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Permanence of an impulsive food web system with Holling-type II functional responses

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Abstract

In this paper, we are studying the property for permanence of a three species food chain system with impulsive perturbations and Holling type II functional response, species which is important concept or property in ecological systems. Specially, we give the conditions for the permanence of this system. To do it, we consider the comparison method which is typical skill happened in impulsive differential inequalities. In addition, we reaffirm our results by using a numerical example.

Keywords: A three-species food chain system, holling-type II functional response, impulsive perturbations.

1. Introduction

Recently, researches on the impulsive prey-predator population systems and simple multi-species systems consisting of a three-species food chain with impulsive control strategy have been discussed in Baek (2008), Liu (2005, 2006), Wang (2007, 2008), and Zhang (2005, 2006a, 2006b). Holling (1965) gave three different kinds of functional response of the predator to the prey, which referred to the change in the density of prey attacked per unit time per predator as the prey density changed, to describe more realistic situations than standard Lotka-Volterra system. According to Hassel et al (1969), the Holling type II functional response is the most common type of functional response among arthropod predators. Thus, based on the predator-prey system with Holling II and group defense, Zhang and et al (2005) proposed an impulsive differential equation to model the process of periodically releasing natural enemies as follows:

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$$
\begin{cases}
x'(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - \dfrac{c_1 x(t)\, y(t)}{e_1 + x(t)}, & \text{t} \neq \text{nT} \\
y'(t) = -d_1 y(t) + \dfrac{c_2 x(t)\, y(t)}{e_1 + x(t)} - \dfrac{c_3 y(t)\, z(t)}{e_2 + y(t)}, & \text{t} \neq \text{nT} \ , \\
z'(t) = -d_2 z(t) + \dfrac{c_4 y(t)\, z(t)}{e_2 + y(t)}, & \text{t} \neq \text{nT} \\
\triangle x(t^+) = 0, & \text{t=nT} \\
\triangle y(t^+) = 0, & \text{t=nT} \ , \\
\triangle z(t^+) = p, & \text{t=nT} \\
(x(0^+), y(0^+), z(0^+)) = (x_0, y_0, z_0) .
\end{cases}
\tag{1.1}
$$

where r is the intrinsic rate of increase, K is the capacity of the prey, c_1 and c_3 are the per-capita rate of predation of the predators, $e_i\,(i = 1, 2)$ are the half-saturation constants, d_1, d_2 denote the death rate of the mid-level predator and the top-level predator, respectively, T is the period of the impulsive immigration or stock of the predator, and q is the size of immigration or stock of the predator.

The authors in Zhang (2005b) studied the stability for the predator-free periodic solution and prey-predator-free periodic solution of the system (1.1). They proved their local stability and showed that the system (1.1) has complicated dynamical behaviors by using numerical simulations.

The main purpose of this paper is to determine the conditions for the permanence of the system (1.1).

2. Preliminaries

First, we shall introduce a few notations and definitions together with a few auxiliary results relating to comparison theorem, which will be useful for our main results.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^3_+ = \left\{\mathbf{x} = (x(t), y(t), z(t)) \in \mathbb{R}^3 : x(t), y(t), z(t) \geq 0\right\}$. Denote $\mathbb{N}$ the set of all nonnegative integers, $\mathbb{R}^*_+ = (0, \infty)$ and $f = (f_1, f_2, f_3)^T$ the right hand of the first three equations in (1.1). Let $V : \mathbb{R}_+ \times \mathbb{R}^3_+ \to \mathbb{R}_+$, then V is said to be in a class V_0 if

(1) V is continuous on $(nT, (n+1)T] \times \mathbb{R}^3_+$, and $\lim_{\substack{(t,\,\mathbf{y}) \to (nT,\,\mathbf{x}) \\ t > nT}} V(t, \mathbf{y}) = V(nT^+, \mathbf{x})$ exists.

(2) V is locally Lipschitzian in $\mathbf{x}$.

Definition 2.1 For $V \in V_0$, we define the upper right Dini derivative of V with respect to the impulsive differential system (1.1) at $(t, \mathbf{x}) \in (nT, (n+1)T] \times \mathbb{R}^3_+$ by

$$
D^+V(t, \mathbf{x}) = \lim_{h \to 0+} \sup \frac{1}{h} \left[V(t + h, \mathbf{x} + hf(t, \mathbf{x})) - V(t, \mathbf{x})\right]
$$

Remark 2.1 (1) The solution of the system (1.1) is a piecewise continuous function $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}^3_+$, $\mathbf{x}(t)$ is continuous on $(nT, (n+1)T]$, $n \in \mathbb{N}$ and $\mathbf{x}(nT^+) = \lim_{t \to nT^+} \mathbf{x}(t)$

exists. (2) The smoothness properties of f guarantee the global existence and uniqueness of solution of the system (1.1). See Bainov (1993) and Lakshmikantha (1989) for the details.

