JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 1, March 2009

ON SOME MEASURE RELATED WITH POISSON INTEGRAL ON THE UNIT BALL

Gye Tak Yang* and Ki Seong Choi**

Abstract. Let μ be a finite positive Borel measure on the unit ball $B \subset \mathbb{C}^n$ and ν be the Euclidean volume measure such that $\nu(B) = 1$. For the unit sphere $S = \{z : |z| = 1\}$, σ is the rotation-invariant measure on S such that $\sigma(S) = 1$. Let $\mathcal{P}[f]$ be the invariant Poisson integral of f. We will show that there is a constant $M > 0$ such that $\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq M \int_B |\mathcal{P}[f](z)|^p d\nu(z)$ for all $f \in L^p(\sigma)$ if and only if $\| \mu \|_r = \sup_{z\in B} \frac{\mu(E(z,r))}{\nu(E(z,r))} < \infty$.

1. Introduction

Throughout this paper, $\mathbb{C}^n (n \geq 1)$ will be the Cartesian product of n copies of $\mathbb{C}$(set of complex numbers). For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in $\mathbb{C}^n$, the inner product is defined by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $|z|^2 = \langle z, z\rangle$. For $w \in \mathbb{C}^n, r > 0$, let $B(w, r) = \{z \in \mathbb{C}^n : |z-w| < r\}$. For simplicity, the unit ball $B(0, 1)$ will be denoted by B. The boundary of B is the unit sphere $S = \{z : |z| = 1\}$.

Let σ be the rotation-invariant measure on S such that $\sigma(S) = 1$. For $1 \leq p \leq \infty$, $L^p(\sigma)$ denote the Lebesgue space of S induced by σ.

For μ a finite positive Borel measure on B and g measurable, we write

$$\| g \|_\mu^p = \int_B |g(z)|^p d\mu(z).$$

Let ν be the usual Euclidean volume measure on $\mathbb{C}^n$ such that $\nu(B) = 1$. For $\mu = \nu$, we suppress the subscript($\| g \|_\nu = \| g \|$).

Suppose $1 \leq p < +\infty$. In this paper, we say that μ is a $L^p(B, d\nu)$-measure if and only if there is a constant $M > 0$ such that

Received January 19, 2009; Accepted February 13, 2009.
2000 Mathematics Subject Classification: Primary 30H05; Secondary 28B15.
Key words and phrases: Bergman metric, Poisson integral and $\mathcal{M}$-harmonic.
Correspondence should be addressed to Ki Seong Choi, ksc@konyang.ac.kr.

$$\int_B |f(z)|^p d\mu(z) \leq M \int_B |f(z)|^p d\nu(z)$$

for all $f \in L^p(B, d\nu)$. In this paper, we will investigate conditions under which μ is a $L^p(B, d\nu)$-measure.

The Kernel

$$\mathcal{P}(z, \zeta) = \left(\frac{1 - |z|^2}{|1 - \langle z, \zeta \rangle|^2} \right)^n \qquad (z \in B, \zeta \in S)$$

is called the invariant Poisson kernel in B. The invariant Poisson integral $\mathcal{P}[f]$ of a function $f \in L^1(\sigma)$ is defined, for $z \in B$, by

$$\mathcal{P}[f](z) = \int_S \mathcal{P}(z, \zeta) f(\zeta) d\sigma(\zeta).$$

In section 2, we will show that if μ is a $L^p(B, d\nu)$-measure, then there is a constant $M > 0$ such that $\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq M \int_S |f(\zeta)|^p d\sigma(\zeta)$ for all $f \in L^p(\sigma)$.

The Bergman metric $\beta(\cdot, \cdot)$ gives the usual topology on B(See [9, p52]). Moreover, the closed metric balls $E(z, r) = \{w : \beta(z, w) \leq r\}$ are compact(See [9, p56]). For any fixed $r > 0$, we define

$$\| \mu \|_r = \sup_{z \in B} \frac{\mu(E(z, r))}{\nu(E(z, r))}.$$

The Bergman Kernel is the function K defined on $B \times B$(or, if preferred, on all of $\mathbb{C}^n \times \mathbb{C}^n$ except where $\langle z, w \rangle = 1$) by

$$K(z, w) = (1 - \langle z, w \rangle)^{-n-1}.$$

Let $L^p_a(B, d\nu)$ be the subspace of $L^p(B, d\nu)$ consisting of analytic functions. It is easily shown that every $f \in L^1_a(B, d\nu)$ satisfies

$$f(z) = \int_B \frac{f(w)}{(1 - \langle z, w \rangle)^{n+1}} d\nu(w)$$

for every $z \in B$(See [8, Theorem 2]).

