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# Marginal distribution of crossing time and renewal numbers related with two-state Erlang process

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### Abstract

In this study, we drive the one dimensional marginal transform function , probability density function and probability distribution function for the random variables $T_{\xi_N}$ (Time taken by the servers during the vacations), $\xi_N$ (Number of vacations taken by the servers) and $\eta_N$ ( Number of customers or units arrive in the system) by controlling the variability of two random variables simultaneously.

*Keywords*: Erlang process, marginal distributions, markov process, renewal process.

## 1. Introduction

Uchida and Aki (1995) considered that the recurrence relations of the probability generating functions (p.g.f.s) of the distributions of the sooner or later waiting time between $F_0$ and $F_1$ by the non overlapping way of counting and by the overlapping way of counting in the Markov chain. They also obtained recurrence relations of the p.g.f.s of the distributions of the sooner or later waiting time by the non overlapping way of counting of "0" runs length r or more and "1" run of length k or more in the Markov chain. The recurrence relations of the p.g.f.s of the waiting time distributions between $F_0$ and $F_1$ by the non overlapping way of counting in Markov chain was also discussed by Feller (1968). The recurrence relations of the p.g.f.s of the sooner and later waiting time distributions between $F_0$ and $F_1$ by the overlapping way of counting in Markov chain was extended from the work of ling (1988). The recurrence relations of the p.g.f.s of the sooner and later waiting time distributions between $F_0$ and $F_1$ by the non overlapping way of counting of "0" runs of length r or more and "1" runs of length k or more in Markov chain was discussed as in Goldstein (1990). Viveros and Balakrishnan (1993) emphasized on few applications of the geometric distribution of order k by using Bernoulli trial with common success probability p the geometric distribution of order k was one of the simplest waiting time distribution. Several waiting time problems had been studied by many authors Li (1980), Gerber and Li (1981), Ling (1981), Aki (1981),

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Aki and Hirano (1993) and Fu (1986, 1995, 1996), Ebneshahrashoob and Sobel (1990), Aki (1992), Balasubramanian et al. (1993), Kutras and Papastavirdis (1993), Godbole (1993), Fu and Kutras (1994), Mohanty (1994).

The purpose was to unify various approaches which have been attempted and to extend the study of waiting time problems from the first order Markov dependent trial to the second order Markov dependent trial. The statistical analysis extended the ideas to include waiting time for the occurrence of events. This is done by replacing the Laplace transform with MGFs and incorporating probabilities into branches. Thus nodes and branches represented events and the waiting time for the occurrence of such events. The standard approach to analyze continuous time Markov chains involved solving the Chapman Kolmogrove equations for the Laplace transform of the transition probabilities or the probability generating functions of the process; it is also used by Talpur and Shi (1994). They found the one dimension marginal distributions of crossing time and renewal numbers related with two poisson processes, using probability arguments and constructing an absorbing Markov process. In this study we extend the same technique for the case of two stage Erlang process. The Joint distribution of the three random variables has been obtained in the Talpur and Iffat (2007). Here for the same case two-stage Erlang distribution associated with two counting processes, we obtain the marginal distributions for three random variables.

### 1.1. Problem description

Searching relevant literature, we found that renewal processes are widely used in reliability theory and models of queuing theory. The two theories are based on counting processes. It is in common practice that one has to deal with the situations where the difference between two or more counting processes is examined. The stochastic processes are found very helpful in analyzing such type of situations. Kroese (1992) showed the difference of the two counting processes as

$$D(t) = N_1(t) - N_2(t)$$

Where $N_1(t)$ and $N_2(t)$ are two counting processes associated with corresponding renewal sequence of $\{X_i\}$ and $\{Y_j\}$. The problem considered for this study is extended from the work of Kroese (1992) and then Talpur and Shi (1994). It is based upon the renewal sequence of two variables $\{X_i\}$ and $\{Y_j\}$ as shown in the Figure 1.1

