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INTEGRAL TRANSFORMS AND INVERSE INTEGRAL TRANSFORMS WITH RELATED TOPICS ON FUNCTION SPACE I

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ABSTRACT. In this paper we establish various relationships among the generalized integral transform, the generalized convolution product and the first variation for functionals in a Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ introduced by Chang and Skoug in [14]. We then derive an inverse integral transform and obtain several relationships involving inverse integral transforms.

1. INTRODUCTION

Let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of real-valued continuous functions x on $[0,T]$ with $x(0)=0$. The concept of the integral transform $\mathcal{F}_{\gamma,\beta}$ was introduced by Lee in [19]. The theory of integral transform was studied extensively and applied to various subjects by many mathematicians. Recently, in [13, 17, 18, 20], the authors studied relationships between the integral transform and the convolution product for functionals in several interesting classes. For certain values of the parameters γ and β and for certain classes of functionals, the Fourier-Wiener transform [1, 2], the modified Fourier-Wiener transform [3], the Fourier-Feynman transform [4] and the Gauss transform are special cases of Lee's integral transform $\mathcal{F}_{\gamma,\beta}$. Also see papers [13, 20] for further work involving integral transform.

The function space $C_{a,b}[0,T]$ induced by generalized Brownian motion was introduced by J. Yeh in [22] and was used extensively by Chang and Chung [9]. In this paper, we establish various interesting relationships among the generalized integral transform, the generalized convolution product and the first variation for functionals in the Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$. Finally, we derive an inverse integral transform

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and obtain several relationships involving inverse integral transforms. The generalized Fourier-Wiener function space transform in [11] and the generalized analytic Fourier-Feynman transform in [8, 10, 14] are special cases of our integral transform.

The Wiener process used in [1-5, 13, 16-19, 20] is stationary in time and is free of drift while the stochastic process used in this paper as well as in [6-12, 14] is nonstationary in time and subject to a drift $a(t)$. This process can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [21].

2. DEFINITIONS AND PRELIMINARIES

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)=0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \leq T$, the n-dimensional random vector $(Y(t_1,\omega), \cdots, Y(t_n,\omega))$ is normally distributed with density function

$$
\tag{2.1}
\begin{aligned}
W_n(\vec{t},\vec{\eta}) &= \Big((2\pi)^n \prod_{j=1}^{n}(b(t_j) - b(t_{j-1}))\Big)^{-1/2} \\
&\quad \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\right\}
\end{aligned}
$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L_2[0,T]$ and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0,T]$.

As explained in [23, p. 18-20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real-valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ where all of the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [23, p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.).

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\}$$

where $|a|(t)$ denotes the total variation of the function $a(\cdot)$ on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)].$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space. Note that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$. Also note that if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then $L^2_{a,b}[0,T] = L^2[0,T]$. In fact,

$$(L^2_{a,b}[0,T], \|\cdot\|_{a,b}) \subset (L^2_{0,b}[0,T], \|\cdot\|_{0,b}) = (L^2[0,T], \|\cdot\|_2)$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent.

Let $\{\phi_j\}_{j=1}^\infty$ be a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j, \phi_k)_{a,b} = \begin{cases} 0 & , j \neq k \\ 1 & , j = k \end{cases},$$

and for each $v \in L^2_{a,b}[0,T]$, let

$$v_n(t) = \sum_{j=1}^n (v, \phi_j)_{a,b}\phi_j(t)$$

for $n = 1, 2, \cdots$. Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund(PWZ) stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty}\int_0^T v_n(t)dx(t)$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists. For a more detailed study of the PWZ integral, see [7, 8, 9, 14].

Let $M(L^2_{a,b}[0,T])$ be the space of complex-valued, countably additive Borel measures on $L^2_{a,b}[0,T]$. The Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ consists of those functionals F on $C_{a,b}[0,T]$ expressible in the form

$$F(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x\rangle\}df(v) \tag{2.2}$$

for s-a.e. $x \in C_{a,b}[0,T]$ where the associated measure f is an element of $M(L^2_{a,b}[0,T])$.

