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Inference on the reliability $P(Y < X)$ in the gamma case

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Abstract

We shall derive a quotient distribution of two independent gamma variables and its moment and reliability are represented by hypergeometric function and Wittaker's function. And we shall consider an inference on the reliability in two independent gamma random variables.

Keywords: Hypergeometric function, reliability, Whittaker function.

1. Introduction

For two independent random variables X and Y and a real number c, the probability $P\{X < cY\}$ induces the following facts ; (i) the probability $P\{X < cY\}$ is the reliability when the real number c equals one, (ii) the probability $P\{X < cY\}$ is the distribution of the ratio $X/(X+Y)$ when $c = t/(1-t)$ for $0 < t < 1$, and (iii) the probability $P\{X < cY\}$ induces the density of a skewed-symmetric random variable if X and Y are symmetric random variables about origin. Woo (2006) introduced the proceeding probability $P\{X < cY\}$. The reliability will increase the need for the industry to perform systematic study for the identifications and reduction of causes of failures. These reliability studies must be performed by persons who (i) can identify and quantify the modes of failures, (ii) know how to obtain and analyze the statistics of failure occurrences, and (iii) can construct mathematical models of the failure that depend on, for example, the parameters of the material strength or the design quality, the fatigue or the wear resistance, and the stochastic nature of the anticipated duty cycle. Many authors have considered properties of the gamma distribution in Johnson and Balakrishnan (1995). McCool (1991) considered an inference on the reliability $P\{X < Y\}$ in the Weibull case. Ali and Woo (2005a, 2005b) studied inferences on reliability $P\{X < Y\}$ in two independent power function distributions and Levy distributions, respectively. Lee and Won(2006) considered an inference on the reliability in an exponentiated uniform distribution. Saunders (2007) introduced the reliability, the life testing, and

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the prediction of service lives for engineers and scientists. Woo (2006) studied the reliability $P\{X < Y\}$, the ratio $X/(X+Y)$, and a skewed-symmetric distribution of two independent random variables. Woo (2007) studied the reliability in a half-triangle distribution and a skew-symmetric distribution.

In this paper, we derive a quotient distribution of two independent gamma variables and its moment and reliability are represented by mathematical special functions, hypergeometric function and Wittaker's function. And we consider an inference on the reliability in two independent gamma random variables having two different scales and known the shape parameter.

2. Quotient and reliability

Let X and Y be independent gamma random variables with the densities :

$$\begin{aligned} f_X(x) &= \frac{1}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} x^{\alpha_1 - 1} e^{-x/\beta_1}, \quad \text{if} \;\; 0 < x < \infty, \\ f_Y(y) &= \frac{1}{\Gamma(\alpha_2)\beta_2^{\alpha_2}} y^{\alpha_2 - 1} e^{-y/\beta_2}, \quad \text{if} \;\; 0 < y < \infty, \end{aligned} \tag{2.1}$$

where $\alpha_i's$ and $\beta_i's$ are positive.

Let $W = Y/X$. Then, from the quotient distribution in Rohatgi(1976) and the formula 3.381(4) in Gradshteyn and Ryzhik(1965), the density of $W = Y/X$ is given by:

$$f_W(w) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (\frac{\beta_1}{\beta_2})^{\alpha_2} \cdot w^{\alpha_2 - 1} (1 + \frac{\beta_1}{\beta_2} \cdot w)^{-\alpha_1 - \alpha_2}, \quad \text{if} \;\; 0 < w < \infty. \tag{2.2}$$

From the density (2.2) and the formula 3.5 in Oberhettinger(1974), the moment generating function(mgf) of W can be obtained as :

$$m_W(t) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)} \left(-\frac{\beta_2}{\beta_1} t \right)^{\frac{\alpha_1 - 1}{2}} exp\left(-\frac{\beta_2}{\beta_1} \frac{t}{2} \right) W_{\frac{1}{2} - \frac{\alpha_1}{2} - \alpha_2 - \frac{\alpha_1}{2}} (-\frac{\beta_2}{\beta_1} t),$$

where $W_{a,b}(x)$ is the Whittaker function.

