Bi-Directional Half-Duplex Relaying Protocols

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(Invited Paper)

Abstract: **The bi-directional relay channel is the natural extension of a three-terminal relay channel where node a transmits to node b with the help of a relay r to allow for two-way communication between nodes a and b. That is, in a bi-directional relay channel, a and b wish to *exchange* independent messages over a shared channel with the help of a relay r. The rates at which this communication may reliably take place depend on the assumptions made on the relay processing abilities. We overview information theoretic limits of the bi-directional relay channel under a variety of conditions, before focusing on *half-duplex* nodes in which communication takes place in a number of temporal phases (resulting in *protocols*), and nodes may forward messages in four manners. The relay-forwarding considered are: *Amplify and forward* (AF), *decode and forward* (DF), *compress and forward* (CF), and *mixed forward*. The last scheme is a combination of CF in one direction and DF in the other. We derive inner and outer bounds to the capacity region of the bi-directional relay channel for three temporal protocols under these four relaying schemes. The first protocol is a two phase protocol where a and b simultaneously transmit during the first phase and the relay r alone transmits during the second. The second protocol considers sequential transmissions from a and b followed by a transmission from the relay while the third protocol is a hybrid of the first two protocols and has four phases. We provide a comprehensive treatment of protocols in Gaussian noise, obtaining their respective achievable rate regions, outer bounds, and their relative performance under different SNR and relay geometries.**

Index Terms: **Achievable rate regions, bi-directional communication, outer bounds, relaying.**

I. INTRODUCTION

A. Motivation

Bi-directional relay channels in which two nodes (a and b)[1] wish to exchange independent messages with the help of relay nodes r's, are both of fundamental and practical interest. Such channels may be relevant to ad hoc networks as well as to networks with a centralized controller through which all messages must pass. From an information theoretic perspective, an understanding of these fundamental bi-directional channels would bring us closer to a coherent picture of multi-user information theory, as they make the interesting conceptual leap from the uni-directional three terminal relay channel to allow for information flow in two directions. Significantly different communication techniques which take into account the bi-directional messages have emerged in recent years.

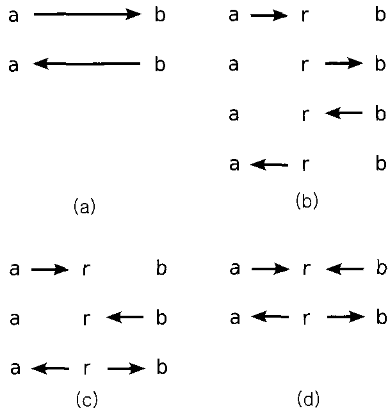

Fig. 1. (a) Traditional approach, (b) naive four phase bi-directional cooperation, (c) coded broadcast three phase protocol, and (d) two phase protocol.

Consider two users, denoted by a and b, who wish to exchange independent messages over a shared channel. This is known as the two-way channel [1], [2], first introduced in [2], where an achievable rate region and an outer bound for the case in which nodes operate in full-duplex were obtained. Non-orthogonal[2] full-duplex operation requires nodes to transmit and receive on the same antenna and frequency simultaneously. While of theoretical interest, with current technology, full-duplex operation may not be practically feasible since the intensity of the near field of the transmitted signal is much higher than that of the far field of the received signal. In this work, we consider half-duplex communication in which a node may transmit or receive at some time, but not both.

In a half-duplex two-way channel without the presence of a relay station, communication between nodes a and b is performed in two steps: First a transmits its message to b followed by similar transmission from b to a (illustrated in Fig. 1(a)), which is known as a "push-to-talk" channel [2]. In the presence of relay node r, one might initially assume that four phases are needed (see Fig. 1(b)). However, by taking advantage of the shared wireless medium, it is known that the third and fourth transmissions may be combined (Fig. 1(c)) into a single trans-

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[1] We call the nodes a and b *terminal* and *source* nodes interchangeably.

[2] By non-orthogonal we mean that no additional space, time, frequency, or coding dimensions are used to separate sent and receive signals.

mission using, for example, ideas from network coding [3], [4]. In particular, if the messages of a and b are w_a and w_b respectively and belong to a group, then it is sufficient for the relay node to successfully transmit $w_a \oplus w_b$ simultaneously to a and b.

B. Past Work on the Bi-Directional Relay Channel

The bi-directional relay channel started receiving attention around 2005–2006 when a number of authors [5]–[12] introduced and sought more efficient ways of communicating in a two-way fashion with the help of a single relay. Since then a large body of work concerning the bi-directional relay channel has emerged [5], [13]–[28], which may be differentiated roughly based on combinations of assumptions that are made on the channel (direct link between terminals or not), on the type of relaying (compress and forward (CF), decode and forward (DF), amplify and forward (AF), de-noise, mixed, structured), on the duplex abilities of nodes (half-duplex or full-duplex). We highlight some of the work under different assumptions before proceeding to describe a number of these channels in more depth.

1. **Relaying type:** The simplest of relaying types is AF, in which relays are not required to do any processing on the received signal besides re-scaling and re-transmiting it. Its benefits lie in its simplicity, at the price of noise being carried forward and are often used as a benchmark against which to compare the performance of more complex relaying types [7], [10], [11], [13], [16], [29]. DF relaying assumes the relay is able to decode all messages before re-transmitting them. Examples of work which assume bi-directional DF relaying include [6], [8], [9], [16], [17], [20]–[24], [27], [30]. Using DF relaying allows the use of network coding at the message level for the broadcast phase and prevents the re-transmission and possible amplification of noise. This comes at the cost of forcing the relays to decode the messages, possibly reducing the permissible transmission rates. CF, as first introduced in the context of the classical relay channel [31], and considered in the bi-directional relaying context in [6], [7], and [24] and the conceptually related de-noise and forward [7], [10], [11], requires the relay to re-transmit a quantized or compressed version of the received signal. This scheme has the advantage that the rate need not be lowered so as to allow the relay to fully decode it, but may still mitigate some of the noise amplification effects seen in AF relaying by judicious choice of the quantizer or compressor. Expanding on the intuition gained from the classical relay channel [32], and as further shown in the bi-directional relaying context [16], when a relay is close (or alternatively sees a weak channel to the source relative to the destination) to the destination it is preferable to perform CF, while if it closer (or sees a better channel to the source relative to the destination) to the source DF is a better choice. Thus, *mixed-forwarding* schemes in general must be considered. Finally, recently, the use of structured codes (lattice, computation, and network codes) [33]–[39] have proven themselves to be powerful tools for improving rates in certain multi-user scenarios, including bi-directional relay channels [26], [40]–[44]. Structured codes intuitively allow for the *sum* (or some function) of the interfering terms to be decoded and stripped off a received signal. This is in contrast to classical random binning schemes in which, due to the lack of structure, each encoded message must be individually decoded, leading to reduced rates.

2. **Duplexing:** Both full-duplex and half-duplex nodes and their corresponding achievable rate regions have been considered for bi-directional relaying. In [5]–[11], [16]–[18], [21], [24], [25], and [28], half-duplex nodes are assumed. This forces communication to take place over a number of phases, using different temporal *protocols*. A temporal protocol specifies which nodes simultaneously transmit at which time. Three of the most commonly considered protocols are depicted in Fig. 1: (b) The naive 4-phase protocol, (c) the 3 phase time-division broadcast channel (TDBC) protocol, as well as the (d) two-phase multiple-access broadcast channel (MABC) protocol. It is interesting to note that in half-duplex protocols, the TDBC allows a destination to obtain *side-information*, or extra overheard knowledge, about the other user's message during the phase in which the message is destined to the relay. This is not possible in 2 phase MABC protocols in whcih both nodes transmit simultaneously to the relay in one phase and are thus unable to overhear any of hte other relay's message. In [16] and [18], it is clearly shown that neither TDBC or MABC dominate the other for all channel gains. In [20]–[23] and [27], the authors have thoroughly analyzed the broadcast phase from a number of angles and assumptions (thus assuming half-duplex, decode and forward relaying) of the bi-directional relay channel. The full-duplex scenarios have been considered somewhat less: In [6], the authors derived achievable rate regions for the *restricted* two-way relay channel using DF, CF and AF schemes, in which the terminals may not cooperate in transmitting their messages. In [26], full-duplex nodes are considered in order to obtain capacity to within a finite number of bits.