Remark 2.1. (1) When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $\mathcal{S}(L^2_{a,b}[0,T])$ reduces to the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [5].

(2) $M(L^2_{a,b}[0,T])$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication.

(3) One can show that the correspondence $f \to F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{S}(L^2_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{L^2_{a,b}[0,T]} |df(v)|.$$

In [5], Cameron and Storvick carried out these arguments in detail for the Banach algebra $\mathcal{S}$. Further works on $\mathcal{S}$ shows that it contains many functionals of interest in Feynman integration theory. For a more detailed see in [5, 15, 16].

The following integration formula is used several times throughout this paper. For any complex number γ and nonzero element $v \in L^2_{a,b}[0,T]$,

$$\int_{C_{a,b}[0,T]} \exp\{\gamma\langle v, x\rangle\}d\mu(x) = \exp\left\{\frac{\gamma^2}{2}(v^2, b') + \gamma(v, a')\right\}, \tag{2.3}$$

where

$$(v^2, b') = \int_0^T v^2(t)b'(t)dt = \int_0^T v^2(t)db(t)$$

and

$$(v, a') = \int_0^T v(t)a'(t)dt = \int_0^T v(t)da(t).$$

Note that (v^2, b') is always positive, while (v, a') may be positive, negative or zero.

In this paper, let $K_{a,b}[0,T]$ be the set of all complex-valued continuous functions $x(t)$ defined on $[0,T]$ which vanish at $t=0$ and whose real and imaginary parts are elements of $C_{a,b}[0,T]$ namely,

$$K_{a,b}[0,T] = \{x : [0,T] \to \mathbb{C} \mid x(0)=0,\ \mathrm{Re}(x) \in C_{a,b}[0,T] \text{ and } \mathrm{Im}(x) \in C_{a,b}[0,T]\}.$$

Thus clearly $C_{a,b}[0,T]$ is a subspace of $K_{a,b}[0,T]$.

We are now ready to state the definition of the generalized integral transform $\mathcal{F}_{\gamma,\beta}$ introduced in [10, 12].

Definition 2.2. Let F be a functional defined on $K_{a,b}[0,T]$. For each pair of nonzero complex numbers γ and β, the generalized integral transform $\mathcal{F}_{\gamma,\beta}F$ of F is defined by

$$\mathcal{F}_{\gamma,\beta}F(y) = \int_{C_{a,b}[0,T]} F(\gamma x + \beta y)\,d\mu(x), \quad y \in K_{a,b}[0,T], \tag{2.4}$$

if it exists.

Remark 2.3. (1) When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $\mathcal{F}_{\gamma,\beta}$ is the integral transform used by Kim and Skoug [18]. In particular, $\mathcal{F}_{1,i}$ is the Fourier-Wiener transform introduced by Cameron in [1] and used by Cameron and Martin in [2]. Also $\mathcal{F}_{\sqrt{2},i}$ is the modified Fourier-Wiener transform used by Cameron and Martin in [3].

(2) When $\gamma = \sqrt{2}$ and $\beta = i$, $\mathcal{F}_{\sqrt{2},i}$ is the generalized Fourier-Wiener function space transform introduced by Chang and Chung in [11].

Next, we state definitions of the generalized convolution product and the first variation on $K_{a,b}[0,T]$.

Definition 2.4. Let F and G be functionals defined on $K_{a,b}[0,T]$. For each nonzero complex number γ, the *generalized convolution product* $(F*G)_\gamma$ of F and G is defined by

$$(F*G)_\gamma(y) = \int_{C_{a,b}[0,T]} F\Big(\frac{y+\gamma x}{\sqrt{2}}\Big) G\Big(\frac{y-\gamma x}{\sqrt{2}}\Big)\,d\mu(x), \quad y \in K_{a,b}[0,T], \tag{2.5}$$

if it exists.

Definition 2.5. Let F be a functional defined on $K_{a,b}[0,T]$ and let w be an element of $K_{a,b}[0,T]$. Then the *first variation* δF of F is defined by

$$\delta F(x|w) = \frac{\partial}{\partial k} F(x+kw)\Big|_{k=0} \tag{2.6}$$

if it exists.