It is the guarantee for an existence of k-th moment of W, and from the formula 3.194(3) in Gradshteyn and Ryzhik (1965), the k-th moment of W is :

$$E(W^k) = \frac{\Gamma(\alpha_1 - k)\Gamma(\alpha_2 + k)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (\frac{\beta_2}{\beta_1})^k, \quad \text{if} \;\; \alpha_1 > k = 1, 2, \cdots .$$

From the density (2.2) and the formulas 3.381(1) and (2) in Gradshteyn and Ryzhik (1965), we can obtain the reliability $P\{X < Y\}$:

Fact 1. Let X and Y be two independent gamma random variables with densities (2.1). Then, for $\rho = \beta_2/\beta_1$,

$$R \equiv P(Y < X) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{\rho^{\alpha_1}}{(1+\rho)^{\alpha_1 + \alpha_2}} {}_2F_1(1, \alpha_1 + \alpha_2 \; , \; 1 + \alpha_2 \; ; \; \frac{1}{1+\rho}),$$

where $F(a, b, c\ ;\ x)$ is the hypergeometric function in Gradshteyn and Ryzhik(1965). From Fact 1 and the formula 15.2.1 in Abramowitz and Stegun(1970), we can obtain the following:

Fact 2. If the mean of Y is greater than that of X, (i.e., $\alpha_2\beta_2 > \alpha_1\beta_1$) in the densities (2.1). then the reliability $R = P(Y < X)$ is a monotone decreasing function of ρ .

Remark. If the mean of X is greater than that of Y, (i.e., $\alpha_1\beta_1 > \alpha_2\beta_2$), then $P(Y < X) = 1 - P(X < Y)$ is a monotone increasing function of ρ, from Fact 2.

3. Inference on the reliability $P\{X < Y\}$

We consider an inference on the reliability $P\{X < Y\}$ when X and Y are two independent gamma variables having two different scale parameters and known the shape parameter in the densities (2.1). Because $R = P\{X < Y\}$ is a monotone function of ρ from Facts 1 and 2, an inference on the reliability is equivalent to an inference on ρ. See McCool (1991). Hence we consider an inference on $\rho \equiv \beta_2/\beta_1$ when the $\alpha_i's$ are known in the densities (2.1).

We introduce the following well-known results in Fact 3:

Fact 3. Let X be a gamma random variable with the mean $\alpha\beta$ and the variance $\alpha\beta^2$. Then

(a) $E(\frac{1}{X}) = \frac{1}{(\alpha-1)\beta}$, if $\alpha > 1$.

(b) $E(\frac{1}{X^2}) = \frac{1}{(\alpha-1)(\alpha-2)\beta^2}$, if $\alpha > 2$.

Assume $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two independent samples from the densities (2.1), respectively. The MLE $\hat{\beta}_i$, $i = 1, 2$, of β_i are

$$\hat{\beta}_1 = \frac{1}{m\alpha_1}\sum_{i=1}^{m} X_i \quad \text{and} \quad \hat{\beta}_2 = \frac{1}{n\alpha_2}\sum_{i=1}^{n} Y_i \ .$$

Therefore, the MLE $\hat{\rho}$ of ρ is $\hat{\rho} = \hat{\beta}_2/\hat{\beta}_1$.

From Fact 3, we can obtain the expectation and variance of $\hat{\rho}$:

$$E(\hat{\rho}) = \frac{m\alpha_1}{m\alpha_1 - 1}\cdot\rho, \quad m\alpha_1 > 1 \tag{3.1}$$

and

$$Var(\hat{\rho}) = \frac{m^2\alpha_1^2(m\alpha_1 + n\alpha_2 - 1)}{(m\alpha_1-1)^2 n\alpha_2(m\alpha_1 - 2)}\cdot\rho^2, \quad m\alpha_1 > 2. \tag{3.2}$$

From the expectation (3.1), we can define an unbiased estimator $\tilde{\rho}$ of ρ :

$$\tilde{\rho} = \frac{m\alpha_1 - 1}{n\alpha_2}\frac{\sum_{j=1}^{n} Y_j}{\sum_{i=1}^{m} X_i}.$$