3. **Direct link between the nodes:** When all communication must pass through the relays, i.e., there are no direct links, the only side-information available at the receiving nodes is that it has a priori: Its own message (nothing overheard from previous phases). Such networks tend to be easier to deal with as the possibility of direct communication between terminal nodes is excluded and thus the tension between how much should be sent through the relay versus how much should be sent directly is eliminated. While much work has been performed on the bi-directional relay channel in which there is a direct link, there are some notable exceptions [19], [26], [28], [42], [45], [46]. Of particular interest is the recent work of [26], an extension and generalization of their earlier work [42], in which a bi-directional, full-duplex relay channel with no link between terminal nodes is considered. Nested lattice coding for the uplink and structured binning for the downlink is proposed, and is shown to lie within 1/2 bit from the outer bound for any/all channel parameters: Noise variances and transmit powers.

C. Past Extensions of the Bi-Directional Relay Channel

The bi-directional relay channel has been extended to include multiple-relays as well as multiple terminals nodes, or multiple information flows. When two terminal nodes wish to exchange messages with the help of multiple relays, the central question is how to employ the relays in the 'best' way. The relays could amplify and forward the received signals in a multi-hop fashion, or may decode and cooperatively re-encode and re-transmit the received signals. These questions have been looked at from different perspective in [30], [47]–[51]. When multiple bi-directional flows co-exist, interference between streams must be handled at the relay and terminal nodes. Multiple bi-directional connections employing a relay have been considered under different assumptions in [41], [43], [46], [52], [53]. In all but [53], the direct links between terminal nodes are eliminated, leading to simpler channels as there is no tradeoff between sending messages directly or through the relay.

D. Outline of This Paper

In this paper, we focus on the fundamental limits of the bi-directional communication under the following assumptions: 1) AF, DF, CF, and mixed relaying types, 2) half-duplex nodes, and 3) direct links between the nodes exist. We will derive inner and outer bounds to the capacity region of the bi-directional relay channel under different protocols and relaying types, which we will compare in Gaussian noise. We thus present a comprehensive overview of bi-directional relay channel which highlights the relative performance and tradeoffs of the different schemes under different channel conditions and relay processing capabilities.

II. BASIC FEATURES

A. Discrete Memoryless Channel

To study the performance bounds of bi-directional relaying protocols we first consider a *discrete-time memoryless* channel. The channel is used countably many times (termed "channel uses") and each time instance is indexed by $k \in \mathbb{N}$. For example, $x_{\mathsf{a}}[k]$ or x_{a}^k represents the channel input of node a at time k. Also, we assume the input/output alphabets, $\mathcal{X}_i/\mathcal{Y}_i$ for node i are finite and that the probability transition which characterizes the channel $p(y_1, \cdots, y_n|x_1, \cdots, x_n)$ is well-defined. The channel is called memoryless if the current channel outputs depend only on the channel inputs at the same time instance.

The *channel capacity* of a discrete-time memoryless channel with a single input and single output is given by:

$$C := \sup_{p(x)} I(X;Y), \tag{1}$$

the supremum of the mutual information over all possible input distributions $p(x)$, where the mutual information $I(X;Y)$ is a quantity which measures the mutual dependence of random variables of X and Y, defined as:

$$I(X;Y) := \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\frac{p(x,y)}{p(x)p(y)}. \tag{2}$$

Channel capacity C is scaled in bits per channel use. If there exists more than one information flow in a channel, the quality of the channel should be measured in a multi dimensional space in which each dimension corresponds to an information flow. Thus the notion of "channel capacity" is extended to "capacity region" defined as the closure of all achievable rate tuples of information flows. In this study, we consider two users, a and b, who exchange two independent messages over a shared channel. Node a transmits messages w_{a} to node b at rate R_{a} and Node b transmits messages w_{b} to node a at rate R_{b}. We are thus interested in determining the two-dimensional capacity region, which consists of the supremum over all achievable rate pairs $(R_{\mathsf{a}}, R_{\mathsf{b}})$.

B. Side Information

In a bi-directional relay channel with two nodes a, b and one relay r, information flows from a $\rightarrow$ r $\rightarrow$ b and from b $\rightarrow$ r $\rightarrow$ a. The quality of the channels between (a and r) and (b and r) will naturally affect the capacity region. However, in bi-directional channels, other information can also be used to enhance the capacity region. We call this information *side information.* One of the main goals of this study is how to efficiently use side information. We consider two kinds of side information, one is *non-causal* side information, while the other is *causal* side information described as:

1. **Non-causal side information:** The messages of node i are side information for node i to decode the received signals from the bi-directional channel. In our case, the messages of a and b, w_{a} and w_{b} are side information for a and b, respectively. One commonly used coding scheme which exploits this type of side information is *Network Coding.* Network coding is used to mean that the relay constructs message $w_{\mathsf{r}} = w_{\mathsf{a}} \oplus w_{\mathsf{b}}$ and encodes it as $x_{\mathsf{r}}(w_{\mathsf{r}})$. The cardinality of w_{r} is reduced to the maximum of the cardinalities of w_{a} and w_{b}. Since a has w_{a} as side information, it can decode $\tilde{w}_{\mathsf{b}}$ by decoding $\tilde{w}_{\mathsf{r}}$ and exploiting its side-information or knowledge of w_{a}. Similarly, b can decode $\tilde{w}_{\mathsf{a}}$.
2. **Causal side information:** Every silent node can overhear and receive information from the channel. This is possible due to the shared half-duplex channel assumption. For example, let us assume when b transmits w_{b} to r that a is in receiving mode. It may not be possible for a to decode w_{b} using only the overheard signals. However, these overheard signals still act as information that can be used when a decodes w_{b} after r transmits to a.

For the remainder of this work, "side information" will refer to causal side-information unless otherwise stated.

C. Relaying Schemes

There may exist many different relaying computations. These differences depend on the required complexity or knowledge at the relay. The relative benefits and merits of the four relaying schemes are summarized in Table 1. The details are:

- *Amplify and forward* (AF): The relay r replicates the received symbol with given constraints, i.e. input power constraints in the Gaussian channel. The AF scheme requires no particular computation for relaying, and amplifies the noise in the receiving period during the relaying period.
- *Decode and forward* (DF): The relay first decodes both messages from nodes a and b then it independently encodes both decoded messages. The DF scheme requires

the full codebooks and huge computation power at the relay r.

- *Compress and forward* (CF): The relay performs something between DF and AF. It *compresses (or quantize)* the received signal. The relay requires the quantizing codebooks or output distribution $p(y_r)$ at the relay.
- *Mixed forward*: The relay exploits DF in one direction (from a $\rightarrow$ b), while it uses CF in the opposite direction (from a $\leftarrow$ b). In the mixed scheme, the relay requires one full codebook and one quantizing codebook.

III. PRELIMINARIES

A. Definitions and Notations

We consider three nodes a, b, and r. We use $R_a (R_b)$ to denote the transmitted data rate of message $W_a (W_b)$ from node a (b) to node b (a), i.e., W_a and W_b are independent and uniformly distributed in $\{0, \cdots, \lfloor 2^{nR_a} \rfloor - 1\} := \mathcal{S}_a$ (and similarly for b). We denote by $\Delta_\ell \geq 0$ the relative time duration of the ℓth phase, where $\sum_\ell \Delta_\ell = 1$. For a given number of total channel uses n, $\Delta_{\ell,n}$ denotes the duration of the ℓth phase, or equivalently $\Delta_{\ell,n} = n_\ell / n$, where n_ℓ is the number of channel uses in phase ℓ. Then, it is obvious that $\Delta_{\ell,n} \rightarrow \Delta_\ell$ with a proper choice of n_ℓ as $n \rightarrow \infty$.