3. GENERALIZED INTEGRAL TRANSFORMS, GENERALIZED CONVOLUTION PRODUCTS AND FIRST VARIATIONS OF FUNCTIONALS IN $\mathcal{S}(L^2_{a,b}[0,T])$

In this section, we establish several interesting relationships among the integral transform, the convolution product and the first variation for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$.

In our first theorem, we establish basic formulas relating integral transforms, convolution products and first variations for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$.

Theorem 3.1. *Let γ and β be nonzero complex numbers and let $F \in \mathcal{S}(L^2_{a,b}[0,T])$ be given by (2.2). Let G be an element of $\mathcal{S}(L^2_{a,b}[0,T])$ of the form*

$$G(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle h, x\rangle\} dg(h) \tag{3.1}$$

for some g in $M(L^2_{a,b}[0,T])$ and let w be an element of $K_{a,b}[0,T]$. Then the generalized integral transform $\mathcal{F}_{\gamma,\beta}F$, the generalized convolution product $(F*G)_\gamma$ and the first variation $\delta F(\cdot|w)$ are given by formulas

$$\mathcal{F}_{\gamma,\beta}F(y) = \int_{L^2_{a,b}[0,T]} \exp\left\{ i\beta\langle v, y\rangle - \frac{\gamma^2}{2}(v^2, b') + i\gamma(v, a')\right\} df(v), \tag{3.2}$$

$$\begin{aligned}(F*G)_\gamma(y) &= \int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle v+h, y\rangle \right.\\ &\quad \left. -\frac{\gamma^2}{4}((v-h)^2, b') + i\frac{\gamma}{\sqrt{2}}(v-h, a')\right\} df(v)dg(h),\end{aligned} \tag{3.3}$$

and

$$\delta F(y|w) = \int_{L^2_{a,b}[0,T]} i\langle v, w\rangle \exp\{i\langle v, y\rangle\} df(v) \tag{3.4}$$

for $y \in K_{a,b}[0,T]$ if these exist.

Proof. Equations (3.2)-(3.4) follow easily from equations (2.3)-(2.6) and the Fubini theorem. □

Remark 3.2. (1) We can rewrite equations (3.2) and (3.4) as

$$\mathcal{F}_{\gamma,\beta}F(y) = \int_{L^2_{a,b}[0,T]} \exp\{i\beta\langle v, y\rangle\} d\phi_1(v) \tag{3.5}$$

and as

$$\delta F(y|w) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, y\rangle\} d\phi_2(v) \tag{3.6}$$

where ϕ_1 and ϕ_2 are set functions defined by

$$\phi_1(E) = \int_E \exp\left\{-\frac{\gamma^2}{2}(v^2, b') + i\gamma(v, a')\right\} df(v) \tag{3.7}$$

and

$$\phi_2(E) = \int_E i\langle v, w\rangle df(v) \tag{3.8}$$

for $E \in \mathcal{B}(L^2_{a,b}[0,T])$, respectively. Also, we can rewrite equation (3.3) as

$$F * G)_\gamma(y) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, y\rangle\} d(\phi_3 \circ \varphi_3^{-1})(v) \tag{3.9}$$

where ϕ_3 is a set function defined by

$$\phi_3(E) = \int_E \exp\left\{-\frac{\gamma^2}{4}((v-h)^2, b') + i\frac{\gamma}{\sqrt{2}}(v-h, a')\right\} df(v)dg(h) \tag{3.10}$$

for $E \in \mathcal{B}(L^2_{a,b}[0,T] \times L^2_{a,b}[0,T])$ and φ_3 is a continuous function on $L^2_{a,b}[0,T] \times L^2_{a,b}[0,T]$ given by

$$\varphi_3(v, h) = \frac{1}{\sqrt{2}}(v+h).$$

Generally, the integrands in equations (3.5)-(3.10) are unbounded as functions of $v \in L^2_{a,b}[0,T]$. Thus $\mathcal{F}_{\gamma,\beta}F(y), \delta F(y|w)$ and $(F*G)_\gamma(y)$ might not exist.

