J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 16, Number 4 (November 2009), Pages 417–425

TOTALLY UMBILIC SPACELIKE SURFACES OF TYPE (I) IN L^n

SEONG-KOWAN HONG

ABSTRACT. In this paper we show that spheres in $E^3 \subset L^n$ and pseudohyperbolic spaces in $L^3 \subset L^n$ are the only totally umbilic spacelike surfaces of type (I) in L^n.

INTRODUCTION

Define $\overline{g}(v,w) = -v_1w_1 + \cdots + v_nw_n$ for v, w in R^n. R^n together with this metric is called the *Lorentzian n-space*, denoted by L^n, whereas E^n means the usual Euclidean n-space. A *spacelike surface* in L^n means an orientable connected 2-dimensional submanifold of L^n equipped with the Riemann metric $g = j^*\overline{g}$, where $j^*\overline{g}$ is the pull back of $\overline{g}$ via the inclusion map $j : M \to L^n$.

Let M be a spacelike in L^n, D the flat Levi-Civita connection on L^n, ∇ the induced connection on M, and h the second fundamental form on M. A point p of M is *umbilic* if there is a normal vector z such that $h(v,w) = g(v,w)z$ for all v, w in T_pM. The z is called the *normal curvature vector* of M at p. A spacelike surface is *totally umbilic* provided every point of M is umbilic. Note that if M is totally umbilic, then there is a smooth normal vector field Z on M, called the *normal curvature vector field* of M such that $h(V,W) = g(V,W)Z$ for all smooth tangent vector fields V, W on M.

A totally umbilic spacelike surface in L^n is called *totally umbilic of type* (I) if for a normal curvature vector field Z on M, $\overline{g}(Z,Z)$ does vanish nowhere on M, and *totally umbilic of type* (II) if $\overline{g}(Z,Z)$ does vanish everywhere on M. Since $\overline{g}(Z,Z)$ is constant on M, which is shown in Lemma 5, the two cases are mutually exclusive.

Our purpose is to show that there are only two kinds of totally umbilic spacelike surface of type (I) in L^n, say spheres and pseudohyperbolic spaces.

Received by the editors June 3, 2009 and, in revised form, November 19, 2009.
2000 *Mathematics Subject Classification.* ,
Key words and phrases. .
This work was supported for two years by Pusan National Research Grants.

MAIN THEOREMS

Theorem 1. *Let V be a k-dimensional subspace of L^n. Then exactly one of the following is true:*

(1) $V = L^k$, *and* $\overline{g} \mid V$ *is nondegenerate,*
(2) $V = E^k$, *and* $\overline{g} \mid V$ *is nondegenerate,*
(3) $\overline{g} \mid V$ *is degenerate, and in this case (and only in this case) we may write* $V = E^{k-1} \oplus Span\xi$, *where* $\overline{g}(\xi, \xi) = 0$ *and* ξ *is orthogonal to* E^{k-1}.

Proof. See [1]. □

Let M be a spacelike surface in L^n. By (x, y) we always denote isothermal coordinates compatible with the orientation on M. Then

$$g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) > 0, \qquad g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0$$

It is well known that (x, y) is defined around each point of M, and we may regard M as a Riemannian surface by introducing a complex local coordinate $z = x + iy$.

Definition. Let $n \geq 2$. A pseudohyperbolic space of radius $r > 0$ in L^{n+1} is the hyperquadric

$$H_0^n(p, r) = \{q \in L^{n+1} \mid \overline{g}(p - q, p - q) = -r^2\}$$

with dimension n and index 0.

Theorem 2. *If M is a connected totally umbilic spacelike surface in L^3, then M is a portion of a spacelike plane or a pseudohyperbolic space.*

Proof. Choose an isothermal parameter (x, y) so that M is defined locally by a map $X(z) = (x_1(z), x_2(z), x_3(z)) \in L^3$, where $z = x + iy$ is a complex local coordinate. Let D be the Levi-Civita connection on M, and ∇ the induced connection on M. Denote X_x by ∂_1 and X_y by ∂_2. Then

$$D_{\partial_i}\partial_j = \nabla_{\partial_i}\partial_j + h(\partial_i, \partial_j).$$