The normalized(in $L^2_a(B, d\nu)$) reproducing kernel is denoted by $k_z(\cdot) = K(z, z)^{-1/2} K(\cdot, z)$. For $r > 0$, there are constants $M(r)$ and $n(r)$ so that

$$\infty > M(r) \geq |k_z(w)|^2 \nu(E(z, r)) \geq n(r) > 0$$

for all $z, w \in B$ with $\beta(z, w) \leq r$(See [1, Lemma 8]).

Using this result, in section 2, we will show that if there is a constant $M > 0$ such that

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq M \int_B |\mathcal{P}[f](z)|^p d\nu(z)$$

for all $f \in L^p(\sigma)(1 \leq p < \infty)$, then

$$\| \mu \|_r = \sup_{z\in B} \frac{\mu(E(z,r))}{\nu(E(z,r))} < \infty.$$

Let $a \in B$ and let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by a, which is given by $P_0 = 0$, and

$$P_a z = \frac{< z, a >}{< a, a >} a, \quad if \quad a \neq 0.$$

Let $Q_a = I - P_a$. Define φ_a on B by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - < z, a >}.$$

It is easily shown that the mapping φ_a belongs to $Aut(B)$ where $Aut(B)$ is the group of all biholomorphic mappings of B onto itself(See [12, Theorem 2.2.2]).

An upper semi-continuous function $f : B \to [-\infty, \infty)$, with $f \neq 0$, is $\mathcal{M}$-subharmonic on B if

$$f(a) \leq \int_S f(\varphi_a(rt)) d\sigma(t)$$

for all $a \in B$ and all r sufficiently small. A continuous function f for which equality holds in above formula is said to $\mathcal{M}$-harmonic on B. It is well known that $\mathcal{P}[f]$ is $\mathcal{M}$-harmonic on B for $f \in L^p(\sigma), 1 \leq p \leq \infty$(See [13, 5.1]). In section 3, we will show that if $\| \mu \|_r < \infty$, then

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq M \int_B |\mathcal{P}[f](z)|^p d\nu(z)$$

for every $f \in L^p(\sigma)$.

2. $L^p(B, d\nu)$-measure

Let $A(B)$ be the class of all $f : B \to \mathbb{C}$ that are continuous on the closed ball $\overline{B}$ and that are holomorphic in its interior B. Equipped with the supremum norm $\| f \|_\infty$, $A(B)$ is a Banach Algebra.

Theorem 2.1. *If $f \in A(B)$, then $f(z) = \mathcal{P}[f](z)$ for all $z \in B$.*

Proof. See [13, Corollary 5.2]. □

THEOREM 2.2. *If* $0 \leq r < 1$, $\zeta \in S$ *and* $\eta \in S$, *then*

$$\mathcal{P}(r\eta, \zeta) = \mathcal{P}(r\zeta, \eta).$$

Also,

$$\int_S \mathcal{P}(r\eta, \zeta) d\sigma(\zeta) = 1 = \int_S \mathcal{P}(r\zeta, \eta) d\sigma(\eta).$$

Proof. See [13, Lemma 5.3]. □

THEOREM 2.3. *The measures* ν *and* σ *are related by the formula*

$$\int_{C^n} f d\nu = 2n \int_0^\infty r^{2n-1} \int_S f(r\zeta) d\sigma(\zeta) dr.$$

In particular,

$$\int_B f d\nu = 2n \int_0^1 r^{2n-1} \int_S f(r\zeta) d\sigma(\zeta) dr.$$

Proof. See [12, Proposition 1.4.3]. □

LEMMA 2.4. *If* μ *is a* $L^p(B, d\nu)$*-measure where* $1 \leq p < \infty$, *then there is a constant* $M > 0$ *such that*

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq M \int_S |f(\zeta)|^p d\sigma(\zeta)$$

for all $f \in L^p(\sigma)$.

Proof. By Hölder inequality and Theorem 2.2,

$$\begin{aligned} |\mathcal{P}[f](z)|^p &= |\int_S f(\zeta)\mathcal{P}(z, \zeta) d\sigma(\zeta)|^p \\ &\leq \int_S |f(\zeta)|^p \mathcal{P}(z, \zeta) d\sigma(\zeta) \int_S \mathcal{P}(z, \zeta) d\sigma(\zeta) \\ &= \int_S |f(\zeta)|^p \mathcal{P}(z, \zeta) d\sigma(\zeta). \end{aligned}$$