We will use a comparison result of impulsive differential inequalities. We suppose that $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the following hypotheses:

(H) g is continuous on $(nT, (n+1)T] \times \mathbb{R}_+$ and the limit $\lim_{(t,y)\to(nT^+,x)} g(t,y) = g(nT^+, x)$ exists and is finite for $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$.

According to Bainov (1993) and Lakshmikantha (1989), we obtain the followin Lemma.

Lemma 2.1 Suppose $V \in V_0$ and

$$\begin{cases} D^+V(t,\mathbf{x}) \le g(t, V(t,\mathbf{x})), & t \ne nT, \\ V(t,\mathbf{x}(t^+)) \le \psi_n(V(t,\mathbf{x})), & t = nT, \end{cases} \tag{2.1}$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies (H) and $\psi_n : \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing for all $n \in \mathbb{N}$. Let $r(t)$ be the maximal solution for the impulsive Cauchy problem

$$\begin{cases} u'(t) = g(t, u(t)), & t \ne nT, \\ u(t^+) = \psi_n(u(t)), & t = nT, \\ u(0^+) = u_0, \end{cases} \tag{2.2}$$

defined on $[0,\infty)$. Then $V(0^+, \mathbf{x}_0) \le u_0$ implies that $V(t, \mathbf{x}(t)) \le r(t)$, $t \ge 0$, where $\mathbf{x}(t)$ is any solution of (2.1).

A similar result can be obtained when all conditions of the inequalities in the Lemma 2.3 are reversed.

Now, we give the basic properties of another impulsive differential equation as follows:

$$\begin{cases} z'(t) = -d_2 z(t), & t \ne nT, \\ z(t^+) = z(t) + p, & t = nT, \\ z(0^+) = z_0. \end{cases} \tag{2.3}$$

The system (2.3) is a periodically forced linear system. It is easy to obtain that

$$z^*(t) = \frac{p\exp(-d_2(t-(n-1)T))}{1-\exp(-d_2T)}, \quad (n-1)T < t \le nT \tag{2.4}$$

$z^*(0^+) = z^*(nT^+) = p/(1-\exp(-d_2T))$, is a positive periodic solution of (2.3). Moreover, we can obtain that

$$z(t) = \left(z\left(0^+\right) - \frac{p}{1-\exp(-d_2T)}\right)\exp(-d_2t) + z^*(t), \quad (n-1)T < t \le nT, \tag{2.5}$$

is a solution of (2.3). From (2.4) and (2.5), we get easily the following result.

Lemma 2.2 Let $z(t)$ be any solution of equation (2.3). Then $z(t)$ tends to the periodic solution $z^*(t)$. i.e., $|z(t) - z^*(t)| \to 0$ as $t \to \infty$.

Note that the logistic equation $x'(t) = rx(t)(1 - (x(t)/K))$ have a globally asymptotically stable equilibrium $x(t) = K$ and an unstable equilibrium $x(t) = 0$. Thus, we obtain the complete expression for the predator eradication periodic solution and the prey predator eradication periodic solution of the system (1.1).

$$(0,\, 0,\, z^*(t)) = \left(0,\, 0,\, \frac{p\, \exp(-d_2(t-(n-1)T))}{1-\exp(-d_2 T)}\right),\quad (n-1)T < t \le nT.$$

$$(K,\, 0,\, z^*(t)) = \left(K,\, 0,\, \frac{p\, \exp(-d_2(t-(n-1)T))}{1-\exp(-d_2 T)}\right),\quad (n-1)T < t \le nT.$$

Now we mention the following useful results in Zhang(2005b) to prove our main Theorem.

Theorem 2.1 There is an $M > 0$ such that $x(t) \le M$, $y(t) \le M$ and $z(t) \le M$ for all t large enough, where $(x(t), y(t), z(t))$ is a solution of the system (1.1).

3. Main theorem

First, we will remark the definition of the permanence before stating our main theorem.

Definition 3.1 The system (1.1) is said to be permanent if there exist $M \ge m > 0$ such that, for any solution $(x(t), y(t), z(t))$ of the system (1.1) with $x_0, y_0, z_0 > 0$, $m \le \lim_{t\to\infty} \infty\ x(t) \le \lim_{t\to\infty} \sup\ x(t) \le M$, $m \le \lim_{t\to\infty} \infty\ y(t) \le \lim_{t\to\infty} \sup\ y(t) \le M$ and $m \le \lim_{t\to\infty} \infty\ z(t) \le \lim_{t\to\infty} \sup\ z(t) \le M$.

We will use the following basic lemmas in Brauer (2001) to show the permanence of the system (1.1).