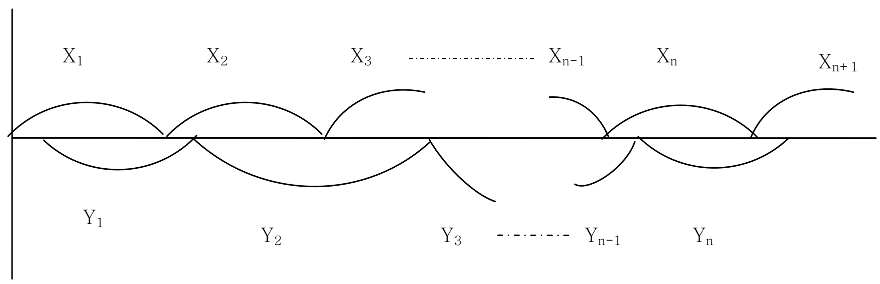


**Figure 1.1**

Let $\xi_N = \min_{n \to \infty} \{n/T_n \geq S_N\}$, where $\xi_N$ is random variable, N is a constant.
$S_0 = 0, S_N = X_1 + X_2 + \ldots + X_n$
$T_0 = 0, T_n = Y_1 + Y_2 + \ldots + Y_n$
$T_{\xi_N} = \sum_{j=1}^{\xi_N} Y_j$, its taking values are t > 0
and $\eta_N = \sum_{i=0}^{\infty} X_i$, its taking values are i = N, N+1, N+2, .......
X represents the inter arrival, Y is the number of vacations performed by the server. Both variables are discrete having renewal processes at each occurrence. The level of absorption was achieved at nth arrival of $X_n$. After nth arrival the nth vacation $Y_n$ of the server would happen. The difference of the time at which the nth vacation happened and the nth customer arrived is the crossing time of the server. The probability generating function, Probability density function, Cumulative probability distribution function for the three random variables, $T_{\xi_N}$ , $\xi_N$ and $\eta_N$ is obtained.

### 1.2. Assumptions

Let $N$ be a constant, $\{X_i\}$ and $\{Y_j\}$ be two sequences of random variables. Suppose that $\{X_i\}$, $i = 1, 2, 3, \ldots;$ independently and identically distributed with finite mean $\lambda^{-1}$ and $\{Y_j\}$, $j = 1, 2, 3, \ldots;$ are independently and identically distributed (i.i.d) with finite mean $\mu^{-1}$
$N_1(t)$ is the Erlang process associated with $\{X_i\}$ in which the distribution of $\{X_i\}$ is 2-stage Erlang distribution. $N_2(t)$ is the Erlang process associated with $\{Y_j\}$ in which the distribution of $\{Y_j\}$ is 2-stage Erlang distribution.
$X_i$ and $Y_j$ are mutually independent.

## 2. Absorbing Markov process and absorbing time distribution

We consider a Markov process $\{X(t),\ t \geq 0\}$ on the state space $E = (0, 1, 2, \ldots;)$. If $E_0$ and $E_1$are two non null sub set of E and they satisfy;
1) $E_0 \cap E_1 = E, E_0 \cup E_1 = \emptyset$, In this case $E_0, E_1$ are called a partition of $E$.
2) $E_0$ is the absorbing state set and $E_1$ is the transient state set.
The absorbing Markov process (A.M.P) is constructed to analyze the problem considering the AMP $\{N_1(t), N_2(t), I(t), J(t)\}$ in which $N_1(t)$, and $N_2(t)$ are the counting process associated with $X_i$ and $Y_j$ respectively.
$E = \{ (i, k, j, l), (i', j')/ i, j = 0, 1, \ldots; k, l = 1, 2; i' = N', N'+1', \ldots;, j' = 1', 2', \ldots;\}$ , Where $(i', j')$ are absorbing states. Transitions of states are shown in the Figure 2.1
Let

$$P_{ij}(k,l,t) = p\{N_1(t) = i, N_2(t) = j, I(t) = k, J(t) = l\}$$
$$and$$
$$P_{ij}(t) = [p_{ij}(1,1,t), ...p_{ij}(1,n,t), ...p_{ij}(m,1,t), ...p_{ij}(m,n,t)].$$

By the transition-rate diagram we can get the system of differential equations as follows