(2) In [8, 9, 10, 14], the authors gave some conditions for the existences of their transforms, convolutions and first variations. Also, we can establish some conditions for the existences of our assertions. For example, we assume that

$$\int_{L^2_{a,b}[0,T]} \int_{L^2_{a,b}[0,T]} \exp\left\{\frac{M^2}{4}\int_0^T (v^2(t) + h^2(t))db(t) + \frac{M}{\sqrt{2}}\int_0^T (|v(t)| + |h(t)|)d|a|(t) + \frac{M^2}{2}\|v\|_{a,b}\|h\|_{a,b}\right\}|df(v)||dg(h)| < \infty$$

for some real number $M > 0$. Then for all nonzero complex number γ with $|\gamma| \le M$, the generalized convolution product $(F*G)_\gamma$ with respect to γ of F and G exists. Moreover, if $Im(y) = 0$, then and $(F*G)_\gamma(y)$ is an element of $\mathcal{S}(L^2_{a,b}[0,T])$ as a function of $y \in C_{a,b}[0,T]$. From now on, for simplicity of our results, we assume that integral transforms, convolution products and first variations in theorems and corollaries below all exist.

Note that using equations (2.4) and (2.5) it follows that

$$(F*1)_\gamma(y) = \mathcal{F}_{\frac{\gamma}{\sqrt{2}},1}F(y/\sqrt{2}) \tag{3.11}$$

and

$$(1 * G)_\gamma(y) = \mathcal{F}_{-\frac{\gamma}{\sqrt{2}},1} G(y/\sqrt{2}) \tag{3.12}$$

for $y \in K_{a,b}[0,T]$.

The following theorem is one of main results in this paper.

Theorem 3.3. *Let γ and β be nonzero complex numbers. Let F and G be given by* (2.2) *and* (3.1), *respectively. Then*

$$\mathcal{F}_{\gamma,\beta}(F * G)_\gamma(z) = \mathcal{F}_{\gamma,\beta}(F * 1)_\gamma(z)\mathcal{F}_{\gamma,\beta}(1 * G)_\gamma(z) \tag{3.13}$$

for $z \in K_{a,b}[0,T]$. Furthermore, both of the expressions in equation (3.13) *are given by the expression*

$$\begin{aligned} \int_{L^2_{a,b}[0,T]} \int_{L^2_{a,b}[0,T]} \exp\Big\{ i\frac{\beta}{\sqrt{2}}\langle v+h, z\rangle \\ -\frac{\gamma^2}{2}(v^2+h^2, b') + i\sqrt{2}\gamma(v,a')\Big\} df(v)dg(h) \end{aligned} \tag{3.14}$$

for $z \in K_{a,b}[0,T]$.

Proof. Using equation (3.2) with F replaced with $(F * G)_\gamma$, the Fubini theorem and equation (2.3), the left-hand side of equation (3.13) is equal to the expression (3.14). On the other hand, using equations (3.2), (3.11) and (3.12) we obtain that

(3.15)

$$\begin{aligned} \mathcal{F}_{\gamma,\beta}(F * 1)_\gamma(z) &= \int_{C_{a,b}[0,T]} (F * 1)_\gamma(\gamma y + \beta z)d\mu(y) \\ &= \int_{C_{a,b}[0,T]} \mathcal{F}_{\frac{\gamma}{\sqrt{2}},1} F\Big(\frac{\gamma y + \beta z}{\sqrt{2}}\Big) d\mu(y) \\ &= \int_{L^2_{a,b}[0,T]} \exp\Big\{ i\frac{\beta}{\sqrt{2}}\langle v, z\rangle - \frac{\gamma^2}{2}(v^2, b') + i\gamma\sqrt{2}(v,a')\Big\} df(v) \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_{\gamma,\beta}(1 * G)_\gamma(z) &= \int_{C_{a,b}[0,T]} (1 * G)_\gamma(\gamma y + \beta z)d\mu(y) \\ &= \int_{C_{a,b}[0,T]} \mathcal{F}_{-\frac{\gamma}{\sqrt{2}},1} G\Big(\frac{\gamma y + \beta z}{\sqrt{2}}\Big) d\mu(y) \\ &= \int_{L^2_{a,b}[0,T]} \exp\Big\{ i\frac{\beta}{\sqrt{2}}\langle h, z\rangle - \frac{\gamma^2}{2}(h^2, b')\Big\} dg(h). \end{aligned} \tag{3.16}$$