We use channel input alphabet $\mathcal{X}_i^* = \mathcal{X}_i \cup \{\varnothing\}$ and channel output alphabet $\mathcal{Y}_i^* = \mathcal{Y}_i \cup \{\varnothing\}$ for node i, ($i \in \{a, b\}$), where $\mathcal{X}_i$ (resp. $\mathcal{Y}_i$) is the useful restricted input alphabet (resp. useful restricted output alphabet). Because of the half-duplex constraint, not all nodes transmit/receive during all phases and we use the dummy symbol $\varnothing$ to denote that there is no input or no output at a particular node during a particular phase. The half-duplex constraint forces either $X_i = \varnothing$ or $Y_i = \varnothing$ for each phase i. We will be constructing CF schemes in which received signals are compressed or quantized before being retransmitted. We let $\hat{Y}_i$ denote the compressed channel output of node i, which lies in the corresponding compressed output alphabet $\hat{\mathcal{Y}}_i$ for node i. We note that $\hat{\mathcal{Y}}_i$ is not necessarily equal to $\mathcal{Y}_i$. In the Gaussian noise channel, we consider the case $\mathcal{X}_i = \mathcal{Y}_i = \hat{\mathcal{Y}}_i = \mathbb{C}, \forall i \in a, b, r$.

For a given block length n, it will be convenient to denote the transmission at time $1 \leq k \leq n$ at node i by X_i^k, the reception at node i at time k by Y_i^k. Note that the distributions of X_i^k and Y_i^k depend on the value of k. During phase ℓ we use $X_i^{(\ell)}$ to denote the random variable with alphabet $\mathcal{X}_i^*$ and input distribution $p^{(\ell)}(x_i)$. It is also convenient to denote by $X_S^k := \{X_i^k | i \in S\}$, the set of transmissions by all nodes in the set S at time k, and by $X_S^{(\ell)} := \{X_i^{(\ell)} | i \in S\}$, a set of random variables with channel input distribution $p^{(\ell)}(x_S)$ for phase ℓ, where $x_S := \{x_i | i \in S\}$. Lower case letters x_i will denote instances of the upper case X_i which lie in the calligraphic alphabets $\mathcal{X}_i^*$. Boldface $\mathbf{x}_i$ represents a vector indexed by time at node i. Finally, we denote $\mathbf{x}_S := \{\mathbf{x}_i | i \in S\}$ as a set of vectors indexed by time. In this study, Q will denote a discrete time-sharing random variable with distribution $p(q)$. For convenience, we drop the notation $\varnothing$ from entropy and the mutual information terms when a node is not transmitting or receiving. For example, $I(X_a^{(1)}; Y_r^{(1)}) = I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)} = X_r^{(1)} = \varnothing)$ when b and r are in the receiving mode during phase 1. Formal definitions for the encoders, decoders, jointly typical sets and associated probabilities of error follow those of [16] and are omitted for brevity.

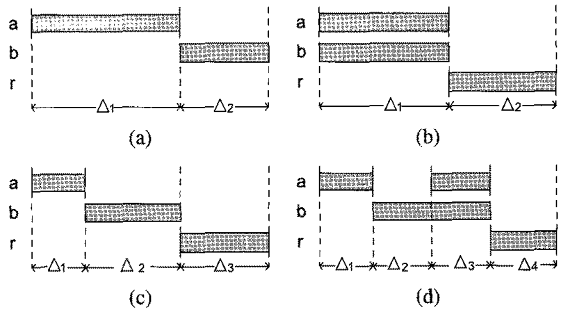

Fig. 2. Proposed protocol diagrams. Shaded areas denote transmission by the respective nodes. It is assumed that all nodes listen when not transmitting: (a) DT protocol, (b) MABC protocol, (c) TDBC protocol, and (d) HBC protocol.

A set of rates $R_{i,j}$ is said to be achievable for a protocol with phase durations $\{\Delta_\ell\}$ if there exist encoders/decoders of block length $n = 1, 2, \cdots$ with both $P[E_{i,j}] \rightarrow 0$ and $\Delta_{\ell,n} \rightarrow \Delta_\ell$ as $n \rightarrow \infty$ for all ℓ. An achievable rate region (resp. capacity region) is the closure of a set of (resp. all) achievable rate tuples for fixed $\{\Delta_\ell\}$.

B. Protocols

The capacity region of the simplest protocol for the bi-directional channel, *direct transmission* (DT) (Fig. 2(a)), is from Lemma 1 of [16] as:

$$R_a \leq \sup_{p^{(1)}(x_a)} \Delta_1 I(X_a^{(1)}; Y_b^{(1)})$$

$$R_b \leq \sup_{p^{(2)}(x_b)} \Delta_2 I(X_b^{(2)}; Y_a^{(2)}),$$

where the distributions are over the alphabets $\mathcal{X}_a$ and $\mathcal{X}_b$, respectively.

With a relay node r, we consider three protocols, which we indicate as *multiple access broadcast* (MABC) protocol, *time division broadcast* (TDBC) and *hybrid broadcast* (HBC). The comparisons of the three protocols are shown in Table 2. In the MABC protocol (Fig. 2(b)), nodes a and b transmit signals during phase 1. Then, the relay r computes the received signals and transmits signals generated depend on the relaying scheme during phase 2. With this scheme, no terminal node can receive any causal side-information during the first phase.

In the TDBC protocol (Fig. 2(c)), node a transmits during the first phase and node b follows during the second phase. In the last phase, the relay r transmits signals based on the received data from the first two phases. Here, node a has two signals to decode the message W_b; one is the transmissions from node b in the second phase and the other is from node r in the third phase. Similarly, node b decodes W_a based on two independent signals from the first and third phases.

Lastly, we consider a HBC protocol (Fig. 2(d)) which is a generalized protocol of the MABC and TDBC protocols with 4 phases.

Table 1. Comparison between four relaying schemes.

Relaying	Complexity	Noise at relay	Relay needs
AF	very low	carried plus noise at rx	nothing
DF	high	perfectly eliminated	full codebooks
CF	low	carried plus distortion	$p(y_r)$
mixed	moderate	partially carried	one codebook, $p(y_r)$

Table 2. Comparison between three protocols for single relay bi-directional channel.

Protocol	Side information	Number of phases	Interference
MABC	not present	2	present
TDBC	present	3	not present
HBC	present	4	present

IV. ACHIEVABLE RATE REGIONS

In this section, we present achievable rate regions in Theorems 1, 3, 5, 7, 9, 12, and 14. Theorems 1, 3, and 5 are for two phase MABC protocols.

The next three theorems 7, 9, and 12 employ three phase TDBC protocols. The last Theorem 14 considers the four phase HBC protocol. In Theorem 14 the relay uses DF when retransmitting both messages.

A. MABC Protocol

In the MABC Protocol, during phase 1 two terminal nodes a and b independently generate and send codewords $\mathbf{x}_a^{(1)}(w_a)$ and $\mathbf{x}_b^{(1)}(w_b)$ to the relay. Since we assume half-duplex nodes, there doesn't exist any side information during phase 1. The relay receives the signal $\mathbf{y}_r^{(1)}$ according to $p(y_r^{(1)}|x_a^{(1)}, x_b^{(1)})$.

A.1 Decode and Forward

In the DF relaying scheme, relay decodes both messages w_a and w_b. Then, it constructs message w_r by network coding, i.e., $w_r = \hat{w}_a \oplus \hat{w}_b$ and the message is mapped into the codeword $\mathbf{x}_r^{(2)}(w_r)$. After decoding $\tilde{w}_r$, since a has its own message as non-causal side information it can subtract w_a from w_r. Similarly, b decodes w_a from $\tilde{w}_r$.

Theorem 1: An achievable rate region of the half-duplex bi-directional relay channel with the decode and forward MABC protocol is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \min\left\{\Delta_1 I(X_a^{(1)}; Y_r^{(1)}|X_b^{(1)}, Q), \Delta_2 I(X_r^{(2)}; Y_b^{(2)})\right\} \tag{3}$$

$$R_b < \min\left\{\Delta_1 I(X_b^{(1)}; Y_r^{(1)}|X_a^{(1)}, Q), \Delta_2 I(X_r^{(2)}; Y_a^{(2)})\right\} \tag{4}$$

$$R_a + R_b < \Delta_1 I(X_a^{(1)}, X_b^{(1)}; Y_r^{(1)}|Q) \tag{5}$$

over all joint distributions, $p^{(1)}(q)p^{(1)}(x_a|q)p^{(1)}(x_b|q)$ $p^{(1)}(y_r|x_a, x_b)p^{(2)}(x_r)p^{(2)}(y_a, y_b|x_r)$ with $|\mathcal{Q}| \leq 3$ over the restricted alphabet $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r \times \mathcal{Y}_a \times \mathcal{Y}_b \times \mathcal{Y}_r$. □

Remark 2: We use a MAC scheme and network coding in Theorem 1. The detailed proof is provided in [16].