Since M is spacelike, there is a C^∞ unit normal vector field N on M. By Theorem 1, it must be timelike. Consider the smooth function $f = \overline{g}(N, Z)$, where Z is a normal curvature vector field on M such that $h(V, W) = g(V, W)Z$ for any smooth tangent vector fields V, W on M. Note that $Z = -fN$. Since $0 = \partial_j\overline{g}(N, N) = 2\overline{g}(N, D_{\partial_j}N)$, $D_{\partial_j}N$ is a local smooth tangent vector field on M. Since

$$\overline{g}(D_{\partial_i}\partial_j, N) = f\overline{g}(\partial_i, \partial_j)$$

and

$$0 = \overline{g}(D_{\partial_i}\partial_j, N) + \overline{g}(\partial_j, D_{\partial_i}N),$$

we have

$$\overline{g}(D_{\partial_i}N, \partial_j) = -f\overline{g}(\partial_i, \partial_j)$$

for $i, \quad j = 1, \quad 2$. Therefore, $D_{\partial_i}N = -f\partial_i$.

Consequently,

$$D_{\partial_j}D_{\partial_i}N = -\partial_j(f)\partial_i - fD_{\partial_j}\partial_i$$

and

$$D_{\partial_i}D_{\partial_j}N = -\partial_i(f)\partial_j - fD_{\partial_i}\partial_j.$$

From the above we have

$$(\partial_j f)\partial_i = (\partial_i f)\partial_j \quad .$$

Then the linear independency of ∂_1 and ∂_2 tells us that $\nabla f = 0$. Hence f is constant.

$f \equiv 0$ implies $D_{\partial_j}N \equiv 0$ for $j = 1, 2$, that is, N is a constant vector field, and in turn M is locally a spacelike plane with a normal vector N.

When $f \equiv c(\neq 0)$, consider N and ∂_i as a smooth function from an open set $U \subset R^2$ to L^3. Since $D_{\partial_i}N = -c\partial_i$, $N = -cX + v_0$ for some $v_0 \in L^3$.

From this we obtain

$$X(x,y) = -\frac{N}{c} + \frac{v_0}{c}$$

and

$$\begin{aligned} \overline{g}\left(X(x,y) - \frac{v_0}{c}, X(x,y) - \frac{v_0}{c}\right) &= \overline{g}\left(-\frac{N}{c}, -\frac{N}{c}\right) \\ &= -\frac{1}{c^2}. \end{aligned}$$

Therefore X lies in a pseudohyperbolic space $H_0^2(\frac{v_0}{c}, \frac{1}{|c|})$. This local argument can be extended to the global argument using continuation along a path in M. □

Now we know there is only one kind of nontrivial totally umbilic spacelike surface in L^3. But what about a totally umbilic spacelike surface in $L^n (n > 3)$?

Theorem 3. *Let M be a totally umbilic spacelike surface of type (I) in $L^n(n > 3)$. Then it is in fact either a pseudohyperbolic space in $L^3 \subset L^n$ or a sphere in $E^3 \subset L^n$.*

To prove the theorem 3, we need several preliminary facts.

Proposition 4. *Let $e^1, \cdots, e^n$ be an orthonormal moving frame on L^n and let ϕ^i's be the dual 1-forms, where e^1 is timelike. Then there exist unique 1-forms ω_{ij} (called the connection forms) such that*

(i) $\omega_{ij} = -\omega_{ji}$,

(ii) $d\phi^i = -\sum_k \varepsilon_i \omega_{ik} \wedge \phi^k$,

(iii) $d\omega_{ij} = -\sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj}$,

where $\varepsilon_1 = -1$ *and* $\varepsilon_j = 1$ *if* $j \neq 1$.

Proof. Define ω_{ij} by $\omega_{ij}(X) = \overline{g}(e^i, D_X e^j)$. Since

$$0 = \overline{g}(e^i, D_X e^j) + \overline{g}(D_X e^i, e^j),$$

we have $\omega_{ij}(X) = -\omega_{ji}(X)$ for any C^∞ vector field X in L^n.