By Theorem 2.3,

$$\begin{aligned} &\int_B |\mathcal{P}[f](z)|^p d\nu(z) \\ &= 2n \int_0^1 r^{2n-1} \int_S |\mathcal{P}[f](r\eta)|^p d\sigma(\eta) dr. \end{aligned}$$

Above two results imply that

$$\begin{aligned}
&\int_B |\mathcal{P}[f](z)|^p d\nu(z) \\
&\le 2n \int_0^1 r^{2n-1} \int_S \left[\int_S |f(\zeta)|^p \mathcal{P}(r\eta, \zeta) d\sigma(\zeta) \right] d\sigma(\eta) dr \\
&= 2n \int_0^1 r^{2n-1} \int_S |f(\zeta)|^p \int_S \mathcal{P}(r\eta, \zeta) d\sigma(\eta) d\sigma(\zeta) dr \\
&= 2n \int_0^1 r^{2n-1} \int_S |f(\zeta)|^p d\sigma(\zeta) dr \\
&= \int_S |f(\zeta)|^p d\sigma(\zeta)
\end{aligned}$$

where the third equality follows from Theorem 2.2.

Since μ is a $L^p(B, d\nu)$-measure, there exists a constant M such that

$$\begin{aligned}
&\int_B |\mathcal{P}[f](z)|^p d\mu(z) \\
&\le M \int_B |\mathcal{P}[f](z)|^p d\nu(z) \\
&\le M \int_S |f(\zeta)|^p d\sigma(\zeta).
\end{aligned}$$

□

THEOREM 2.5. *(a) $f \in C(S)$ and F is defined on $\overline{B}$ so that $F = f$ on S and $F = \mathcal{P}[f]$ in B, then $F \in C(\overline{B})$ and $\| F \|_\infty = \| f \|_\infty$.*

(b) If $1 \le p \le \infty$, $f \in L^p(\sigma)$, $F = \mathcal{P}[f]$ and

$$F_r(\zeta) = F(r\zeta) \ (0 \le r < 1, \zeta \in S),$$

then $\| F_r \|_p \le \| f \|_p$. If also $1 \le p < \infty$, then

$$\lim_{r \to 1} \| F_r - f \|_p = 0.$$

Proof. See [13, Theorem 5.5 and Proposition 5.6]. □

LEMMA 2.6. *For $r > 0$, there are constants $M(r)$ and $n(r)$ so that*

$$\infty > M(r) \ge |k_z(w)|^2 \nu(E(z, r)) \ge n(r) > 0$$

for all $z, w \in B$ with $\beta(z, w) \le r$.

Proof. See [1, Lemma 8]. □

THEOREM 2.7. *If there is a constant $M > 0$ such that*

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \leq M \int_B |\mathcal{P}[f](z)|^p d\nu(z)$$

for all $f \in L^p(\sigma)(1 \leq p < \infty)$, then

$$\| \mu \|_r = \sup_{z \in B} \frac{\mu(E(z,r))}{\nu(E(z,r))} < \infty.$$

Proof. For the following function f such that

$$f(\zeta) = \left(\frac{(1-|w|^2)^{n+1}}{(1-\langle \zeta, w \rangle)^{2(n+1)}} \right)^{1/p},$$

$$\begin{aligned} \mathcal{P}[f](z) &= \int_S \mathcal{P}(z,\zeta) \left(\frac{(1-|w|^2)^{n+1}}{(1-\langle \zeta, w \rangle)^{2(n+1)}} \right)^{1/p} d\sigma(\zeta) \\ &= \left(\frac{(1-|w|^2)^{n+1}}{(1-\langle z, w \rangle)^{2(n+1)}} \right)^{1/p} \end{aligned}$$

by Theorem 2.1. This implies that

$$\begin{aligned} &\int_B \frac{(1-|w|^2)^{n+1}}{|1-\langle z, w \rangle|^{2(n+1)}} d\mu(z) \\ &= \int_B |\mathcal{P}[f](z)|^p d\mu(z) \\ &\leq M \int_B |\mathcal{P}[f](z)|^p d\nu(z) \\ &= M \int_B \frac{(1-|w|^2)^{n+1}}{|1-\langle z, w \rangle|^{2(n+1)}} d\nu(z). \end{aligned}$$

Since $k_w(z) = \frac{(1-|w|^2)^{\frac{n+1}{2}}}{(1-\langle z,w \rangle)^{(n+1)}}$ is unit vector in $L^2_a(B, d\nu)$,

$$\int_B |k_w(z)|^2 d\mu(z) = \int_B \frac{(1-|w|^2)^{n+1}}{|1-\langle z, w \rangle|^{2(n+1)}} d\mu(z) \leq M.$$