Hence, using equations (3.15) and (3.16), the right-hand side of equation (3.13) also is equal to the expression (3.14). □

Corollary 3.4. *Let γ, β, F and G be as in Theorem 3.3. Then*

$$\mathcal{F}_{\gamma,\beta}(F * G)_\gamma(z) = \mathcal{F}_{\gamma,\beta}F\Big(\frac{z}{\sqrt{2}} - \frac{\gamma}{\beta}(1-\sqrt{2})a\Big)\mathcal{F}_{\gamma,\beta}G\Big(\frac{z}{\sqrt{2}} - \frac{\gamma}{\beta}a\Big) \tag{3.17}$$

for $z \in K_{a,b}[0,T]$. In particular when $\gamma = \sqrt{2}$ and $\beta = i$,

$$\mathcal{F}_{\sqrt{2},i}(F * G)_{\sqrt{2}}(z) = \mathcal{F}_{\sqrt{2},i}F\Big(\frac{z}{\sqrt{2}} + i(\sqrt{2}-2)a\Big)\mathcal{F}_{\sqrt{2},i}G\Big(\frac{z}{\sqrt{2}} + i\sqrt{2}a\Big)$$

for $z \in K_{a,b}[0,T]$.

Proof. Equation (3.17) follows easily from equations (2.4) and (2.5). □

Next, we give another expression for $\mathcal{F}_{\gamma,\beta}(F * G)_\gamma$ in Theorem 3.3. The equation (3.18) is used several times in theorems and corollaries throughout Sections 3 and 4.

Theorem 3.5. *Let γ, β, F and G be as in Theorem 3.3. Then*

$$\mathcal{F}_{\gamma,\beta}(F * G)_\gamma(z) = \mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}\mathcal{F}_{\frac{\gamma}{\sqrt{2}},1}F(z/\sqrt{2})\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}\mathcal{F}_{-\frac{\gamma}{\sqrt{2}},1}G(z/\sqrt{2}) \tag{3.18}$$

for $z \in K_{a,b}[0,T]$.

Proof. Taking the integral transform $\mathcal{F}_{\gamma,\beta}$ of each side of equations (3.11) and (3.12), and using equation (2.4), we obtain equation (3.18) as desired. □

Remark 3.6. On the whole, our integral transforms are not commutative. But for special parameters, we can obtain the commutativity of our integral transforms. Let α, β, γ and η be nonzero complex numbers such that

$$\mathcal{F}_{\alpha,\beta}\mathcal{F}_{\gamma,\eta}F(z) = \mathcal{F}_{\gamma,\eta}\mathcal{F}_{\alpha,\beta}F(z) \tag{3.19}$$

for $z \in K_{a,b}[0,T]$. Then the equation (3.19) reduces to a system

$$\begin{cases} \gamma^2 + \alpha^2\eta^2 = \alpha^2 + \beta^2\gamma^2, \\ \gamma + \alpha\eta = \alpha + \beta\gamma \end{cases}.$$

However the set of solutions of this system is given by

$$\{(\alpha, 1, \gamma, 1) : \alpha, \gamma \in \mathbb{C} - \{0\}\} \cup \{(\alpha, \beta, \gamma, \eta) : \alpha = \gamma, \beta = \eta\}.$$

Hence we know that two integral transforms $\mathcal{F}_{\alpha,\beta}$ and $\mathcal{F}_{\gamma,\eta}$ are equal or

$$\mathcal{F}_{\alpha,1}\mathcal{F}_{\gamma,1}F(z) = \mathcal{F}_{\gamma,1}\mathcal{F}_{\alpha,1}F(z) \tag{3.20}$$

for $z \in K_{a,b}[0,T]$.

Next we give an interesting formula involving integral transforms and convolution products. Corollary 3.7 follows easily from equations (3.2), (3.3), (3.11), (3.12) and (3.20).

