) \leq 1/r_0(\delta||z - x_0|| + ||\Lambda z - \Lambda x_0||) \leq 1/r_0(\delta||z - x_0|| + ||\Lambda z|| + ||\Lambda x_0||) \leq \delta\delta_1 + 2\delta_2/r_0$. This shows that Γ is equicontinuous. It is straightforward to show that $B = X$. $\square$

Lemma 3.5. *Let (X, d_1) and (Y, d_2) be metric spaces and Λ be a map from X into Y. The set $G = \{(x, \Lambda x) : x \epsilon X\}$ is closed if and only if, for every sequence $(x_n) \in X$ with $x_n \to x$ and $\Lambda(x_n) \to y$, we have $\Lambda(x) = y$.*

Proof. See [7]. $\square$

Lemma 3.6. *Suppose that X is a normed space, $\delta > 1$ and E a subset of X such that, for every x in the unit ball $B(0,1)$ of X, there exists $a \in E$ such that $||x - a|| < 1/\delta$. Then there exists a sequence $(a_n) \in E$, such that*:

$$\left|\left| x - \sum_{j=0}^{n} a_j/\delta^j \right|\right| < 1/\delta^{n+1}.$$

Proof. See [7]. $\square$

Theorem 3.7. *Let X be a normed space, Y a complete normed space and $T : X \to Y$ a homogeneous nearly additive map with closed graph. Then T is continuous.*

Proof. For every $\alpha > 0$, set $V_\alpha = \{x \in X : ||Tx|| \leq \alpha\}$. We claim that there exists $\alpha > 0$ such that $B_1 \subseteq V_\alpha$, where $B_1 = B(0,1)$ is a unit ball of X. We have $X = \bigcup_{n=1}^{\infty} \overline{V_n}$. Now by the Bair's Theorem there exist $n_0 \in \mathbb{N}$, $x \in X$ and $r > 0$ such that $B(x,r) \subseteq \overline{V_{n_0}}$, so $B(0,r) \subseteq \overline{V_{2n_0}}$, and $B_1 \subseteq \overline{V_{2n_0/r}}$. So for every $x \in B_1$, there exists $a \in V_{2n_0/r}$ such that $||x - a|| < 1/2$. By Lemma 3.6 there exists a sequence (a_n) in $V_{2n_0/r}$ such that for every $x \in B_1$, $||x - b_n|| < 1/2^{n+1}$, where $b_n = \sum_{j=0}^{n} a_j/2^j$. Now for $n = 0, 1, 2, \ldots$ we get:

$$||T(b_n) - T(b_{n-1})|| \leq \delta||b_n - b_{n-1}|| + ||T(b_n - b_{n-1})|| \leq 2n_0(\delta + 1)/(r2^n).$$

So, $T(b_n)$ is a Cauchy sequence in Y, and therefore there exists $y \in Y$ such that $T(b_n) \to y$. Also $b_n \to x$, and $T(b_n) \to y$. Since T has a closed graph, so by Lemma 3.5, $T(x) = y$ and it follows that:

$$\begin{aligned} ||T(x)|| = \lim_{n\to\infty} ||T(b_n)|| &\leq \lim_{n\to\infty} \left(\delta|| \sum_{j=0}^{n} a_j/2^j|| + || \sum_{j=0}^{n} T(a_j/2^j)|| \right) \\ &\leq 4n_0(\delta + 1)/r. \end{aligned}$$

Consequently $B_1 \subseteq V_{4n_0(\delta+1)/r}$. □

Theorem 3.8. *Let X be a normed space, Y a complete normed space, K a compact convex set in X, Γ a collection of homogeneous continuous nearly additive mappings of X into Y, and the orbits $\Gamma(x) = \{\Lambda x : \Lambda \in \Gamma\}$ are bounded subsets of Y for every $x \in K$. Then there exists a bounded set B of Y such that $\Lambda(K) \subseteq B$ for every $\Lambda \in \Gamma$.*