A.2 Compress and Forward

In the CF relaying scheme, relay encodes the received $\mathbf{y}_r^{(1)}$ into a signal $\hat{\mathbf{y}}_r^{(1)}$ based on the compressing codebook it has. Then the relay constructs the codeword $\mathbf{x}_r^{(2)}$ from $\hat{\mathbf{y}}_r^{(1)}$ which is broadcast in phase 2. The challenge is to construct effective compressing codebook such that just enough information is delivered to the destination nodes to decode the original messages. A key observation is that the nodes may use their own phase 1 transmitted messages as non-causal side information in the decoding of phase 2 signals. We now present our achievable rate region for the CF MABC, presented in [16]. Similar results were obtained independently in [24].

Theorem 3: An achievable rate region of the half-duplex bi-directional relay channel with the compress and forward MABC protocol is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \Delta_1 I(X_a^{(1)}; \hat{Y}_r^{(1)}|X_b^{(1)}, Q) \tag{6}$$

$$R_b < \Delta_1 I(X_b^{(1)}; \hat{Y}_r^{(1)}|X_a^{(1)}, Q) \tag{7}$$

subject to

$$\Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)}|X_b^{(1)}, Q) < \Delta_2 I(X_r^{(2)}; Y_b^{(2)}) \tag{8}$$

$$\Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)}|X_a^{(1)}, Q) < \Delta_2 I(X_r^{(2)}; Y_a^{(2)}) \tag{9}$$

over all joint distributions,

$$p(q, x_a, x_b, x_r, y_a, y_b, y_r, \hat{y}_r) = p^{(1)}(q, x_a, x_b, y_r, \hat{y}_r|q)p^{(2)}(x_r, y_a, y_b) \tag{10}$$

where

$$p^{(1)}(q, x_a, x_b, y_r, \hat{y}_r) = p^{(1)}(q)p^{(1)}(x_a|q)p^{(1)}(x_b|q)p^{(1)}(y_r|x_a, x_b)p^{(1)}(\hat{y}_r|y_r, q) \tag{11}$$

$$p^{(2)}(x_r, y_a, y_b) = p^{(2)}(x_r)p^{(2)}(y_a, y_b|x_r) \tag{12}$$

with $|\mathcal{Q}| \leq 4$ over the restricted alphabet $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r \times \mathcal{Y}_a \times \mathcal{Y}_b \times \mathcal{Y}_r \times \hat{\mathcal{Y}}_r$. □

Remark 4: Strong typicality is required for the proof of Theorem 3 in order to apply the Markov lemma to $(X_a^{(1)}, X_b^{(1)}) \to Y_r^{(1)} \to \hat{Y}_r^{(1)}$ for each given q. Since strong typicality is defined for discrete alphabets, Theorem 3 cannot be directly extended to continuous alphabets. However, the extended Markov lemma (see Remark 30 of [54] as well as Lemma 3 of [55]) shows that for Gaussian distributions, the Markov lemma still applies.

A.3 Mixed Forward

In the case that the channel is asymmetric between the two nodes at the relay, it may be more effective to have the better channel use a DF scheme while the worse channel uses a CF scheme. In the mixed forwarding, the relay exploits a novel strategy when it constructs codeword $\mathbf{x}_r^{(2)}$. The a $\rightarrow$ r $\rightarrow$ b link uses decode and forward while a $\leftarrow$ r $\leftarrow$ b link uses compress and forward. Furthermore, the relay applies a Gelfand-Pinsker coding scheme to avoid interference when broadcasting w_a in the r $\rightarrow$ b link. In this case, an achievable rate region is given by Theorem 5.

Theorem 5: An achievable rate region of the half-duplex bi-directional relay channel with the mixed forward MABC protocol, where a $\rightarrow$ b link uses decode and forward and b $\rightarrow$ a link uses compress and forward, is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \min\Big\{\Delta_1 I(X_a^{(1)}; Y_r^{(1)}|Q), \Delta_2 I(U_r^{(2)}; Y_b^{(2)}|Q) - \Delta_2 I(U_r^{(2)}; U_b^{(2)}|Q)\Big\} \quad (13)$$

$$R_b < \Delta_1 I(X_b^{(1)}; \hat{Y}_r^{(1)}|X_a^{(1)}, Q) \quad (14)$$

subject to

$$\Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)}|X_a^{(1)}, Q) < \min\{\Delta_2 I(U_r^{(2)}, U_b^{(2)}; Y_a^{(2)}|Q), \Delta_2 I(U_b^{(2)}; U_r^{(2)}, Y_a^{(2)}|Q)\} \quad (15)$$

over all joint distributions,

$$p(q, x_a, x_b, x_r, u_a, u_b, u_r, y_a, y_b, y_r, \hat{y}_r) = p(q)p^{(1)}(x_a, x_b, y_r, \hat{y}_r|q)p^{(2)}(u_b, u_r, x_r, y_a, y_b|q) \quad (16)$$

where

$$p^{(1)}(x_a, x_b, y_r, \hat{y}_r|q) = p^{(1)}(x_a|q)p^{(1)}(x_b|q)p^{(1)}(y_r|x_a, x_b)p^{(1)}(\hat{y}_r|y_r, q) \quad (17)$$

$$p^{(2)}(u_b, u_r, x_r, y_a, y_b|q) = p^{(2)}(u_b, u_r|q)p^{(2)}(x_r|u_b, u_r, q)p^{(2)}(y_a, y_b|x_r) \quad (18)$$

with $|\mathcal{Q}| \leq 7$ over the alphabet $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r \times \mathcal{U}_b \times \mathcal{U}_r \times \mathcal{Y}_a \times \mathcal{Y}_b \times \mathcal{Y}_r \times \hat{\mathcal{Y}}_r$. □

Remark 6: In the second phase, the relay broadcasts the received signals from the first phase. In contrast to the DF and CF schemes, one of the terminal nodes (in this case, node a) has perfect information of interference at node a, while there remains unknown interference at the other side (at node b). We use a Gel'fand-Pinsker coding scheme [56], [57] for the link r $\rightarrow$ b which yields the second term of (13). From the side information w_a available at node a, node a is able to reduce the interference, yielding (15).

B. TDBC Protocol

The TDBC protocol consists of sequential three phases. The MABC protocol takes benefits of the multiple access gains during phase 1. However, there is no causal side information. The main motivation of the TDBC protocol is to exploit the causal side information available at both terminal nodes.

As in the previous section, the relay processes its received signals in one of three ways. In Theorem 7 the relay decodes both messages w_a and w_b, while in Theorem 9 the relay uses a compress and forward scheme (hence named CF TDBC) in which the signals received are not decoded but compressed for transmission during phase 3. In Theorem 12 the a $\rightarrow$ r $\rightarrow$ b link uses decode and forward and the a $\leftarrow$ r $\leftarrow$ b link uses CF (hence named mixed TDBC). We now proceed to state and prove the achievable rate regions for these three protocols.

B.1 Decode and Forward

Theorem 7: An achievable rate region of the half-duplex bi-directional relay channel with the decode and forward TDBC protocol is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \min\Big\{\Delta_1 I(X_a^{(1)}; Y_r^{(1)}), \Delta_1 I(X_a^{(1)}; Y_b^{(1)}) + \Delta_3 I(X_r^{(3)}; Y_b^{(3)})\Big\} \quad (19)$$

$$R_b < \min\Big\{\Delta_2 I(X_b^{(2)}; Y_r^{(2)}), \Delta_2 I(X_b^{(2)}; Y_a^{(2)}) + \Delta_3 I(X_r^{(3)}; Y_a^{(3)})\Big\} \quad (20)$$

over all joint distributions, $p^{(1)}(x_a)p^{(1)}(y_b, y_r|x_a)$ $p^{(2)}(x_b)p^{(2)}(y_a, y_r|x_b)p^{(3)}(x_r)p^{(3)}(y_a, y_b|x_r)$ over the restricted alphabet $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r \times \mathcal{Y}_a^2 \times \mathcal{Y}_b^2 \times \mathcal{Y}_r^2$. □

Remark 8: We use random binning to exploit causal side information in Theorem 7. The detailed proof is provided in [16].