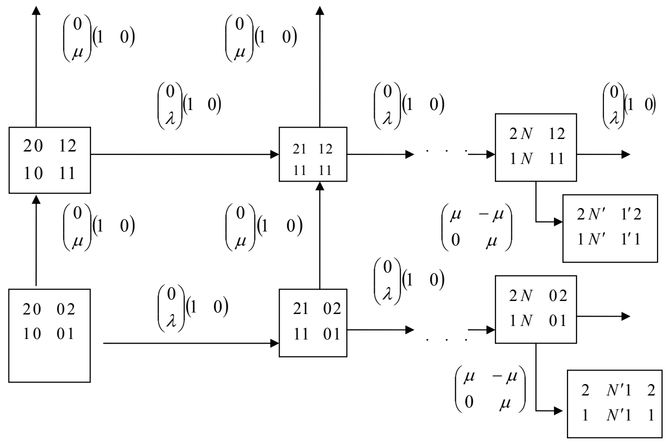

**Figure 2.1** Transition rate diagram

$$P'_{ij}(t) = p_{ij}(t)\left\{-\left\{\begin{pmatrix}\lambda & -\lambda\\ 0 & \lambda\end{pmatrix} + \begin{pmatrix}\mu & -\mu\\ 0 & \mu\end{pmatrix}\right\}\right\} + p_{i-1,j}(t)\begin{pmatrix}0 & 0\\ \lambda & 0\end{pmatrix} + p_{ij-1}(t)\begin{pmatrix}0\\ \mu\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}, i = 0, 1, ...N-1; , j = 0, 1, 2, ...; \tag{2.1}$$

$$P'_{ij}(t) = p_{ij}(t)\left\{-\left\{\begin{pmatrix}\lambda & -\lambda\\ 0 & \lambda\end{pmatrix} + \begin{pmatrix}\mu & -\mu\\ 0 & \mu\end{pmatrix}\right\}\right\} + p_{i-1j}(t)\begin{pmatrix}0 & 0\\ \lambda & 0\end{pmatrix}, i = N, N+1, ...; , j = 0, 1, ...; \tag{2.2}$$

From these differential difference equations we have obtained the joint distribution for three random variables in a different paper (2007). The one dimension marginal distributions for the same case are obtained in this study.

## 3. Methodology

The one dimensional marginal probability generating functions (probability transform functions), one dimensional probability density functions and cumulative probability distribution functions for random variables $T_{\xi_N}$, $\xi_N$ and $\eta_N$ are obtained by controlling the variability of two random variables simultaneously and find the effect of individual variable at one time.

### 3.1. Probability generating function for $T_{\xi_N}$

**Theorem 3.1** The one dimensional probability transform function of the random variable$T_{\xi_N}$ is given by

$$f^*(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -\mu & S+\lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -\lambda & s+\lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

**Proof**: The one dimensional marginal probability generating function (transform function ) for the random variable $T_{\xi_N}$ is computed from the joint probability generating function of three random variables $T_{\xi_N}$, $\xi_N$ and $\eta_N$ see Iffat (2004).

$$f^*(s,u,z) = u \begin{pmatrix} 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -u\mu & S+\lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda z & 0 \end{pmatrix} \right\}^N \times \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -\lambda z & s+\lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

The effect of number of vacations made by server at service channels and the number of arriving customers are controlled to get the probability generating effect for the time taken by the number of vacations made by servers. So let z and u close to 1 to find the one dimensional marginal probability generating function for the random variable $T_{\xi_N}$.

$$f^*(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -\mu & S+\lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \times \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -\lambda & s+\lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \tag{3.1}$$

**Theorem 3.2** The one dimensional marginal probability density function of random variable $T_{\xi_N}$ is given by

$$p\left\{T_{\xi_N} \le t\right\} = \begin{pmatrix} N+j-2 \\ j-1 \end{pmatrix} \mu^j \lambda^i \left[ \frac{(\lambda+\mu)^{j+i-1}\, t^{2j+2i-2}}{(2j+2i-2)!} + \frac{(\lambda+\mu)^{j+i}\, t^{2j+2i-1}}{(2j+2i-1)!} \right] e^{-(\lambda+\mu)t}$$