By Lemma 2.6,

$$\begin{aligned} n(r) \frac{\mu(E(w,r))}{\nu(E(w,r))} &\leq \int_{E(w,r)} \frac{n(r)}{\nu(E(w,r))} d\mu(z) \\ &\leq \int_{E(w,r)} |k_w(z)|^2 d\mu(z). \end{aligned}$$

This implies that

$$\| \mu \|_r = \sup_{w \in B} \frac{\mu(E(w,r))}{\nu(E(w,r))} < \frac{1}{n(r)} M.$$

□

3. Notes on measure related with Poisson Integral

THEOREM 3.1. *Let ψ be a biholomorphic mapping of B onto itself and $a = \psi^{-1}(0)$. The determinant $J_R\psi$ of the real Jacobian matrix of ψ satisfies the following identity:*

$$J_R\psi(z) = |J\psi(z)|^2 = \left(\frac{1 - |a|^2}{|1 - < z, a > |^2} \right)^{n+1} = \left(\frac{1 - |\psi(z)|^2}{1 - |z|^2} \right)^{n+1}.$$

Proof. See [10, Theorem 2.2.6]. □

THEOREM 3.2. *For a, b in Ω with $\beta(a,b) \le R$ and $r, s > 0$, we have*

$$0 < m(R,r,s) \le \frac{\nu(E(a,r))}{\nu(E(b,s))} \le M(R,r,s) < \infty.$$

Proof. See [1, Lemma 6]. □

THEOREM 3.3. *For fixed $r > 0$, there is a sequence $\{w_j\}$ in B such that*

(1) $\cup_{j=1}^{\infty} E(w_j, r) = B$,

(2) there is a positive integer N_0 such that, for any z in B, z is contained in at most N_0 of the sets $E(w_k, 2r)$.

For the above sequence $\{w_j\}$ and any positive Borel measure m, we have

$$\sum_{k=1}^{\infty} m(E(w_k, 2r)) \le N_0 m(B).$$

Proof. See [2, Lemma 5 and Lemma 6]. □

THEOREM 3.4. *If $f \in L^p(\sigma), 1 \le p \le \infty$,*

$$\mathcal{P}[f] \circ \psi = \mathcal{P}[f \circ \psi]$$

for every $\psi \in Aut(B)$.

Proof. See [12, Theorem 3.3.8]. □

THEOREM 3.5. *For $f \in L^p(\sigma), 1 \le p \le \infty$, $\mathcal{P}[f]$ is $\mathcal{M}$-harmonic on B and*

$$\sup_{0<r<1} \int_S |\mathcal{P}[f](r\zeta)|^p d\sigma(\zeta) \le \int_S |f(\zeta)|^p d\sigma(\zeta).$$

Proof. See [13, 5.1]. □

Recall that $\varphi_a(z) = \frac{a - P_a z - \sqrt{1-|a|^2} Q_a z}{1 - <z,a>}$. It is easily shown that

$$\varphi_a(0) = a, \ \varphi_a(a) = 0 \quad and \quad \varphi_a(\varphi_a(z)) = z.$$

Furthermore, for all $z, w \in \overline{B}$, we have

$$1 - < \varphi_a(z), \varphi_a(w) > = \frac{(1-|a|^2)(1 - < z, w >)}{(1 - < z, a >)(1 - < a, w >)}.$$

In particular, for $a \in B, z \in \overline{B}$,

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1 - < z, a > |^2}$$

(See [12, Theorem 2.2.2]).

THEOREM 3.6. *For $f \in L^p(\sigma), 1 \le p \le \infty$,*

$$|\mathcal{P}[f](w)|^p \le \frac{C_r}{\nu(E(z,r))} \int_{E(w,r)} |\mathcal{P}[f](z)|^p d\nu(z).$$

Proof. Since $\mathcal{P}[f]$ is $\mathcal{M}$-harmonic by Theorem 3.5,

$$\mathcal{P}[f](a) = \int_S \mathcal{P}[f](\varphi_a(rt)) d\sigma(t).$$

By Hölder inequality,

$$|\mathcal{P}[f](a)|^p \le \int_S |\mathcal{P}[f](\varphi_a(rt))|^p d\sigma(t).$$

Since

$$|\mathcal{P}[f](\varphi_a(a))|^p \le \int_S |\mathcal{P}[f](\varphi_a(\varphi_a(rt)))|^p d\sigma(t)$$

by Theorem 3.4,

$$|\mathcal{P}[f](0)|^p \le \int_S |\mathcal{P}[f](rt)|^p d\sigma(t).$$

By Theorem 2.3,

$$2n\int_0^{\alpha} r^{2n-1}|\mathcal{P}[f](0)|^p dr \le 2n\int_0^{\alpha} r^{2n-1}\int_S |\mathcal{P}[f](rt)|^p d\sigma(t)dr$$
$$= \int_{B(0,\alpha)} |\mathcal{P}[f](z)|^p d\nu(z).$$