Proof. Let $B = \bigcup_{x \in K} \Gamma(x)$. Pick balanced neighborhoods $U = B(0, r_1)$ and $W = B(0, r_2)$ of Y. Put $E = \bigcap_{\Lambda\in\Gamma} \Lambda^{-1}(\overline{U})$. By the Bair's Theorem there exists a positive integer n such that $\mathrm{int}_K(K \cap nE) \neq \emptyset$. Let $x_0 \in \mathrm{int}_K(K \cap nE)$. We choose a balanced neighborhood $V = B(0, r_3)$ such that $K \cap (x_0 + V) \subseteq nE$. We have $K \subseteq \bigcup_{\lambda>0}(x_0 + \lambda V)$. Now, since K is compact there exist scalars $\lambda_1, \ldots, \lambda_m$ such that $K \subseteq \bigcup_{i=1}^{m} (x_0 + \lambda_i V)$. Suppose that $p > \max\{\lambda_1, \ldots, \lambda_m, 1\}$. Then $x_0 + \lambda_i V \subseteq x_0 + pV$ for every $1 \leq i \leq n$. If x is any point of K and $z = (1/p)x + (1 - 1/p)x_0$, then $z \in K$ and $z - x_0 \in V$. Hence $z \in nE$. K is compact and hence it is bounded. So, for every $k \in K$ there exists r_4 such that $||k|| \leq r_4$. Since $x = pz - (p - 1)x_0$, it follows that:

$$\begin{aligned} ||\Lambda x|| &\leq p||\Lambda z|| + (p - 1)||\Lambda x_0|| + \delta||pz - (p - 1)x_0|| \\ &\leq pnr_1 + (p - 1)nr_1 + \delta pr_3 + \delta r_4 \end{aligned}$$

which proves that B is bounded. □

4. Stability

Our starting point in this section is the following theorem.

Theorem 4.1. *Let A be a normed algebra, B a complete normed algebra and $f : A \to B$ a mapping such that $f(tx)$ is continuous in t for each fixed $x \in A$. Assume that there exist $\delta > 0$ and $p \neq 1$ such that $||f(x+y) - f(x) - f(y)|| \leq \delta||x+y||^p$ for every $x, y \in A$. Then, there exist a unique linear mapping $T : A \to B$ and $\beta > 0$ such that $||f(x) - T(x)|| \leq \beta||x||^p$ for every $x \in A$.*

Proof. For $p < 1$ the function $\varphi(x, y) = \delta||x+y||^p$ is an admissible control function, so by Theorem 2.2, the proof is complete.

For $p > 1$, define the mapping T by the formula $T(x) = \lim_{n\to\infty} 2^n f(x/2^n)$ for all $x \in A$. Obviously, one has to verify the convergence of the sequence $2^n f(\frac{x}{2^n})$. Putting $x/2$ in place of x and y in inequality $||f(x+y)-f(x)-f(y)|| \leq \delta||x+y||^p$, we obtain $||f(x) - 2f(x/2)|| \leq \delta||x||^p$ for all $x \in A$. Hence for each $n \in \mathbb{N}$ and every $x \in A$, we have:

$$\begin{aligned} ||f(x) - 2^n f\left(\frac{x}{2^n}\right)|| &\leq ||f(x) - 2f(x/2)|| + 2||f(x/2) - 2f(x/2^2)|| + \cdots \\ &\quad + 2^{n-1}||f(x/2^{n-1}) - 2f(x/2^n)|| \\ &\leq \left(1 + \frac{2}{2^p - 2}\right)\delta||x||^p = \beta||x||^p, \end{aligned}$$

where $\beta = (1 + \frac{2}{2^p-2})\delta$. Now fix an $x \in A$ and choose arbitrary $m, n \in \mathbb{N}$ with $m > n$. Then $||2^m f(x/2^m) - 2^n f(x/2^n)|| \leq 2^{n(1-p)}\beta||x||^p$, which becomes arbitrary small as $n \to \infty$. On the account of the completeness of the algebra B, this implies that the sequence $\{2^n f(x/2^n) : n \in N\}$ is convergent for each $x \in A$. Thus T is well-defined. Moreover, it satisfies in the condition $||f(x) - T(x)|| \leq \beta||x||^p$ as $n \to \infty$. It is sufficient to show that T is additive. Replacing x by $x/2^n$ and y by $y/2^n$ in $||f(x+y) - f(x) - f(y)|| \leq \delta||x+y||^p$ and then multiplying both sides of the resulting inequality by 2^n, we get