**Proof**: The definition of L transform can be expressed by the following equation as shown by Pipes (1970)

$$f^*(s) = \int_0^\infty \exp(-st) dp\left\{T_{\xi_N} \le t\right\} \tag{3.2}$$

The value of $f^*(s)$ from equation (3.1) is placed in equation (3.2) yields

$$\int_0^\infty \exp(-st) dp\left\{T_{\xi_N} \le t\right\} = \begin{pmatrix} 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -\mu & S+\lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix} \right\}^N \times \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -\lambda & s+\lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

Let $a = s + \lambda + \mu$ and multiplying and dividing by $a$ we get

$$f^*(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \left\{ \left\{ I - \begin{pmatrix} 0 & \frac{(\lambda+\mu)}{a} \\ \frac{\mu}{a} & 0 \end{pmatrix} \right\}^{-1} \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{a} & 0 \end{pmatrix} \right\}^N \frac{1}{a} \left\{ I - \begin{pmatrix} 0 & \frac{(\lambda+\mu)}{a} \\ \frac{\lambda}{a} & 0 \end{pmatrix}^{-1} \right\} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

The rule of power series is applied as in Pipes and Harwil (1970) and Talpur and Shi (1994).

$$f^*(s) = \frac{1}{a} \begin{pmatrix} 1 & 0 \end{pmatrix} \left\{ \sum_{k=0}^{\infty} \begin{pmatrix} 0 & \frac{(\lambda+\mu)}{a} \\ \frac{\mu}{a} & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{a} & 0 \end{pmatrix} \right\}^N \left\{ \sum_{l=0}^{\infty} \begin{pmatrix} 0 & \frac{(\lambda+\mu)}{a} \\ \frac{\lambda}{a} & 0 \end{pmatrix}^l \right\} \begin{pmatrix} \mu \\ \mu \end{pmatrix} \qquad (3.3)$$

Putting these results of two set of series $\left\{ \sum_{k=0}^{\infty} \begin{pmatrix} 0 & \frac{(\lambda+\mu)}{a} \\ \frac{\mu}{a} & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 \\ \frac{\lambda}{a} & 0 \end{pmatrix} \right\}^N$ and $\left\{ \sum_{l=0}^{\infty} \begin{pmatrix} 0 & \frac{(\lambda+\mu)}{a} \\ \frac{\lambda}{a} & 0 \end{pmatrix}^l \right\}$ is placed in equation (3.3).

$$f^*(s) = \frac{1}{a} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \left\{ \frac{\lambda}{a} \sum_{k=0}^{\infty} \left(\frac{\mu}{a}\right)^k \left(\frac{\lambda+\mu}{a}\right)^{k+1} \right\}^N & 0 \\ \left\{ \frac{\lambda}{a} \sum_{k=0}^{\infty} \left(\frac{\mu}{a}\right)^k \left(\frac{\lambda+\mu}{a}\right)^k \right\}^N \left(\frac{\lambda+\mu}{a}\right)^{N-1} & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} \sum_{l=0}^{\infty} \left(\frac{\lambda}{a}\right)^l \left(\frac{\lambda+\mu}{a}\right)^l & \sum_{l=0}^{\infty} \left(\frac{\lambda}{a}\right)^l \left(\frac{\lambda+\mu}{a}\right)^{l+1} \\ \sum_{l=0}^{\infty} \left(\frac{\lambda}{a}\right)^{l+1} \left(\frac{\lambda+\mu}{a}\right)^l & \sum_{l=0}^{\infty} \left(\frac{\lambda}{a}\right)^l \left(\frac{\lambda+\mu}{a}\right)^l \end{pmatrix} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

After some algebraic manipulations with the application of the negative binomial distribution we obtain the following expression, as Bailey (1964).

$$f^*(s) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \begin{pmatrix} N+k-1 \\ k \end{pmatrix} \left(\frac{\mu}{a}\right)^{k+1} \left(\frac{\lambda}{a}\right)^{l+N} \left(\frac{\lambda+\mu}{a}\right)^{k+l+N} \left\{ 1 + \left(\frac{\lambda+\mu}{a}\right) \right\}$$