B.2 Compress and Forward

In the CF TDBC protocol, we use two different broadcasting schemes for relay transmission; one is Marton's broadcast scheme of [58] and the other is an assumption of a compound channel. In Marton's broadcast scheme two independent messages are transmitted to the two distinct receivers, while in a compound channel, a common message is transmitted to both receivers which may have different side information. We denote the first relay-broadcast phase as phase 3 and the second as phase 4.

Theorem 9: An achievable rate region of the half-duplex bi-directional relay channel with the CF TDBC protocol is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \Delta_1 I(X_a^{(1)}; \hat{Y}_r^{(1)}, Y_b^{(1)}|Q) \quad (21)$$

$$R_b < \Delta_2 I(X_b^{(2)}; \hat{Y}_r^{(2)}, Y_a^{(2)}|Q) \quad (22)$$

subject to

$$\alpha_a \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)}|Q) < \Delta_3 I(U_a^{(3)}; Y_b^{(3)}|Q) \quad (23)$$

$$\alpha_b \Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)}|Q) < \Delta_3 I(U_b^{(3)}; Y_a^{(3)}|Q) \quad (24)$$

$$\alpha_a \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)}|Q) + \alpha_b \Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)}|Q) < \Delta_3 I(U_a^{(3)}; Y_b^{(3)}|Q) + \Delta_3 I(U_b^{(3)}; Y_a^{(3)}|Q) - \Delta_3 I(U_a^{(3)}; U_b^{(3)}|Q) \quad (25)$$

$$(1-\alpha_a)\Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)}|Q) + \Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)}|X_b^{(2)}, Q) < \Delta_4 I(X_r^{(4)}; Y_b^{(4)}) + \Delta_1 I(\hat{Y}_r^{(1)}; Y_b^{(1)}|Q) \quad (26)$$

$$(1-\alpha_b)\Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)}|Q) + \Delta_1 I(Y_r^{(1)}; \hat{Y}_r^{(1)}|X_a^{(1)}, Q) < \Delta_4 I(X_r^{(4)}; Y_a^{(4)}) + \Delta_2 I(\hat{Y}_r^{(2)}; Y_a^{(2)}|Q) \quad (27)$$

where $0 < \alpha_a, \alpha_b < 1$ over all joint distributions,

$$p(q, x_a, x_b, x_r, y_a, y_b, y_r, \hat{y}_r) = p(q)p^{(1)}(x_a, y_b, y_r, \hat{y}_r|q)p^{(2)}(x_b, y_a, y_r, \hat{y}_r|q) \times p^{(3)}(u_a, u_b, x_r, y_a, y_b|q)p^{(4)}(x_r, y_a, y_b) \quad (28)$$

where

$$p^{(1)}(x_a, y_b, y_r, \hat{y}_r|q) = p^{(1)}(x_a|q)p^{(1)}(y_b, y_r|x_a)p^{(1)}(\hat{y}_r|y_r, q) \quad (29)$$

$$p^{(2)}(x_b, y_a, y_r, \hat{y}_r|q) = p^{(2)}(x_b|q)p^{(2)}(y_a, y_r|x_b)p^{(2)}(\hat{y}_r|y_r, q) \quad (30)$$

$$p^{(3)}(u_a, u_b, x_r, y_a, y_b|q) = p^{(3)}(u_a, u_b|q)p^{(3)}(x_r|u_a, u_b, q)p^{(3)}(y_a, y_b|x_r) \quad (31)$$

$$p^{(4)}(x_r, y_a, y_b) = p^{(4)}(x_r)p^{(4)}(y_a, y_b|x_r) \quad (32)$$

with $|\mathcal{Q}| \leq 13$ over the alphabet $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r^2 \times \mathcal{Y}_a^3 \times \mathcal{Y}_b^3 \times \mathcal{Y}_r^2 \times \hat{\mathcal{Y}}_r^2$. □

Remark 10: If side information is very limited, then with $\Delta_4 \rightarrow 0$, $\alpha_a, \alpha_b \rightarrow 1$ the relay phase acts as a classical broadcast channel. At the opposite extreme, the side information cancels out all interference, i.e., $I(Y_r^{(1)}; \hat{Y}_r^{(1)}|X_a^{(1)}) = I(Y_r^{(2)}; \hat{Y}_r^{(2)}|X_b^{(2)}) = 0$. Then, we set $\Delta_3 \rightarrow 0$, $\alpha_a, \alpha_b \rightarrow 0$.[3]

Remark 11: We use random binning and a Marton's coding scheme in Theorem 9. The detailed proof is provided in [15].

B.3 Mixed Forward

Similar to the Mixed MABC protocol, we provide a rate region for a TDBC protocol in which the $a \rightarrow r \rightarrow b$ link uses DF and the reverse link exploits CF.

Theorem 12: An achievable rate region for the half-duplex bi-directional relay channel with a mixed TDBC protocol, where the $a \rightarrow r \rightarrow b$ link uses decode and forward and the $b \rightarrow r \rightarrow a$ link uses compress and forward, is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \min\Big\{\Delta_1 I(X_a^{(1)}; Y_r^{(1)}), \Delta_1 I(X_a^{(1)}; Y_b^{(1)}) + \Delta_3 I(U_r^{(3)}; Y_b^{(3)}|Q) - \Delta_3 I(U_r^{(3)}; U_b^{(3)}|Q)\Big\} \quad (33)$$

$$R_b < \Delta_2 I(X_b^{(2)}; \hat{Y}_r^{(2)}, Y_a^{(2)}|Q) \quad (34)$$

subject to

$$\Delta_2 I(Y_r^{(2)}; \hat{Y}_r^{(2)}|Y_a^{(2)}, Q) < \min\{\Delta_3 I(U_r^{(3)}, U_b^{(3)}; Y_a^{(3)}|Q), \Delta_3 I(U_b^{(3)}; U_r^{(3)}, Y_a^{(3)}|Q)\} \quad (35)$$

over all joint distributions,

$$p(q, x_a, x_b, x_r, u_b, u_r, y_a, y_b, y_r, \hat{y}_r) = p(q)p^{(1)}(x_a, y_b, y_r)p^{(2)}(x_b, y_a, y_r, \hat{y}_r|q)p^{(3)}(u_b, u_r, x_r, y_a, y_b|q) \quad (36)$$

[3]This choice of $\Delta_3, \alpha_a, \alpha_b$ is on the boundary of the closure of the achievable rate region.

where

$$p^{(1)}(x_a, y_b, y_r) = p^{(1)}(x_a)p^{(1)}(y_b, y_r|x_a) \quad (37)$$

$$p^{(2)}(x_b, y_a, y_r, \hat{y}_r|q) = p^{(2)}(x_b|q)p^{(2)}(y_a, y_r|x_b)p^{(2)}(\hat{y}_r|y_r, q) \quad (38)$$

$$p^{(3)}(u_a, u_b, u_r, x_r, y_a, y_b|q) = p^{(3)}(u_b, u_r|q)p^{(3)}(x_r|u_b, u_r, q)p^{(3)}(y_a, y_b|x_r) \quad (39)$$

with $|\mathcal{Q}| \leq 6$ over the alphabet $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r \times \mathcal{U}_b \times \mathcal{U}_r \times \mathcal{Y}_a^2 \times \mathcal{Y}_b^2 \times \mathcal{Y}_r^2 \times \hat{\mathcal{Y}}_r$. □

Remark 13: We use random binning and a Gel'fand-Pinsker coding scheme in Theorem 12. The detailed proof is provided in [15].

C. HBC Protocol

We have looked at the two-phase MABC and three-phase TDBC protocols. As we will see, each of these protocols sees gains over the other depending on the channel conditions. To take advantage of the benefits of multiple-access (seen in the MABC) and information in direct link between a and b (seen in the TDBC), the HBC protocol combines both protocols into a new four phase "hybird" protocol: During phase 1, node a is the sole node to transmit, while both the relay and node b receive this transmission. During phase 2, node b transmits while the relay and node a receive. During phase 3, both a and b multiple access to r. After phase 3, the relay processes the signals received during the first three phases and proceeds to broadcast to nodes a and b during the fourth phase.