$$\left|\left|2^n f\left(\frac{x+y}{2^n}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right|\right| \leq 2^{n(1-p)}\delta||x+y||^p$$

for all $x, y \in A$. Since the right-hand side of this inequality tends to zero as $n \to \infty$, it becomes apparent that the mapping T is additive. The function $f(tx)$ is continuous relative to t, so $T(tx)$ is continuous in t, and it is linear. For the uniqueness of T, suppose that there exists another S such that $||f(x)-S(x)|| \leq \beta||x||^p$ and $S(x) \neq T(x)$. For any integer $n > \frac{2\beta||x||^p}{||T(x)-S(x)||}$, it is obvious that $||T(nx) - S(nx)|| > 2\beta||x||^p$, which contradicts with the inequalities $||T(x) - f(x)|| \leq \beta||x||^p$ and $||S(x) - f(x)|| \leq \beta||x||^p$. Hence T is the unique linear map such that satisfies in $||f(x) - T(x)|| \leq \beta||x||^p$. □

Baak and Moslehian [4] investigate the stability of J^*-homomorphisms. In 2003, Park [14] established the stability of algebra $*$-homomorphisms on a Banach $*$-algebra and the stability of automorphism of a unital C^*-algebra for admissible control functions. A similar argument as in the proofs of Park's theorems, we prove the following theorems.

Theorem 4.2. *Let A be a Banach $*$-algebra and $f : A \to A$ a mapping with $f(0) = 0$ such that, for $p \notin [1, 2]$ and $\delta > 0$ we have*:

(i) $||f(\mu x + \mu y) - \mu f(x) - \mu f(y)|| \leq \delta ||x + y||^p$,

(ii) $||f(x^*) - f(x)^*|| \leq \delta ||x||^p$, *and*

(iii) $||f(zw) - f(z)f(w)|| \leq \delta ||z + w||^p$

for all $\mu \in T_1 = \{\lambda \in C : |\lambda| = 1\}$, all self adjoint elements z, w and all $x, y \in A$. Then, there exists a unique algebra $*$-homomorphism $T : A \to A$ and $\beta > 0$ such that $||f(x) - T(x)|| \leq \beta ||x||^p$ for all $x \in A$.

Proof. For $p < 1$, $\varphi(x, y) = ||x + y||^p$ is an admissible control function and it is proved by Park (see [14]). Now for $p > 2$, put $\mu = 1$. It follows from Theorem 4.1 that there exists a unique additive mapping $T : A \to A$ such that the inequality $||f(x) - T(x)|| \leq \beta ||x||^p$ holds. The additive mapping T is given by $T(x) = \lim_{n\to\infty} 2^n f(x/2^n)$ for all $x \in A$. By the assumption, for each $\mu \in T_1$,

$$\begin{aligned} &||f(\mu 2^{-n-1}x + \mu 2^{-n-1}x) - \mu f(2^{-n-1}x) - \mu f(2^{-n-1}x)|| \\ &\leq \delta ||2^{-n-1}x + 2^{-n-1}x||^p = \delta ||2^{-n}x||^p \end{aligned}$$

for all $x \in A$. Then one can show that

$$||\mu f(2^{-n}x) - 2\mu f(2^{-n-1}x)|| \leq |\mu| ||f(2^{-n}x) - 2f(2^{-n-1}x)|| \leq \delta ||2^{-n}x||^p$$

for all $\mu \in T_1$ and all $x \in A$. So

$$\begin{aligned} &||f(2^{-n}\mu x) - \mu f(2^{-n}x)|| \\ &\leq ||f(2^{-n}\mu x) - 2\mu f(2^{-n-1}x)|| + ||2\mu f(2^{-n-1}x) - \mu f(2^{-n}x)|| \\ &\leq 2\delta ||2^{-n}x||^p \end{aligned}$$