Lemma 3.1 Consider the following competitive Lotka-Volterra predator-prey system:

$$\begin{cases} u'(t) = ru(t)(1 - du(t)K) - \alpha u(t)v(t), \\ v'(t) = -dv(t) + \beta u(t)v(t). \end{cases} \tag{3.1}$$

Then the system (3.1) has a unique positive equilibrium point $(u^*, v^*) = ((d/\beta), (r/\alpha)(1 - u^*/K))$ which is globally asymptotically stable if $u^* < K$.

It follows from Liu (2003) that the following Lemma holds.

Lemma 3.2 Consider the following prey-dependent consumption system:

$$\begin{cases} u'(t) = ru(t)\left(1 - \dfrac{u(t)}{K}\right) - \alpha u(t)v(t). \\ v'(t) = -dv(t) + \dfrac{\beta u(t)v(t)}{\gamma + u(t)}. \end{cases} \tag{3.2}$$

Then the system (3.1) has a unique positive equilibrium point $(u^*, v^*) = (d\gamma/(\beta - d), (r/\alpha)(1 - u^*/K))$ which is globally asymptotically stable if $u^* < K$ and $\beta > d$.

Theorem 3.1 The system (1.1) is permanent if $0 < Kc_2 - d_1e_1 < (Kc_1c_2d_2e_2)/(rc_4e_1)$ and

$$p < \min\left\{\frac{e_2\left(c_2K - d_1e_1 - d_1K\right)\left(1 - \exp\left(\rho T\right)\right)}{c_3\left(e_1 + K\right)}, \frac{e_2\left(c_2 - d_1\right)\left(1 - \exp\left(\rho T\right)\right)}{c_3}\right\}, \tag{3.3}$$

where $\rho = (c_4\, e_1 r\, (Kc_2 - d_1e_1) - Kc_1c_2d_2e_2)/(Kc_1c_2e_2)$.

Proof: Let $(x\,(t)\,, y\,(t)\,, z\,(t))$ be the solution of the system (1.1) with a positive initial value (x_0, y_0, z_0) . By Theorem 2.1, the solution $(x\,(t)\,, y\,(t)\,, z\,(t))$ has an upperbound $M > 0$. Now, we will find a lowerbound $m > 0$. To do this, consider the following system:

$$\begin{cases} x_1'\,(t) = rx_1\,(t)\left(1 - \dfrac{x_1\,(t)}{K}\right) - \dfrac{c_1}{e_1}x_1\,(t)\,y_1\,(t)\,, \\ y_1'\,(t) = -d_1y_1\,(t) + \dfrac{c_2}{e_1}x_1\,(t)\,y_1\,(t)\,, \\ (x_1\,(0^+)\,, y_1\,(0^+)) = (x_0, y_0)\,. \end{cases} \tag{3.4}$$

From Lemma 2.1, we get $x\,(t) \geq x_1\,(t)$ and $y\,(t) \leq y_1\,(t)$. By Lemma 3.2 and $0 < Kc_2 - d_1e_1$, we obtain $(x_1^*,\, y_1^*) = (d_1e_1/c_2, (e_1r/c_1)(1 - (x_1^*/K)))$ is globally asymptotically stable. Thus $\lim_{t\to\infty} x_1\,(t) = x_1^*$ and $\lim_{t\to\infty} y_1\,(t) = y_1^*$. Since $Kc_2 - d_1e_1 < (Kc_1c_2d_2e_2)/(rc_4e_1)$, we can choose a sufficiently small $\epsilon_1 > 0$ such that $\eta \equiv -d_2 + (c_4/e_2)\,(y_1^* + \epsilon_1) < 0$. Further, since $\lim_{t\to\infty} x_1\,(t) = x_1^*$ and $\lim_{t\to\infty} y_1\,(t) = y_1^*$, there exists a $T_1 > 0$ such that $x\,(t) > x_1^* - \epsilon_1 \equiv m_1$ and $y\,(t) < y_1^* + \epsilon_1$ for all $t > T_1$. For the simplicity, we may assume that $x\,(t) > x_1^* - \epsilon_1 \equiv m_1$ and $y\,(t) < y_1^* + \epsilon_1$ for all $t > 0$. By the third equation of (1.1), Lemmas 2.1 and 2.2, we have $z\,(t) \leq w\,(t)$ and $|w\,(t) - w^*\,(t)| \to \infty$ as $t \to \infty$, where $w\,(t)$ is the solution of

$$\begin{cases} w'\,(t) = \eta w\,(t)\,, & t \neq nT, \\ \triangle w\,(t) = p, & t = nT, \\ w\,(0^+) = z_0 \end{cases} \tag{3.5}$$