Theorem 14: An achievable rate region of the half-duplex bi-directional relay channel with the decode and forward HBC protocol is the closure of the set of all points (R_a, R_b) satisfying

$$R_a < \min\Big\{\Delta_1 I(X_a^{(1)}; Y_r^{(1)}) + \Delta_3 I(X_a^{(3)}; Y_r^{(3)}|X_b^{(3)}, Q), \Delta_1 I(X_a^{(1)}; Y_b^{(1)}) + \Delta_4 I(X_r^{(4)}; Y_b^{(4)})\Big\} \quad (40)$$

$$R_b < \min\Big\{\Delta_2 I(X_b^{(2)}; Y_r^{(2)}) + \Delta_3 I(X_b^{(3)}; Y_r^{(3)}|X_a^{(3)}, Q), \Delta_2 I(X_b^{(2)}; Y_a^{(2)}) + \Delta_4 I(X_r^{(4)}; Y_a^{(4)})\Big\} \quad (41)$$

$$R_a + R_b < \Delta_1 I(X_a^{(1)}; Y_r^{(1)}) + \Delta_2 I(X_b^{(2)}; Y_r^{(2)}) + \Delta_3 I(X_a^{(3)}, X_b^{(3)}; Y_r^{(3)}|Q) \quad (42)$$

over all joint distributions, $p^{(1)}(x_a)p^{(1)}(y_b, y_r|x_a)p^{(2)}(x_b)$ $p^{(2)}(y_a, y_r|x_b)p^{(3)}(q)p^{(3)}(x_a|q)p^{(3)}(x_b|q)p^{(3)}(y_r|x_a, x_b)p^{(4)}(x_r)$ $p^{(4)}(y_a, y_b|x_r)$ over the restricted alphabet $\mathcal{X}_a^2 \times \mathcal{X}_b^2 \times \mathcal{X}_r \times \mathcal{Y}_a^2 \times \mathcal{Y}_b^2 \times \mathcal{Y}_r^3$ with $|\mathcal{Q}| \leq 3$. □

Remark 15: Relay r receives data from terminal nodes during phases 1–3, which is decoded by the relay using a MAC protocol to recover w_a, w_b. Theorem 14 then follows the same argument as the proof of Theorem 7. The rigorous proof is provided in [16].

V. OUTER BOUNDS

We derive outer bounds for the MABC, TDBC and HBC protocols using the cut-set bound Lemma 1 of [16]. We state the outer bounds, proven in [16] which will be numerically evaluated and discussed in the following sections.

A. MABC Protocol

Theorem 16: (Outer bound) The capacity region of the half-duplex bi-directional relay channel with the MABC protocol is outer bounded by

$$R_a < \min\left\{\Delta_1 I(X_a^{(1)};Y_r^{(1)}|X_b^{(1)},Q), \Delta_2 I(X_r^{(2)};Y_b^{(2)})\right\} \quad (43)$$

$$R_b < \min\left\{\Delta_1 I(X_b^{(1)};Y_r^{(1)}|X_a^{(1)},Q), \Delta_2 I(X_r^{(2)};Y_a^{(2)})\right\} \quad (44)$$

over all joint distributions $p^{(1)}(q)p^{(1)}(x_a|q)p^{(1)}(x_b|q)p^{(2)}(x_r)$ with $|\mathcal{Q}| \le 2$ over the restricted alphabet $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r$. □

B. TDBC Protocol

Theorem 17: (Outer bound) The capacity region of the half-duplex bi-directional relay channel with the TDBC protocol is outer bounded by

$$R_a \le \min\left\{\Delta_1 I(X_a^{(1)};Y_r^{(1)},Y_b^{(1)}), \Delta_1 I(X_a^{(1)};Y_b^{(1)}) + \Delta_3 I(X_r^{(3)};Y_b^{(3)})\right\} \quad (45)$$

$$R_b \le \min\left\{\Delta_2 I(X_b^{(2)};Y_r^{(2)},Y_a^{(2)}), \Delta_2 I(X_b^{(2)};Y_a^{(2)}) + \Delta_3 I(X_r^{(3)};Y_a^{(3)})\right\} \quad (46)$$

for some choice of the joint distribution $p^{(1)}(x_a)\,p^{(2)}(x_b)p^{(3)}(x_r)$ over the restricted alphabet $\mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r$. □

Remark 18: To prove Theorem 17, we follow the same argument as the proof of Theorem 16.

C. HBC Protocol

Theorem 19: (Outer bound) The capacity region of the half-duplex bi-directional relay channel with the HBC protocol is outer bounded by the union

$$R_a \le \min\left\{\Delta_1 I(X_a^{(1)};Y_r^{(1)},Y_b^{(1)}) + \Delta_3 I(X_a^{(3)};Y_r^{(3)}|X_b^{(3)},Q), \Delta_1 I(X_a^{(1)};Y_b^{(1)}) + \Delta_4 I(X_r^{(4)};Y_b^{(4)})\right\} \quad (47)$$

$$R_b \le \min\left\{\Delta_2 I(X_b^{(2)};Y_r^{(2)},Y_a^{(2)}) + \Delta_3 I(X_b^{(3)};Y_r^{(3)}|X_a^{(3)},Q), \Delta_2 I(X_b^{(2)};Y_a^{(2)}) + \Delta_4 I(X_r^{(4)};Y_a^{(4)})\right\} \quad (48)$$

over all joint distributions $p^{(1)}(x_a)p^{(2)}(x_b)p^{(3)}(q)\,p^{(3)}(x_a,x_b|q)$ $p^{(4)}(x_r)$ with $|\mathcal{Q}| \le 2$ over the restricted alphabet $\mathcal{X}_a^2 \times \mathcal{X}_b^2 \times \mathcal{X}_r$. □

Remark 20: We use the same arguments as in the proofs of Theorems 16.

VI. GAUSSIAN NOISE CHANNEL

We now apply the derived bounds to the bi-directional relaying channel subject to independent, identically distributed white Gaussian noise. Since with continuous alphabets strong typicality cannot be used, the achievable rate regions of CF and mixed schemes do not directly computed in continuous domains. However, for the Gaussian distributions, the Markov lemma of [55], which generalizes the Markov lemma to the continuous domains, allows to extend the previous results in the discrete memoryless channel to the Gaussian case.

The corresponding Gaussian channel model is:

$$Y_a[m] = h_{ra}X_r[m] + h_{ba}X_b[m] + Z_a[m] \quad (49)$$

$$Y_b[m] = h_{rb}X_r[m] + h_{ab}X_a[m] + Z_b[m] \quad (50)$$

$$Y_r[m] = h_{ar}X_a[m] + h_{br}X_b[m] + Z_r[m] \quad (51)$$

where $X_i[m]$'s follow the input distributions $X_i^{(\ell)} \sim \mathcal{CN}(0,P_i)$ during transmitting, where $m \in [n\sum_{j=0}^{\ell-1}\Delta_{j,n} + 1, n\sum_{j=0}^{\ell}\Delta_{j,n}]$ and $\mathcal{CN}(\mu,\sigma^2)$ indicates a complex Gaussian random variable with mean μ and variance σ^2, and ℓ corresponds to the phase number. If node i is in the transmitting mode, the transmit power is less than or equal to P_i, i.e., $E[X_i^2] \le P_i$. $h_{ij} \in \mathbb{C}$ is the effective channel gain between transmitter i and receiver j. The channel is assumed to be reciprocal such that $h_{ij} = h_{ji}$ and each node has fill CSI, i.e., fully aware of h_{ar}, h_{br}, and h_{ab}. The noise at all receivers is of unit power, additive, white Gaussian, complex and circularly symmetric. we also denotes the function $C(x) := \log_2(1+x)$.