Consider the moving frames $e^1, \cdots, e^n$, with a little abuse of notation, as an R^n-valued function $e^i : R^n \longrightarrow R^n$. Then we can consider dI and de^i's as R^n-valued 1-forms. Since $d^2 = 0$, we have

$$\begin{aligned} 0 &= d^2 I \\ &= d\Big(\sum_i \phi^i \wedge e^i\Big) \\ &= \sum_i (d\phi^i) e^i - \sum_k \phi^k \wedge de^k \\ &= \sum_i (d\phi^i) e^i - \sum_k \phi^k \wedge \sum_i \varepsilon_i \omega_{ik} e^i \\ &= \sum_i \Big(d\phi^i - \sum_k \varepsilon_i \phi^k \wedge \omega_{ik}\Big) e^i \end{aligned}$$

Setting the coefficient of each e^i equal to 0, we obtain

$$d\phi^i = -\sum_k \varepsilon_i \omega_{ik} \wedge \phi^k.$$

We also have

$$\begin{aligned} 0 &= d^2 e^j \\ &= \sum_i \varepsilon_i d\omega_{ij} e^i - \sum_k \varepsilon_k \omega_{kj} \wedge de^k \\ &= \sum_i \Big(\varepsilon_i d\omega_{ij} - \sum_k \varepsilon_i \varepsilon_k \omega_{kj} \wedge \omega_{ik}\Big) e^i, \end{aligned}$$

from which we immediately deduce

$$d\omega_{ij} = -\sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj}.$$

□

Lemma 5. *Let Z be a normal curvature vector field of a totally umbilic spacelike surface M in L^n. Then $\overline{g}(Z,Z)$ is constant everywhere on M.*

Proof. If $\overline{g}(Z,Z)$ vanishes everywhere, then there is nothing to prove. Therefore we assume $\overline{g}(Z,Z)$ does not vanish everywhere. There are two possibilities.

(Case 1) There is p in M where $\overline{g}(Z,Z)$ at p is negative.

Let $p \in M$. In a neighborhood of p, we choose an orthonomal moving frame e^2, e^3 on M, and complete to an adapted orthonormal moving frame $e^1, \cdots, e^n$ with the unit timelike vector field e^1 in Z-direction. Then, for $j = 2$, 3, and X tangent to M, we have

$$\begin{aligned}\omega_{ij}(X) &= \overline{g}(e^i, D_X e^j)\\ &= \begin{cases} -\sqrt{\overline{g}(Z,Z)}\overline{g}(X,e_j), & i=1\\ 0, & i>3,\end{cases}\end{aligned}$$

which means that on TM we have

$$\begin{aligned}\omega_{1j} &= -\varepsilon_j\sqrt{-\overline{g}(Z,Z)}\phi^j\\ \omega_{ij} &= 0, \quad \text{if } i>3.\end{aligned} \tag{1}$$

Denote $\sqrt{-\overline{g}(Z,Z)}$ by λ. From (1) and the second structural equation, we find that on TM we have

$$\begin{aligned}d\omega_{1j} &= -\varepsilon_j d\lambda\wedge\phi^j - \varepsilon_j\lambda d\phi^j\\ &= -\sum_k \varepsilon_k\omega_{1k}\wedge\omega_{kj}\\ &= \lambda\sum_{k=2}^{3}\phi^k\wedge\omega_{kj},\end{aligned}$$

while the first structual equation gives

$$\begin{aligned}d\phi^j &= -\sum_{k=1}^{3}\varepsilon_j\omega_{jk}\wedge\phi^k\\ &= \sum_{k=1}^{3}\varepsilon_j\phi^k\wedge\omega_{jk}\\ &= -\varepsilon_j\sum_{k=2}^{3}\phi^k\wedge\omega_{kj},\end{aligned}$$

since $\phi^1 \equiv 0$ on TM, so we find that

$$-\varepsilon_j d\lambda\wedge\phi^j = 0 \quad \text{for } j=2,3.$$

Hence $d\lambda \equiv 0$ and so $\sqrt{-\overline{g}(Z,Z)}$ is constant around p. This means $\{q \in M \mid \overline{g}(Z,Z)(q) = \overline{g}(Z,Z)(p)\}$ is a nonempty open and closed set in M. Therefore $\overline{g}(Z,Z)$ is constant everywhere since M is connected.

(Case 2) There is p in M where $\overline{g}(Z,Z)$ at p is positive.