Corollary 3.7. *Let γ, F and G be as in Theorem 3.3, and let α be a complex number satisfying the condition* (3.20). *Then*

$$(\mathcal{F}_{\frac{\gamma}{\sqrt{2}},1}F * 1)_\alpha(z) = (\mathcal{F}_{\frac{\alpha}{\sqrt{2}},1}F * 1)_\gamma(z)$$

for $z \in K_{a,b}[0,T]$.

The following theorem tells us that the integral transform $\mathcal{F}_{\gamma,\beta}$ and the first variation δ are commutative.

Theorem 3.8. *Let γ, β, F and w be as in Theorem 3.1. Then*

$$\beta\mathcal{F}_{\gamma,\beta}\delta F(\cdot|w)(z) = \delta\mathcal{F}_{\gamma,\beta}F(z|w) \tag{3.21}$$

for $z \in K_{a,b}[0,T]$.

Proof. Using equation (3.2) with F replaced with δF and using equation (3.4) with F replaced with $\mathcal{F}_{\gamma,\beta}F$, we can easily prove Theorem 3.8. □

Next, we establish interesting relationships among the integral transform, the convolution product and the first variation for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$. These relationships are given by (3.22) through (3.26) below. Equations (3.22)-(3.26) now follow from (3.2)-(3.4), (3.18) and (3.21).

Corollary 3.9. *Let α, γ, β, F and G be as in Corollary 3.7 and let w be as in Theorem 3.1. Then*

$$\delta(F * G)_\gamma(z|w) = (\delta F(\cdot|w/\sqrt{2}) * G)_\gamma(z) + (F * \delta G(\cdot|w/\sqrt{2}))_\gamma(z), \tag{3.22}$$

$$\mathcal{F}_{\alpha,1}\mathcal{F}_{\gamma,1}\delta F(\cdot|w)(z) = \delta\mathcal{F}_{\gamma,1}\mathcal{F}_{\alpha,1}F(z|w), \tag{3.23}$$

$$\begin{aligned}&\beta^2\mathcal{F}_{\gamma,\beta}(\delta F(\cdot|w) * \delta G(\cdot|w))_\gamma(z)\\ &= \delta\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}\mathcal{F}_{\frac{\gamma}{\sqrt{2}},1}F(z/\sqrt{2}|w)\delta\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}\mathcal{F}_{-\frac{\gamma}{\sqrt{2}},1}G(z/\sqrt{2}|w),\end{aligned} \tag{3.24}$$

$$\begin{aligned}&\beta\mathcal{F}_{\gamma,\beta}\delta(F * G)_\gamma(\cdot|w)(z)\\ &= \delta(\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}\mathcal{F}_{\frac{\gamma}{\sqrt{2}},1}F(\cdot/\sqrt{2})\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}\mathcal{F}_{-\frac{\gamma}{\sqrt{2}},1}G(\cdot/\sqrt{2}))(z|w)\\ &= \delta\mathcal{F}_{\gamma,\beta}(F * G)_\gamma(z|w)\end{aligned} \tag{3.25}$$

and

$$\beta^2(\mathcal{F}_{\gamma,\beta}\delta F(\cdot|w) * \mathcal{F}_{\gamma,\beta}\delta G(\cdot|w))_\gamma(z) = (\delta\mathcal{F}_{\gamma,\beta}F(\cdot|w) * \delta\mathcal{F}_{\gamma,\beta}(\cdot|w))_\gamma(z) \tag{3.26}$$

for $z \in K_{a,b}[0,T]$.

4. Inverse Integral Transforms

In this section we derive a version of the inverse integral transform. We then obtain several relationships involving the inverse integral transform.

The following theorem is one of main results in this paper.