and $w^*(t) = p\,\exp(\eta(t - nT))/(1 - \exp(\eta T))\,, t \in ((n-1)T, nT]\ (w^*(0^+) = p/(1 - \exp(\eta T)))$ is the periodic solution of (3.5). It is easy to see from the choice of ϵ_1 that the periodic solution $w^*\,(t)$ is asymptotically stable. Therefore, for $\epsilon_2 > 0$, there exists a $T_2 > 0$ such that $z\,(t) \leq w\,(t) < w^*\,(t) + \epsilon_2 < p/(1 - \exp(\eta T)) + \epsilon_2 \equiv \sigma$ for $t > T_2$. Without loss of generality, we may suppose that $z\,(t) \leq \sigma$ for all $t > 0$. From Lemma 2.1 and Theorem 2.1, we have $x_2\,(t) \geq x\,(t)$ and $y_2\,(t) \leq y\,(t)$, where $(x_2\,(t)\,,\, y_2\,(t))$ is a solution of the following differential equation:

$$\begin{cases} x_2'\,(t) = rx_2\,(t)\left(1 - \dfrac{x_2\,(t)}{K}\right) - \dfrac{c_1}{e_1 + M}x_2\,(t)\,y_2\,(t)\,, \\ y_2'\,(t) = -d_1y_2\,(t) + \dfrac{c_2x_2\,(t)\,y_2\,(t)}{e_1 + x_2\,(t)} - \dfrac{c_3\sigma}{e_2}y_2\,(t)\,, \\ (x_2\,(0^+)\,, y_2\,(0^+)) = (x_0, y_0)\,. \end{cases} \tag{3.6}$$

It is inferred from Lemma 3.2 that the unique positive equilibrium point

$$(x_2^*, y_2^*) = \left(\frac{(d_1 e_2 + c_3 \sigma) e_1}{e_2 (c_2 - d_1) - c_3 \sigma}, \frac{r (e_1 + M)}{c_1} \left(1 - \frac{x_2^*}{K} \right) \right)$$

is globally asymptotically stable since the condition (3.3) holds. Thus, for $\epsilon_3 > 0$, there exists a $T_3 > 0$ such that $x(t) < x_2^* + \epsilon_3$ and $y(t) < y_2^* - \epsilon_3 \equiv m_2$ for all $t > T_3$. On the other hand, from Lemma 2.1 and 2.2 we get $z(t) \geq w_1(t)$ and $|w_1(t) - w_1^*(t)| \to \infty$ as $t \to \infty$, where $w_1(t)$ and $w_1^*(t)$ are the solution and the periodic solution of the following system, respectively.

$$\begin{cases} w_1'(t) = -d_2 w_1(t), & t \neq nT, \\ \triangle w_1(t) = p, & t = nT, \\ w_1(0^+) = z_0. \end{cases} \tag{3.7}$$

Thus $z(t) \geq w_1^*(t) - \epsilon_3 > p \exp(-d_2 T)/(1 - \exp(-d_2 T)) - \epsilon_3 \equiv m_3$ for large enough $t > 0$. Let $m = \min\{m_1, m_2, m_3\}$. Then we have $x(t), y(t), z(t) \geq m > 0$ for sufficiently large $t > 0$. This completes the proof. $\square$

Example 3.1 To illustrate an numerical example related to Theorem 3.1, let γ =1.1, c_1 =1.0, c_2 =1.5, c_3 =0.9, c_4 =0.01, d_1 =0.1, d_2 =0.6, e_1 =0.2, e_2 =0.7, K =3.0, p =1 and T =10. These parameters satisfy the conditions of Theorem 3.1. It implies that the system (1.1) is permanent. Figure 3.1 shows this phenomenon.

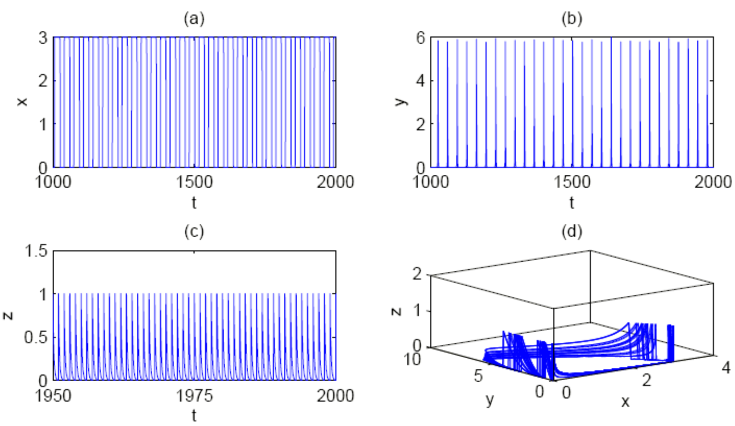

Figure 3.1 The dynamical behavior of the system (1.1). Figs. (a)-(c) show that a tragjectory with a starting point $(x_0, y_0, z_0) = (1, 1, 1)$ approaches to the periodic orbit $(K, 0, z^*(t))$, (d) displays the phase portrait of the system (1.1).

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