Let $j = k + 1$ and $i = l + N$, and substituting the value of $a$ we get

$$f^*(s) = \sum_{j=1}^{\infty} \sum_{i=N}^{\infty} \begin{pmatrix} N+j-2 \\ j-1 \end{pmatrix} \left(\frac{\mu}{s+\lambda+\mu}\right)^j \left(\frac{\lambda}{s+\lambda+\mu}\right)^i$$
$$\times \left\{ \left(\frac{\lambda+\mu}{s+\lambda+\mu}\right)^{j+i-1} + \left(\frac{\lambda+\mu}{s+\lambda+\mu}\right)^{j+i} \right\}$$

Taking inverse of Laplace transform one can obtain the probability density function for continuous random variable as performed by Kreyszig (1999).

$$dp\left\{T_{\xi_N} \le t\right\} = \begin{pmatrix} N+j-2 \\ j-1 \end{pmatrix} \frac{\mu^j \lambda^i (\lambda+\mu)^{j+i-1} t^{2j+2i-2}}{(2j+2i-2)!} e^{-(\lambda+\mu)t} dt$$
$$+ \frac{\mu^j \lambda^i (\lambda+\mu)^{j+i} t^{2j+2i-1}}{(2j+2i-1)!} e^{-(\lambda+\mu)t} dt$$

The one dimension marginal probability density function of random variable $T_{\xi_N}$ time taken by vacations of servers is obtained as

$$p\{T_{\xi_N} \le t\} = \binom{N+j-2}{j-1} \mu^j \lambda^i \times \left[ \frac{(\lambda+\mu)^{j+i-1} t^{2j+2i-2}}{(2j+2i-2)!} + \frac{(\lambda+\mu)^{j+i} t^{2j+2i-1}}{(2j+2i-1)!} \right] e^{-(\lambda+\mu)t} \tag{3.4}$$

**Theorem 3.3** The one dimensional marginal cumulative probability distribution function of the random variable$T_{\xi_N}$.

$$p\{T_{\xi_N} \le t\} = \binom{N+j-2}{j-1} \left( \frac{\mu^j \lambda^i}{(\lambda+\mu)^{j+i}} \right) \times \left\{ \sum_{r=0}^{(2j+2i-N-1)} \frac{[(\lambda+\mu)t]^r}{r!} + \sum_{r=0}^{(2j+2i-N)} \frac{[(\lambda+\mu)t]^r}{r!} \right\} e^{-(\lambda+\mu)t} \tag{3.5}$$

**Proof**: The cumulative probability function can be defined as

$$p\{T_{\xi_N} \le t\} = \int_t^\infty p\{T_{\xi_N} \le t\} dt$$

Substituting the value of $p\{T_{\xi_N} \le t\}$ from equation no.(2.1.4) we get

$$p\{T_{\xi_N} \le t\} = \int_t^\infty \binom{N+j-2}{j-1} \mu^j \lambda^i \left[ \frac{(\lambda+\mu)^{j+i-1} t^{2j+2i-2}}{(2j+2i-2)!} + \frac{(\lambda+\mu)^{j+i} t^{2j+2i-1}}{(2j+2i-1)!} \right] e^{-(\lambda+\mu)t} dt$$

Integration by parts is used for finding the cumulative probability distribution function of time spent in vacations taken by servers as done by Medhi (1982), so the one dimensional cumulative distribution function for the random variable $T_{\xi_N}$ is established.

### 3.2. One dimensional marginal probability distribution functions for $\eta_N$

The effect of number of arriving customers represented by $\eta_N$ is studied by controlling the time taken by the number of vacations made and the number of vacations made by different service channels.