Theorem 4.1. *Let* γ, β *and* F *be as in Theorem* 3.1. *Then*

$$\mathcal{F}_{-i\gamma,1}\mathcal{F}_{i\gamma,1}\mathcal{F}_{-\frac{\gamma}{\beta},\frac{1}{\beta}}\mathcal{F}_{\gamma,\beta}F(x) = \mathcal{F}_{\gamma,\beta}\mathcal{F}_{-i\gamma,1}\mathcal{F}_{i\gamma,1}\mathcal{F}_{-\frac{\gamma}{\beta},\frac{1}{\beta}}F(x) = F(x) \tag{4.1}$$

for $x \in K_{a,b}[0,T]$. *That is to say, the inverse integral transform is given by*

$$\mathcal{F}^{-1}_{\gamma,\beta} = \mathcal{F}_{-i\gamma,1}\mathcal{F}_{i\gamma,1}\mathcal{F}_{-\frac{\gamma}{\beta},\frac{1}{\beta}}.$$

Proof. Using equation (3.2) several times, we can prove Theorem 4.1. □

Remark 4.2. Let $\gamma_1, \gamma_2, \beta_1$ and β_2 be nonzero complex numbers. By careful examination we see that there are no nonzero complex numbers γ' and β' such that

$$\mathcal{F}_{\gamma_1,\beta_1}\mathcal{F}_{\gamma_2,\beta_2} = \mathcal{F}_{\gamma',\beta'}.$$

Hence our inverse integral transform $\mathcal{F}^{-1}_{\gamma,\beta}$ can not be expressed by one pair of any parameters, that is to say,

$$\mathcal{F}^{-1}_{\gamma,\beta} \neq \mathcal{F}_{\gamma',\beta'}$$

for any γ', β' in $\mathbb{C} - \{0\}$. But $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, we have

$$\mathcal{F}_{-i\gamma,1}\mathcal{F}_{i\gamma,1}\mathcal{F}_{-\frac{\gamma}{\beta},\frac{1}{\beta}} = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}.$$

In this case, we can write

$$\mathcal{F}^{-1}_{\gamma,\beta} = \mathcal{F}_{i\frac{\gamma}{\beta},\frac{1}{\beta}}.$$

Our next goal is to obtain other versions for our inverse integral transform.

Theorem 4.3. *Let* γ, β, F *and* G *be as in Theorem* 3.3. *Then*

$$\begin{aligned}(\mathcal{F}^{-1}_{\frac{\gamma}{\sqrt{2}},1}F * \mathcal{F}^{-1}_{-\frac{\gamma}{\sqrt{2}},1}G)_\gamma(z) \\ = \mathcal{F}^{-1}_{\gamma,\beta}(\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}F(\cdot/\sqrt{2})\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}G(\cdot/\sqrt{2}))(z)\end{aligned} \tag{4.2}$$

for $z \in K_{a,b}[0,T]$.

Proof. Using equations (3.18) and (4.1), it follows that

$$\begin{aligned}&\mathcal{F}_{\gamma,\beta}(\mathcal{F}^{-1}_{\frac{\gamma}{\sqrt{2}},1}F * \mathcal{F}^{-1}_{-\frac{\gamma}{\sqrt{2}},1}G)_\gamma(z) \\ &= \mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}\mathcal{F}_{\frac{\gamma}{\sqrt{2}},1}\mathcal{F}^{-1}_{\frac{\gamma}{\sqrt{2}},1}F(z/\sqrt{2})\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}\mathcal{F}_{-\frac{\gamma}{\sqrt{2}},1}\mathcal{F}^{-1}_{-\frac{\gamma}{\sqrt{2}},1}G(z/\sqrt{2}) \\ &= \mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}F(z/\sqrt{2})\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}G(z/\sqrt{2}).\end{aligned} \tag{4.3}$$

Now taking the inverse integral transform $\mathcal{F}_{\gamma,\beta}^{-1}$ of each side of equation (4.3), we obtain equation (4.2) as desired. □

Theorem 4.4. *Let γ, β, F and G be as in Theorem 3.3. Then*

$$\mathcal{F}_{\sqrt{2}\gamma,\beta}(\mathcal{F}_{\gamma,1}^{-1}F * \mathcal{F}_{-\gamma,1}^{-1}G)_{\sqrt{2}\gamma}(z) = \mathcal{F}_{\gamma,\beta}F(z/\sqrt{2})\mathcal{F}_{\gamma,\beta}G(z/\sqrt{2}) \tag{4.4}$$

for $z \in K_{a,b}[0,T]$.