Hence, from Fact 3, we can obtain the variance of $\tilde{\rho}$:

$$Var(\tilde{\rho}) = \frac{m\alpha_1 + n\alpha_2 - 1}{n\alpha_2(m\alpha_1 - 2)}\cdot\rho^2, \quad m\alpha_1 > 2. \tag{3.3}$$

From the results (3,1), (3.2) and (3,3), we can compare mean squares errors(MSE) of $\hat{\rho}$ and $\tilde{\rho}$:

Fact 4. An unbiased estimator $\tilde{\rho}$ is more efficient than the MLE $\hat{\rho}$ in a sense of MSE, if $m\alpha_1 > 2$.

To consider an interval estimator of ρ, especially if the shape parameter $\alpha_i's$ in the densities (2.1) are positive integers, i.e. Erlang distributions, then $\rho \cdot \hat{\beta}_1/\hat{\beta}_2$ is a pivot quantity having the following distribution :

$$\frac{\hat{\beta}_1}{\hat{\beta}_2} \cdot \rho \quad \text{has a F-distribution with} \quad df(2m\alpha_1, 2n\alpha_2).$$

Therefore, an $(1-\gamma)100\%$ confidence interval of ρ can be obtained as:

$$\left(\frac{1}{F_{\gamma/2}(2n\alpha_2, 2m\alpha_1)} \cdot \frac{\hat{\beta}_2}{\hat{\beta}_1},\ F_{\gamma/2}(2m\alpha_1, 2n\alpha_2) \cdot \frac{\hat{\beta}_2}{\hat{\beta}_1} \right),$$

where $\int_c^\infty h(t)dt = \gamma/2$, $c \equiv F_{\gamma/2}(2m\alpha_1, 2n\alpha_2)$, and $h(t)$ is the density of a F-distribution with df $(2m\alpha_1,\ 2n\alpha_2)$.

Next, we consider a test with the following statistical hypothesis :

$$H_o\ ;\ \beta_1 = \beta_2 = \beta\ (\rho = 1) \quad \text{against} \quad H_1\ ;\ \beta_1 \neq \beta_2\ (\rho \neq 1).$$

Applying the likelihood ratio test in Rohatgi(1976, p.436),

$$\lambda(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n) = \frac{f_X(x_1, x_2, \cdots, x_m; \hat{\beta}\ ) \cdot f_Y(y_1, y_2, \cdots, y_n; \hat{\beta}\ )}{f_X(x_1, x_2, \cdots, x_m; \hat{\beta}_1\ ) \cdot f_Y(y_1, y_2, xdots, y_n; \hat{\beta}_2\ )} \leq c,$$

where $\hat{\beta} = \dfrac{1}{m\alpha_1 + n\alpha_2}(\sum_{i=1}^m X_i + \sum_{j=1}^n Y_j)$. From the proceeding function $\lambda(x_1, \cdots, x_m;$ $y_1, \cdots, y_n)$ in the likelihood ratio test. we can obtain the following equivalent:

$$\lambda(x_1, \cdots, x_m; y_1, \cdots, y_n) <= c \ \Leftrightarrow\ \frac{\sum_{i=1}^n Y_i}{\sum_{j=1}^m X_j} >= c_1 \quad or \quad \frac{\sum_{i=1}^n Y_i}{\sum_{j=1}^m X_j} <= c_2 \tag{3.4}$$

If H_0 is true and the shape parameter $\alpha_i's$ in the densities (2.1) are positive integers, i.e. Erlang distributions, then, from the relations (3.4), we can obtain the critical region " C " of the test :

$$C\ ;\ \frac{\hat{\beta}_2}{\hat{\beta}_1} < \frac{1}{F_{\gamma/2}(2m\alpha_1, 2n\alpha_2)} \quad \text{or} \quad \frac{\hat{\beta}_2}{\hat{\beta}_1} > F_{\gamma/2}(2n\alpha_2, 2m\alpha_1).$$

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