**Theorem 3.4** The one dimensional probability generating function (probability transform function) for the random variable $\eta_N$ is given by

$$f(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -\mu & \lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ z\lambda & 0 \end{pmatrix} \right\}^N \begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -z\lambda & \lambda+\mu \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu \end{pmatrix}$$

**Proof**: The one dimensional probability generating function (probability transform function) for the random variable $\eta_N$ is obtained from the joint probability generating function for three random variables $T_{\xi_N}$, $\xi_N$ and $\eta_N$ see Iffat (2004).

$$f^*(s,u,z) = u\begin{pmatrix}1 & 0\end{pmatrix}\left\{\begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -u\mu & S+\lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ \lambda z & 0\end{pmatrix}\right\}^N \times \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -\lambda z & s+\lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

The effect of the time taken by the number of vacations made and that of the random variable number of vacations made are controlled by putting s and u close to 0 and 1 respectively. The one dimensional marginal probability generating function is computed as

$$f(z) = \begin{pmatrix}1 & 0\end{pmatrix}\left\{\begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -\mu & \lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ z\lambda & 0\end{pmatrix}\right\}^N \begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -z\lambda & \lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix} \tag{3.6}$$

**Theorem 3.5** The one dimensional probability density function for the random variable $\eta_N$ is given by

$$p\{\eta_N = i\} = \sum_{j=1}^{\infty}\binom{N+j-2}{j-1}2\left(\frac{\mu}{\lambda+\mu}\right)^j\left(\frac{\lambda}{\lambda+\mu}\right)^i$$

**Proof**: The definition of z transform can be expressed by the following equation as given by Talpur and Shi (1994).

$$f(z) = \sum_{i=N}^{\infty} p\{\eta_N = i\}z^i$$

So by putting the value of $f(z)$ from the equation No. (3.6) we yields

$$\sum_{i=N}^{\infty} p\{\eta_N = i\}z^i = \begin{pmatrix}1 & 0\end{pmatrix}\left\{\begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -\mu & \lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ z\lambda & 0\end{pmatrix}\right\}^N \begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -z\lambda & \lambda+\mu\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

Let $a = \lambda+\mu$, and dividing by $a$ one can get

$$\begin{aligned} f(z) &= \begin{pmatrix}1 & 0\end{pmatrix}\left\{\begin{pmatrix} 1 & -1 \\ -\frac{\mu}{a} & 1\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ \frac{\lambda z}{a} & 0\end{pmatrix}\right\}^N \frac{1}{a}\begin{pmatrix} 1 & -1 \\ -\frac{z\lambda}{a} & 1\end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix} \\ &= \begin{pmatrix}1 & 0\end{pmatrix}\left\{I - \begin{pmatrix} 0 & 1 \\ \frac{\mu}{a} & 0\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ \frac{\lambda z}{a} & 0\end{pmatrix}\right\}^N \frac{1}{a}\left\{I - \begin{pmatrix} 0 & 1 \\ \frac{z\lambda}{a} & 0\end{pmatrix}^{-1}\right\}\begin{pmatrix}\mu\\ \mu\end{pmatrix} \end{aligned}$$

The rule of geometric series is applied as expressed by Saaty (1961).

$$f(z) = \begin{pmatrix}1 & 0\end{pmatrix}\left\{\sum_{k=0}^{\infty}\begin{pmatrix} 0 & 1 \\ \frac{\mu}{a} & 0\end{pmatrix}^k\begin{pmatrix}0 & 0\\ \frac{\lambda z}{a} & 0\end{pmatrix}\right\}^N \frac{1}{a}\left\{\sum_{l=0}^{\infty}\begin{pmatrix} 0 & 1 \\ \frac{z\lambda}{a} & 0\end{pmatrix}^l\right\}\begin{pmatrix}\mu\\ \mu\end{pmatrix} \tag{3.7}$$

The results of two set of series $\left\{\sum_{k=0}^{\infty}\begin{pmatrix} 0 & 1 \\ \frac{\mu}{a} & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 \\ \frac{\lambda z}{a} & 0 \end{pmatrix}\right\}^N$ and $\left\{\sum_{l=0}^{\infty}\begin{pmatrix} 0 & 1 \\ \frac{z\lambda}{a} & 0 \end{pmatrix}^l\right\}$ is placed in equation (3.7) and after some simplification we gets

$$f(z) = \frac{1}{a}\left\{\frac{\lambda z}{a}\sum_{k=0}^{\infty}\left(\frac{\mu}{a}\right)^k\right\}^N \left\{\mu\sum_{l=0}^{\infty}\left(\frac{\lambda z}{a}\right)^l + \mu\sum_{l=0}^{\infty}\left(\frac{\lambda z}{a}\right)^l\right\}$$