In a neighborhood of p, we choose an orthonormal moving frame e^{n-1}, e^n on M, and complete to an adapted orthonomal moving frame $e^1, \cdots, e^n$ with e^{n-2} in Z-direction. Note that e^{n-2} is spacelike. Then, for $j = n-1, n$ and X tangent to M, we have

$$\begin{aligned} \omega_{ij}(X) &= \overline{g}(e^i, D_X e^j) \\ &= \begin{cases} \sqrt{\overline{g}(Z,Z)}\overline{g}(X, e_j) & i = n-2 \\ 0 & i < n-2, \end{cases} \end{aligned}$$

which means that on TM we have

$$\begin{aligned} \omega_{ij} &= \sqrt{\overline{g}(Z,Z)}\phi^j \quad \text{if } i = n-2, \\ \omega_{ij} &= 0, \qquad\qquad\quad \text{if } i < n-2. \end{aligned} \tag{2}$$

From (2) and the second structural equation, we find that on TM we have

$$\begin{aligned} d\omega_{ij} &= d\lambda \wedge \phi^j + \lambda d\phi^j \\ &= -\sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} \\ &= -\lambda \sum_{k=n-1}^{n} \phi^k \wedge \omega_{kj}, \end{aligned}$$

while the first structual equation gives

$$\begin{aligned} d\phi^j &= -\sum_{k=n-2}^{n} \varepsilon_j \omega_{jk} \wedge \phi^k \\ &= \sum_{k=n-2}^{n} \varepsilon_j \phi^k \wedge \omega_{jk} \\ &= -\sum_{k=n-1}^{n} \phi^k \wedge \omega_{kj}, \end{aligned}$$

since $\phi^{n-2} \equiv 0$ on TM, so we find that

$$d\lambda \wedge \phi^j = 0 \quad \text{for } j = n-1, n.$$

Hence $d\lambda \equiv 0$ and so $\sqrt{\overline{g}(Z,Z)}$ is constant around p in M and consequently on the whole M. □

Lemma 6. *Let Δ be a k-dimensional distribution along the curve $c : [a, b] \to L^n$ with $\frac{dc}{dt} \in \Delta(t)$. Suppose Δ is parallel along c. Then c is a curve in some k-dimensional plane $W \subset L^n$, and W is just* $\exp(\Delta(t))$ *for any t.*

Proof. Let $W = \Delta(a)$, considered as a k-dimensional subspace in L^n. Then W is $SO(1, n-1)$-equivalent to L^k, E^k, or $E^{k-1} \oplus span\{\xi\}$, where ξ is a nonzero lightlike vector in L^n. Without loss of generality we may assume W is L^k, E^k or $H^k = \{(x, x, y_1, \cdots, y_{k-1}, 0, \cdots, 0) \in L^n \mid x, y_i \in \mathbb{R}\}$.

Case 1. $W = L^k$.

If c does not lie entirely in W, then by the mean value theorem some tangent vector $c'(t)$ has a nonzero i-th component for some $i > k$. But this is impossible, since $c'(t) \in \Delta(t)$ and $\Delta(t)$ is parallel to $W = \Delta(a)$. Since each $\Delta(t)$ is parallel to $W = \Delta(a)$ and also contains the points c(t) in W, each $\Delta(t)$ must be equal to W. In other words, $W = \exp(\Delta(t))$ for all t.

Case 2. $W = E^k$.

The exact same proof as in case 1 may be applied here with $W = 0 \oplus E^k$.

Case 3. $W = H^k \subset L^{k+1}$.

Since $c'(t) \in \Delta(t)$ and $\Delta(t)$ is parallel to $W = \Delta(a)$, $c_1'(t) = c_2'(t)$ for any t and $c_i'(t) = 0$ for $i > k+1$, and result is proved in this case. □

We also need the converse assertion of this.

Lemma 7. *Let Δ be a smooth k-dimensional distribution along $c : [a, b] \to L^n$. Suppose the induced covariant derivative $\frac{dV}{dt}$ belongs to Δ whenever V is a smooth vector field along c belonging to Δ. Then Δ is parallel along c.*

Proof. The proof given in [5, p. 41-42] works here. □

Lemma 8. *Let M be a connected spacelike surface in L^n and let Δ be a smooth k-dimensional distribution along M such that $T_pM \subset \Delta(p)$ for all $p \in M$. Suppose that Δ is parallel along every curve c in M. Then M lies in some k-dimensional plane $W \subset L^n$.*

Proof. Choose a point $p \in M$ and let W be the k-dimensional plane of L^n with $\exp(\Delta(p)) = W$. For any $q \in M$, choose a curve $c : [0, 1] \to M \subset L^n$ with $c(0) = p$, and $c(1) = q$. Since $T_pM \subset \Delta(p)$ for all $p \in M$, $c'(t) \in \Delta(c(t))$ for all $t \in [0, 1]$. Hence, Lemma 6 applied to the distribution $t \to \Delta(c(t))$ along c, implies that c lies in the k-dimensional plane $W = \exp(\Delta(0)) \subset L^n$, because $\exp \Delta(c(t)) = W$ for all t. (Of course W may be degenerate.) □

Now we are ready to prove the Theorem 3.