For the CF scheme, we denote $\hat{Y}_r^{(\ell)}$ as zero mean Gaussians and $P_y^{(\ell)} := E[(Y_r^{(\ell)})^2]$, $P_{\hat{y}}^{(\ell)} := E[(\hat{Y}_r^{(\ell)})^2]$ and $\sigma_y^{(\ell)} := E[\hat{Y}_r^{(\ell)}Y_r^{(\ell)}]$. The relationship between $Y_r[m]$ and $\hat{Y}_r[m]$ are given by the following equivalent channel model:

$$\hat{Y}_r[m] = h_{r\hat{r}}[m]Y_r[m] + Z_{\hat{r}}[m] \quad (52)$$

where $Y_r[m]$, $\hat{Y}_r[m]$ and $Z_{\hat{r}}[m]$ follow the distributions $Y_r^{(\ell)} \sim \mathcal{CN}(0,P_y^{(\ell)})$, $\hat{Y}_r^{(\ell)} \sim \mathcal{CN}(0,P_{\hat{y}}^{(\ell)})$ and $Z_{\hat{r}}^{(\ell)} \sim \mathcal{CN}(0,P_{\hat{y}}^{(\ell)} - \frac{(\sigma_y^{(\ell)})^2}{P_y^{(\ell)}})$ and $h_{r\hat{r}}[m] = \frac{\sigma_y^{(\ell)}}{P_y^{(\ell)}}$, where $m \in [n\sum_{j=0}^{\ell-1}\Delta_{j,n} + 1, n\sum_{j=0}^{\ell}\Delta_{j,n}]$. We will numerically optimize, $P_{\hat{y}}^{(\ell)}$ and $\sigma_y^{(\ell)}$ in the following section.

We consider four different relaying schemes for each MABC and TDBC bi-directional protocol as we considered the previous sections. In addition to achievable rate regions, we apply outer bounds of the MABC and TDBC protocols to the Gaussian channel. Also, we consider the DF scheme for the HBC protocol.

VII. NUMERICAL EVALUATIONS

To show relative benefits of each protocol, we plot the bounds described in Section VI in this section for a number of different channel conditions. We compare the rate regions and outer bounds in three different geometric cases; one symmetric case ($h_{ar} = h_{br} = 1$, $h_{ab} = 0.2$) as well as two asymmetric cases ($h_{ar} = 0.6, h_{br} = 20, h_{ab} = 0.5$, and $h_{ar} = 20, h_{br} = 0.6, h_{ab} = 0.5$) for two different transmit powers (SNRs of 0 and 20 dB). The main results of this section are that relative strengths are different under different channel conditions.

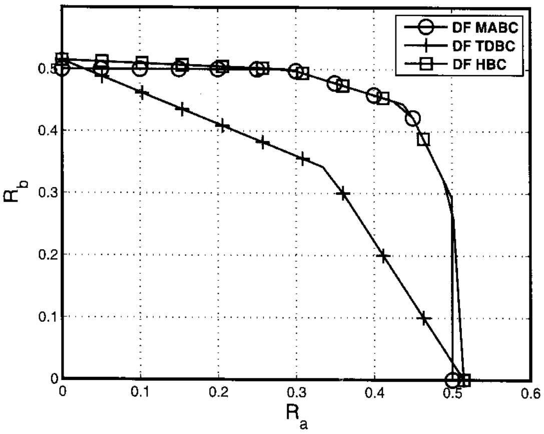

Fig. 3. Comparison of bi-directional regions with $h_{ar} = h_{br} = 1$, $h_{ab} = 0.2$, $P_a = P_b = P_r = 0$ dB.

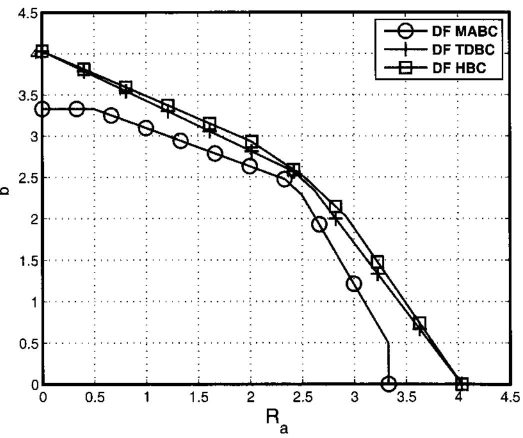

Fig. 4. Comparison of bi-directional regions with $h_{ar} = h_{br} = 1$, $h_{ab} = 0.2$, $P_a = P_b = P_r = 20$ dB.

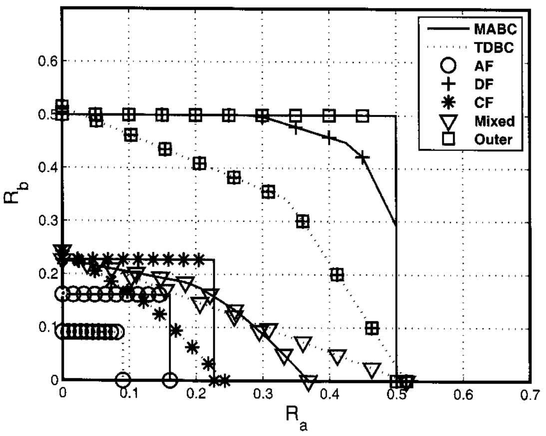

Fig. 5. Comparison of bi-directional regions with $h_{ar} = h_{br} = 1$, $h_{ab} = 0.2$, $P_a = P_b = P_r = 0$ dB.

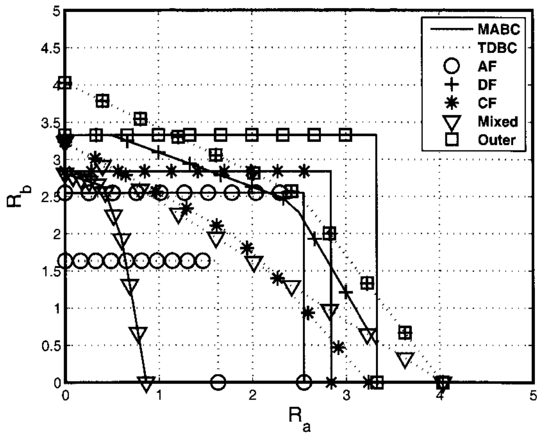

Fig. 6. Comparison of bi-directional regions with $h_{ar} = h_{br} = 1$, $h_{ab} = 0.2$, $P_a = P_b = P_r = 20$ dB.

A. Achievable Rate Region Comparisons with the DF Relaying Schemes

In this case $h_{ar} = h_{br} = 1$, $h_{ab} = 0.2$. Achievable rate regions of the three DF protocols are plotted in Figs. 3 and 4 (in the low and the high SNR regime, respectively). In the low SNR regime, the MABC protocol dominates the TDBC protocol, while the TDBC protocol is better in the high SNR regime. Since the HBC protocol is generalized from two protocols, the region of the HBC protocol is always larger than or equal to those of the other protocols. Surprisingly, the rate pairs of the HBC protocol cannot be achieved by any convex combinations of the other protocols in some regimes. This implies that the HBC protocol generates a new protocol deferent from the MABC or TDBC protocols in general.

B. Achievable Rate Region Comparisons with the MABC and TDBC Protocols

We compare the achievable rate regions and outer bounds of the MABC and TDBC protocols for three channel conditions at transmit SNRs of 0 and 20 dB.

B.1 Symmetric Case

In this case $h_{ar} = h_{br} = 1$ (Figs. 5 and 6). In the low SNR regime, the DF MABC protocol achieves the best performance. The MABC protocol in general dominates the TDBC protocol since the advantage of side information and reduced interference are relatively smaller in this regime than using less temporal phase and the multiple access gain. The DF scheme is better than the other schemes since the noise cancelation is important in this regime. In contrast, the DF TDBC protocol outperforms the other protocols at high SNR since the direct link is strong enough to exploit side information in the high SNR regime.

In the TDBC protocol, the CF scheme is outperformed by the DF scheme since the DF uses two independent channels in two different phases, while the CF uses one common channel with two receivers. This implies that $R_a^{DF} < \Delta_1 C(\cdot) + \Delta_3 C(\cdot)$ for the DF, while $R_a^{CF} < \Delta_1 C(\sum \cdot)$ for the CF scheme. However, under the MABC protocol, the CF scheme achieves larger region than the DF scheme in the high SNR regime since the interference of the multiple access to the relay affects the DF MABC scheme but not the CF scheme.