Using applications of the negative binomial distribution as Hogg and Craig (1995) one can obtained.

$$f(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{N+k-1}{k}2\left(\frac{\mu}{a}\right)^{k+1}\left(\frac{\lambda}{a}\right)^{l+N} z^{l+N}$$

By taking $j = k+1$ and $i = l+N$, and substituting the value of $a$ and comparing the coefficient of z we yields

$$p\{\eta_N = i\} = \sum_{j=1}^{\infty}\binom{N+j-2}{j-1}2\left(\frac{\mu}{\lambda+\mu}\right)^j\left(\frac{\lambda}{\lambda+\mu}\right)^i \quad (3.8)$$

This is looks like as a negative binomial distribution, which is the convolution of the geometric distribution as expressed by Feller (1970). The vacation of the servers will vanish at this stage, as the crossing time is attained by reaching the absorbing state.

**Theorem 3.6** The one dimensional cumulative probability distribution function for the random variable $\eta_N$ is given by

$$p\{\eta_N = i\} = \sum_{j=1}^{\infty}\sum_{i=N}^{\infty}\binom{N+j-2}{j-1}2\left(\frac{\mu}{\lambda+\mu}\right)^j\left(\frac{\lambda}{\lambda+\mu}\right)^i \quad (3.9)$$

**Proof**: The one dimensional cumulative probability distribution function for random variable $\eta_N$ is obtained by summing the one dimensional marginal probability density function for the discrete random variable $\eta_N$ the number of arriving customers from equation (3.8) the proof is obvious.

The cdf for the number of customers arrived act as negative binomial distribution, where the number of vacations considered until crossing time for having the absorbing state and the number of customers are taken after the achievement of absorption state.

### 3.3. One dimensional marginal probability distribution functions for $\xi_N$

The effect of number of vacations made by service channels represented by $\xi_N$ is studied by controlling the time taken by the number of vacations made and the number of arriving customers or units. The one dimensional marginal probability generating function (probability transform function), density function and cumulative distribution function for the random variable $\xi_N$.

**Theorem 3.7** The one dimensional marginal probability generating function (probability transform function) for the random variable $\xi_N$ is given by

$$f(u) = u\begin{pmatrix} 1 & 0 \end{pmatrix}\left\{\begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -u\mu & \lambda+\mu \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}\right\}^N \begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -\lambda & \lambda+\mu \end{pmatrix}^{-1}\begin{pmatrix} \mu \\ \mu \end{pmatrix} \quad (3.10)$$

**Proof**: The one dimensional probability generating function for the random variable $\xi_N$ is obtained from the joint probability generating function for three random variables $T_{\xi_N}$, $\xi_N$ and $\eta_N$ see Iffat (2004).

$$f^*(s,u,z) = u\begin{pmatrix}1 & 0\end{pmatrix}\left\{\begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -u\mu & S+\lambda+\mu \end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ \lambda z & 0\end{pmatrix}\right\}^N \times \begin{pmatrix} s+\lambda+\mu & -(\lambda+\mu) \\ -\lambda z & s+\lambda+\mu \end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

The effect of the random variable $T_{\xi_N}$ time taken by the number of vacations made and that of the random variable $\eta_N$ number of customers arrived are controlled by putting s and z close to 0 and 1 respectively the proof is obvious.

**Theorem 3.8** The one dimensional marginal probability density function (pdf) for the random variable $\xi_N$ is given by

$$p\{\xi_N = j\} = \sum_{i=N}^{\infty}\binom{N+j-2}{j-1}2\left(\frac{\mu}{\lambda+\mu}\right)^j\left(\frac{\lambda}{\lambda+\mu}\right)^i$$