Proof of Theorem 3. Let Z be a normal curvature vector field on M.

(Case 1) Z is timelike.

Denote the constant function $\sqrt{-g(Z,Z)}$ by λ. (1) gives

$$\begin{aligned} D_X e^1 &= \sum_{k=1}^{n} \varepsilon_k \omega_{k1}(X) e^k \\ &= \sum_{k=2}^{3} \omega_{k1}(X) e^k \\ &= -\lambda X. \end{aligned} \tag{3}$$

We also have

$$D_X e^j = \sum_{k=1}^{3} \varepsilon_k \omega_{kj} e^k, \qquad \text{for } j = 1, 2. \tag{4}$$

Let Δ be the 3-dimensional C^∞ distribution on M with $\Delta(P) = M_p + \mathbb{R} \cdot e^1(p)$. Equation(3), (4) and Lemma 3 shows that Δ is parallel along every curve lying in M. So Lemma 4 implies that M lies in a 3-dimensional plane W of L^n. Since $\Delta(p) \cong L^3$ and $\exp(\Delta(p)) = W$, we know that W has an index 1 and therfore $W \cong L^3 \subset L^n$.

Next, we have to show that M lies in a pseudohyperbolic space of radius $\frac{1}{\lambda}$. Let P be the position vector field on L^n. Then $D_X P = X$ for all tangent vector field X to M in L^n, and so we can rewrite (3) as

$$D_X(e^1 - \lambda P) = 0 \quad .$$

Thus the vector field $e^1 - \lambda P$ is parallel along M. Identifying tangent vectors of M with elements of L^n, this means that $e^1 - \lambda P$ is a constant vector v_0 on M, so we have $p = \frac{e^1(p) - v_0}{\lambda}$ fof all $p \in M$, which means that M lies in pseudohyperbolic space with radius $\frac{1}{\lambda}$, center $-\frac{v_0}{\lambda}$.

(Case 2) Z is spacelike.

Denote the constant function $\sqrt{\overline{g}(Z,Z)}$ by λ. By (2),

$$D_X e^j = \sum_{k=n-1}^{n} \omega_{kj}(X) e^k = \lambda X. \tag{5}$$

for $j = n-2$. We also have

$$D_X e^j = \sum_{k=n-2}^{n} \omega_{kj} e^k, \qquad \text{for} \qquad j = n-1, n. \tag{6}$$

Let Δ be the 3-dimensional C^∞ distribution on M with $\Delta(P) = M_p + \mathbb{R} \cdot e^{n-2}(p)$. Then we conclude that M lies in an 3-dimensional plane W of L^n. But, in this case, $\Delta(p) \cong E^3$ implies $W \cong E^3 \subset L^n$. In the similar way to the case 1, we can show $e^{n-2} - \lambda P$ is a constant vector $v_0 \in M \subset L^n$ and $p = \frac{e^{n-2} - v_0}{\lambda}$ for all $p \in M$, which means that M lies in a sphere of radius $\frac{1}{\lambda}$, center $-\frac{v_0}{\lambda}$. This completes the proof. □

REFERENCES

1. Graves, L.: Codimension one isometric immersions betweem Lorentzian spaces. *Trans. Amer. Math. Soc.* **252** (1979), 367-392.
2. ______: On codimension one isometric immersions between indefinite space forms. *Tsukuba J. Math.* **3** (1979), no. 2, 17-29.
3. Hoffman, D. & R. Osserman: The geometry of the generalized Gauss map. *Memoires Amer. Math. Soc.* **28**, 1980.
4. O'Neil, B.: *Semi-Riemannian Geometry with application to Relativity.* Academic Press, NY, 1983.
5. Spivak, M.: *Introduction to Differential Geometry* IV, Publish or Perish Inc., Berkeley, CA, 1979.

DEPARTMENT OF MATHEMATICS EDUCATION, PUSAN NATIONAL UNIVERSITY, BUSAN 609-735, KOREA

Email address: `aromhong@hanafos.com`