for all $\mu \in T_1$ and all $x \in A$. Thus $\lim_{n\to\infty} 2^n ||f(2^{-n}\mu x) - \mu f(2^{-n}x)|| = 0$ for all $\mu \in T_1$ and all $x \in A$. Hence

$$T(\mu x) = \lim_{n\to\infty} 2^n f(2^{-n}\mu x) = \lim_{n\to\infty} 2^n \mu f(2^{-n}x) = \mu T(x)$$

for all $\mu \in T_1$ and all $x \in A$. Now, let $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and let M be an integer greater than $4|\lambda|$. Then $|\lambda/M| < 1/4 < 1/3$. By [12], there exist three elements $\mu_1, \mu_2, \mu_3 \in T_1$ such that $3\lambda/M = \mu_1 + \mu_2 + \mu_3$. Also, $T(x) = T(3 \times 1/3x) = 3T(1/3x)$ for all $x \in A$. So $T(1/3x) = 1/3T(x)$ for all $x \in A$. Thus

$$T(\lambda x) = T\left(\frac{M}{3} 3 \frac{\lambda}{M} x\right) = M/3T(\mu_1 x + \mu_2 x + \mu_3 x) = \lambda T(x)$$

for all $x \in A$. Hence the unique additive mapping is a $\mathbb{C}$-linear mapping. Also, $||f(2^{-n}x^*) - f(2^{-n}x)^*|| \leq \delta 2^{(1-n)p} ||x^* + x||^p$. We get

$$T(x^*) = \lim_{n\to\infty} 2^n f(2^{-n}x^*) = \lim_{n\to\infty} 2^n f(2^{-n}x)^*$$

for all $x \in A$. Now, it follows that $4^n||f(2^{-n}z2^{-n}w) - f(2^{-n}z)f(2^{-n}w)|| \leq 4^n 2^{(1-n)p}||z+w||^p$ for all self-adjoint elements z, w. Therefore

$$\lim_{n\to\infty} 4^n f(2^{-n}z2^{-n}w) - f(2^{-n}z)f(2^{-n}w)|| = 0.$$

So $T(zw) = \lim_{n\to\infty} 4^n f(4^{-n}zw) = T(z)T(w)$ for all self-adjoint elements z, w. For elements $x, y \in A$, we have $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$ and $y = \frac{y+y^*}{2} + i\frac{y-y^*}{2i}$. Put $x_1 = \frac{x+x^*}{2}$, $x_2 = \frac{x-x^*}{2i}$, $y_1 = \frac{y+y^*}{2}$ and $y_2 = \frac{y-y^*}{2i}$. It is clear that x_1, x_2, y_1, y_2 are self-adjoint elements. The map T is $\mathbb{C}$-linear, so $T(xy) = T(x_1y_1 - x_2y_2 + ix_1y_2 + ix_2y_1) = T(x_1 + ix_2)T(y_1 + iy_2) = T(x)T(y)$ for all $x, y \in A$. Hence, the additive mapping T is an algebra $*$-homomorphism satisfying the inequality as desired. $\square$

Theorem 4.3. *Let A be a Banach $*$-algebra and $f : A \to A$ a mapping with $f(0) = 0$ such that for $p \notin [1, 2]$ and $\delta > 0$ we have:*

(i) $||f(\mu x + \mu y) - \mu f(x) - \mu f(y)|| \leq \delta(||x||^p + ||y||^p)$,
(ii) $||f(x^*) - f(x)^*|| \leq \delta||x||^p$, *and*
(iii) $||f(zw) - f(z)f(w)|| \leq \delta(||z||^p + ||w||^p)$

for all $\mu \in T_1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, all self adjoint elements z, w and all $x, y \in A$. Then, there exists a unique algebra $*$-homomorphism $T : A \to A$ and $\beta > 0$ such that $||f(x) - T(x)|| \leq \beta||x||^p$ for all $x \in A$.

Proof. The proof is similar to that of Theorem 4.2 and is omitted. $\square$

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