**Proof**: The definition of z transform can be expressed by the following equation as given by Talpur and Shi (1994).

$$f(u) = \sum_{j=1}^{\infty} p\{\xi_N = j\}u^j$$

So by putting the value of f(u) from equation (3.10)

$$\begin{aligned} f(u) &= \sum_{j=1}^{\infty} p\{\xi_N = j\}u^j \\ &= u\begin{pmatrix}1 & 0\end{pmatrix}\left\{\begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -u\mu & \lambda+\mu \end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ \lambda & 0\end{pmatrix}\right\}^N \begin{pmatrix} \lambda+\mu & -(\lambda+\mu) \\ -\lambda & \lambda+\mu \end{pmatrix}^{-1}\begin{pmatrix}\mu\\ \mu\end{pmatrix} \end{aligned}$$

Taking $a = s+\lambda+\mu$, and dividing by $a$ one can get

$$f(u) = \frac{u}{a}\begin{pmatrix}1 & 0\end{pmatrix}\left\{I - \begin{pmatrix}0 & 1\\ \frac{u\mu}{a} & 0\end{pmatrix}^{-1}\begin{pmatrix}0 & 0\\ \frac{\lambda}{a} & 0\end{pmatrix}\right\}^N\left\{I - \begin{pmatrix}0 & 1\\ \frac{\lambda}{a} & 0\end{pmatrix}^{-1}\right\}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

Applying rule of the geometric series it can be expressed as

$$f(u) = \frac{u}{a}\begin{pmatrix}1 & 0\end{pmatrix}\left\{\sum_{k=0}^{\infty}\begin{pmatrix}0 & 1\\ \frac{u\mu}{a} & 0\end{pmatrix}^k\begin{pmatrix}0 & 0\\ \frac{\lambda}{a} & 0\end{pmatrix}\right\}^N\left\{\sum_{l=0}^{\infty}\begin{pmatrix}0 & 1\\ \frac{\lambda}{a} & 0\end{pmatrix}^l\right\}\begin{pmatrix}\mu\\ \mu\end{pmatrix}$$

By simplifying this and applying negative binomial distribution as done by Saaty (1961), Hogg and Craig (1995) we get

$$f(u) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{N+k-1}{k}2\left(\frac{\mu}{a}\right)^{k+1}\left(\frac{\lambda}{a}\right)^{l+N}u^{k+1}$$

By putting $j = k + 1$ and $i = l + N$ and substituting the value of $a$and comparing the coefficient of u we obtained the proof.

$$p\{\xi_N = j\} = \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda+\mu}\right)^j \left(\frac{\lambda}{\lambda+\mu}\right)^i \tag{3.11}$$

The pdf for the discrete random variable of the number of vacations performed by service channels until having the absorbing state is expressed as a negative binomial distribution.

**Theorem 3.9** The one dimensional marginal cumulative probability distribution function for the random variable $\xi_N$ is given by

$$p\{\xi_N = j\} = \sum_{j=1}^{\infty}\sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda+\mu}\right)^j \left(\frac{\lambda}{\lambda+\mu}\right)^i$$

**Proof**: The one dimensional cumulative probability distribution function for random variable $\xi_N$ is obtained by summing the one dimensional marginal probability density function for the discrete random variable $\xi_N$ the number of vacations made by service channels from equation (3.11).

$$p\{\xi_N = j\} = \sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda+\mu}\right)^j \left(\frac{\lambda}{\lambda+\mu}\right)^i$$

The one dimensional cumulative probability distribution function (cdf) for the random variable $\xi_N$.

$$p\{\xi_N = j\} = \sum_{j=1}^{\infty}\sum_{i=N}^{\infty} \binom{N+j-2}{j-1} 2 \left(\frac{\mu}{\lambda+\mu}\right)^j \left(\frac{\lambda}{\lambda+\mu}\right)^i \tag{3.12}$$

The cdf for the number of vacations customers arrived accomplishes the negative binomial distribution, where the observed number of vacations made by service channels is related to the crossing time for having the absorption state and the number of customers is taken after the achievement of absorption state.

## 4. Results and discussion

Above theorems show that the probability density functions are related to discrete random variables and could be linked to the Poisson process as shown by Medhi (1982). The crossing time shows the two stage Erlang distribution, the number of vacations related to crossing time follows the negative binomial distribution and the number of arrivals for the absorption state also satisfies the negative binomial distribution.

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