The achievable rate region of the Mixed TDBC protocol is

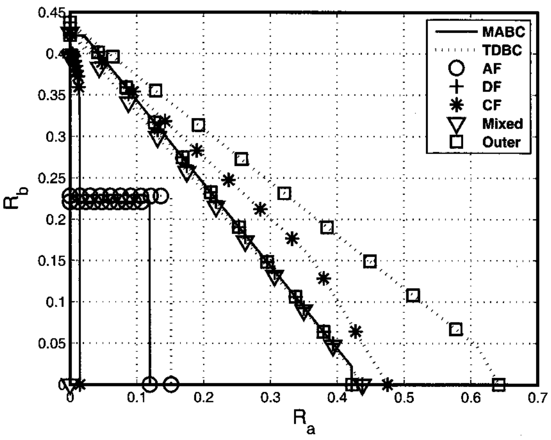

Fig. 7. Comparison of bi-directional regions with $h_{\mathsf{ar}} = 0.6$, $h_{\mathsf{br}} = 20$, $h_{\mathsf{ab}} = 0.5$, $P_{\mathsf{a}} = P_{\mathsf{b}} = P_{\mathsf{r}} = 0$ dB.

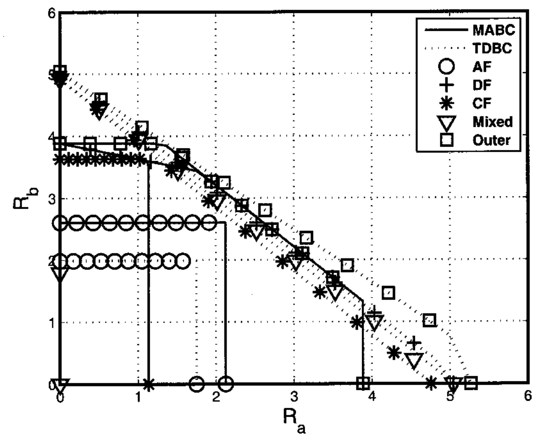

Fig. 8. Comparison of bi-directional regions with $h_{\mathsf{ar}} = 0.6$, $h_{\mathsf{br}} = 20$, $h_{\mathsf{ab}} = 0.5$, $P_{\mathsf{a}} = P_{\mathsf{b}} = P_{\mathsf{r}} = 20$ dB.

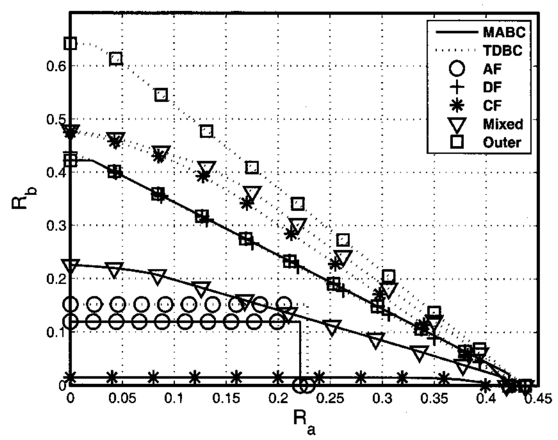

Fig. 9. Comparison of bi-directional regions with $h_{\mathsf{ar}} = 20$, $h_{\mathsf{br}} = 0.6$, $h_{\mathsf{ab}} = 0.5$, $P_{\mathsf{a}} = P_{\mathsf{b}} = P_{\mathsf{r}} = 0$ dB.

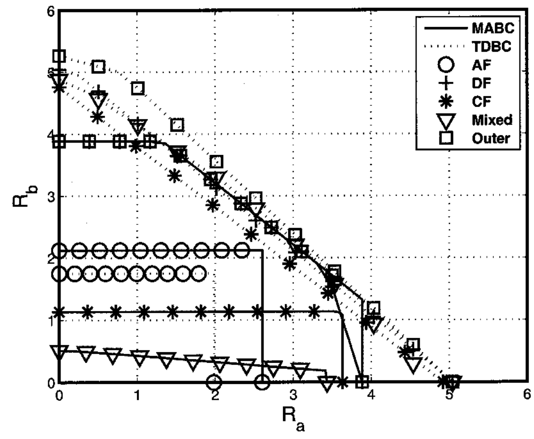

Fig. 10. Comparison of bi-directional regions with $h_{\mathsf{ar}} = 20$, $h_{\mathsf{br}} = 0.6$, $h_{\mathsf{ab}} = 0.5$, $P_{\mathsf{a}} = P_{\mathsf{b}} = P_{\mathsf{r}} = 20$ dB.

in between the CF TDBC protocol and the DF TDBC protocol. In the TDBC protocol, $\max_{R_{\mathsf{b}}} R_{\mathsf{a}}^{MIX} = \max_{R_{\mathsf{b}}} R_{\mathsf{a}}^{DF}$, where R_{a}^{MIX} is the rate of node a in the mixed scheme. Here the max is the maximum over all rates in the achievable rate regions. The $\max R_{\mathsf{a}}^{MIX}$ is achieved with $\Delta_2 = 0$. The rate R_{a}^{DF} is similarly described and $\max R_{\mathsf{a}}^{DF}$ is achieved in a similar way. Thus, the point $(\max R_{\mathsf{a}}^{DF}, 0)$ exists in both the mixed scheme and the DF scheme. In the TDBC scheme we can make a particular rate to 0 by setting the corresponding Δ_i to 0. However, in the MABC protocol this cannot be achieved. In the MABC protocol, even in the case $R_{\mathsf{a}} = 0$ or $R_{\mathsf{b}} = 0$, the transmit power (P_{a} or P_{b}) remains constant in the transmitting phase and does not decrease to 0, acting as interference for the opposite transmission. Therefore, $\max_{R_{\mathsf{b}}} R_{\mathsf{a}}^{MIX} \leq \max_{R_{\mathsf{b}}} R_{\mathsf{a}}^{DF}$. This interference seen in MABC protocols can be seen more explicitly in the high SNR regime, where the gap between the intercept points of the mixed scheme and the DF scheme grows as the interference increases. We note that this effect is because no power optimization is performed in our scenarios. If we were to allow the transmission power optimization, larger achievable rate regions for the mixed MABC protocol could be achieved.

In the low SNR regime, the achievable rate region of the DF MABC protocol is tight to the outer bound. However, in the high SNR regime, the achievable rate region of the CF MABC protocol is tighter. For the TDBC protocol, there is little gap between the achievable rate region of the DF TDBC protocol and the outer bound. This is because there is no interference between transmissions in the TDBC protocol and hence decoding is almost optimal.

B.2 Asymmetric Case

In asymmetric cases we assume $h_{\mathsf{ar}} = 0.6$, $h_{\mathsf{br}} = 20, h_{\mathsf{ab}} = 0.5$ (Figs. 7 and 8) and $h_{\mathsf{ar}} = 20$, $h_{\mathsf{br}} = 0.6, h_{\mathsf{ab}} = 0.5$ (Figs. 9 and 10). In the low SNR regime, the CF TDBC and mixed TDBC protocol perform the best in Figs. 7 and 9, respectively, while in the high SNR regime, the DF MABC protocol and the DF TDBC protocol achieve the largest regions. In contrast to the symmetric case, the CF MABC protocol does not outer bound the DF MABC protocol.

The performance of the mixed forwarding scheme differs in the two asymmetric cases. In the mixed MABC protocol, if $h_{\mathsf{br}} > h_{\mathsf{ar}}$ then the channel seen at relay r when it decodes $\tilde{w}_{\mathsf{a}}$ is worse than when $h_{\mathsf{br}} < h_{\mathsf{ar}}$. Thus, the corresponding achievable rate region is also getting relatively smaller. In the mixed TDBC protocol, if $h_{ar} > h_{br}$, then we have a larger region since the link between source and relay is crucial for the DF performance. As the SNR increases, the regions of the two asymmetric cases is getting similar.

VIII. CONCLUSION

In conclusion, in bi-directional relay channels, a number of factors come into play including the relaying type, the protocol (series of temporal phases indicating which nodes transmit when) as well as how the nodes exploit the *side-information* available to them. We have surveyed the half-duplex bi-directional relay channel, providing inner and outer bounds for the MABC, TDBC, and HBC protocols using a combination of CF, DF, AF, and mixed relaying schemes. The exact capacity region of the bi-directional relay channel is known in several scenarios, but remains unknown in general. The single relay bi-directional channel has been extended to include multiple relays and multiple terminal nodes, where new challenges have emerged. It has also been considered recently from a degrees of freedom and diversity-multiplexing tradeoff perspective.

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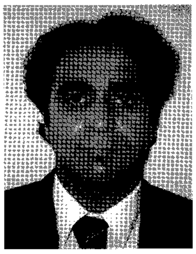

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