Proof. Proceeding as in the proof of Theorem 4.3, we can obtain equation (4.4). □

Corollary 4.5. *Let γ, β and F be as in Theorem 3.5. Then*

$$(\mathcal{F}_{\frac{\gamma}{\sqrt{2}},1}^{-1}F * \mathcal{F}_{-\frac{\gamma}{\sqrt{2}},1}^{-1}F)_{\gamma}(z) \quad = \mathcal{F}_{\gamma,\beta}^{-1}([\mathcal{F}_{\frac{\gamma}{\sqrt{2}},\beta}F(\cdot/\sqrt{2})]^2)(z)$$

and

$$\mathcal{F}_{\sqrt{2}\gamma,\beta}(\mathcal{F}_{\gamma,1}^{-1}F * \mathcal{F}_{-\gamma,1}^{-1}F)_{\sqrt{2}\gamma}(z) = [\mathcal{F}_{\gamma,\beta}F(z/\sqrt{2})]^2$$

for $z \in K_{a,b}[0,T]$.

Remark 4.6. In view of Theorems 4.3 and 4.4, we see that there is an interesting viewpoint. In Theorem 4.3, if we take γ replaced with $\sqrt{2}\gamma$ then equation (4.2) becomes

$$\begin{aligned}&(\mathcal{F}_{\gamma,1}^{-1}F * \mathcal{F}_{-\gamma,1}^{-1}G)_{\sqrt{2}\gamma}(z)\\ &= \mathcal{F}_{\sqrt{2}\gamma,\beta}^{-1}(\mathcal{F}_{\gamma,\beta}F(\cdot/\sqrt{2})\mathcal{F}_{\gamma,\beta}G(\cdot/\sqrt{2}))(z)\end{aligned} \tag{4.5}$$

for $z \in K_{a,b}[0,T]$. Now taking the integral transform $\mathcal{F}_{\sqrt{2}\gamma,\beta}$ of each side of equation (4.5), we obtain equation (4.4). Hence we can obtain various alternative forms of equations (4.2) and (4.4).

In our last theorem, we establish a formula involving inverse integral transforms.

Theorem 4.7. *Let γ, F and G be as in Theorem 3.3. Then*

$$(\mathcal{F}_{\gamma,1}^{-1}F * 1)_{\sqrt{2}\gamma}(z)(1 * \mathcal{F}_{-\gamma,1}^{-1}G)_{\sqrt{2}\gamma}(z) = F(z/\sqrt{2})G(z/\sqrt{2}) \tag{4.6}$$

for $z \in K_{a,b}[0,T]$.

Proof. Using equations (3.11) and (3.12), we obtain equation (4.6) as desired. □

We finish this section by establishing several relationships involving inverse integral transforms. These relationships are given by (4.7)-(4.9) below. We omit the proof of our next corollary because the proof is similar to the proof of Corollary 3.9.

Corollary 4.8. *Let γ, β, F and G be as in Theorem 4.4 and let w be as in Theorem 3.8. Then*

$$\mathcal{F}_{\gamma,\beta}^{-1}\delta F(\cdot|w)(z) = \beta\delta\mathcal{F}_{\gamma,\beta}^{-1}F(z|w), \tag{4.7}$$

$$\beta\delta\mathcal{F}_{\gamma,\beta}^{-1}(F * G)_\gamma(z|w) = \mathcal{F}_{\gamma,\beta}^{-1}\delta(F * G)_\gamma(\cdot|w)(z) \tag{4.8}$$

and

$$(\mathcal{F}_{\gamma,\beta}^{-1}\delta F(\cdot|w) * \mathcal{F}_{\gamma,\beta}^{-1}\delta G(\cdot|w))_\gamma(z) = \beta^2(\delta\mathcal{F}_{\gamma,\beta}^{-1}F(\cdot|w) * \delta\mathcal{F}_{\gamma,\beta}^{-1}G(\cdot|w))_\gamma(z) \tag{4.9}$$

for $z \in K_{a,b}[0,T]$.

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