Since the Bergman metric induces the usual Euclidean topology on B, $E(0,r)$ contains Euclidean ball $B(0,\alpha) = \{z \in \mathbb{C}^n : |z| < \alpha\}$ for some r,

$$|\mathcal{P}[f](0)|^p \le \frac{1}{\alpha^{2n}}\int_{E(0,r)} |\mathcal{P}[f](z)|^p d\nu(z).$$

Replacing f by $f \circ \varphi_w$, we have

$$|\mathcal{P}[f \circ \varphi_w](0)|^p \le \frac{1}{\alpha^{2n}}\int_{E(0,r)} |\mathcal{P}[f \circ \varphi_w](z)|^p d\nu(z).$$

By Theorem 3.4,

$$|\mathcal{P}[f](w)|^p \le \frac{1}{\alpha^{2n}}\int_{E(0,r)} |\mathcal{P}[f] \circ \varphi_w(z)|^p d\nu(z).$$

This implies that

$$\begin{aligned}|\mathcal{P}[f](w)|^p &\le \frac{1}{\alpha^{2n}}\int_{E(0,r)} |\mathcal{P}[f] \circ \varphi_w(z)|^p d\nu(z)\\ &\le \frac{1}{\alpha^{2n}}\int_{E(w,r)} |\mathcal{P}[f](z)|^p |k_w(z)|^2 d\nu(z)\\ &\le \frac{M(r)}{\alpha^{2n}\nu(E(w,r))}\int_{E(w,r)} |\mathcal{P}[f](z)|^p d\nu(z)\end{aligned}$$

by Theorem 3.1 and Lemma 2.6. □

THEOREM 3.7. *If* $\| \mu \|_r < \infty$, *then*

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \le M\int_B |\mathcal{P}[f](z)|^p d\nu(z)$$

where $1 \le p \le \infty$.

Proof. By Theorem 3.3,

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \le \sum_{n=1}^{\infty} \int_{E(w_n,r)} |\mathcal{P}[f](z)|^p d\mu(z)$$
$$\le \sum_{n=1}^{\infty} \mu(E(w_n,r)) \sup\{|\mathcal{P}[f](z)|^p : z \in E(w_n,r)\}.$$

By Theorem 3.6,

$$|\mathcal{P}[f](z)|^p \le \frac{C_r}{\nu(E(z,r))} \int_{E(z,r)} |\mathcal{P}[f](w)|^p d\nu(w)$$
$$\le \frac{C_r}{\nu(E(z,r))} \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)$$

for $z \in E(w_n,r)$. By Theorem 3.2,

$$\frac{1}{\nu(E(z,r))} \le \frac{M}{\nu(E(w_n,r))}$$

for some constant M. This implies that

$$\sup\{|\mathcal{P}[f](z)|^p : z \in E(w_n,r)\} \le \frac{C_r M}{\nu(E(w_n,r))} \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w).$$

Since

$$\| \mu \|_r = \sup_{w\in B} \frac{\mu(E(w,r))}{\nu(E(w,r))} < \infty,$$

$$\int_B |\mathcal{P}[f](z)|^p d\mu(z) \le C_r M \sum_{n=1}^{\infty} \frac{\mu(E(w_n,r))}{\nu(E(w_n,r))} \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)$$
$$\le C_r M \| \mu \|_r \sum_{n=1}^{\infty} \int_{E(w_n,2r)} |\mathcal{P}[f](w)|^p d\nu(w)$$
$$\le C_r M \| \mu \|_r N_0 \int_B |\mathcal{P}[f](w)|^p d\nu(w)$$

by Theorem 3.3. □

References

[1] C.A. Berger, L.A. Coburn and K.H. Zhu, *Function theory on Cartan domains and the Berezin-Toeplitz symbols calculus*, Amer. J. Math. **110** (1988), 921-953.

[2] D. Bekolle, C.A. Berger, L.A. Coburn and K.H. Zhu, *BMO in the Bergman metric on bounded symmetric domain*, J. Funct. Anal. **93** (1990), 310-350.

[3] K. S. Choi, *Lipschitz type inequality in Weighted Bloch spaces* $\mathfrak{B}_q$, J. Korean Math. Soc. **39** (2002), no.2, 277-287.

[4] K.S. Choi, *little Hankel operators on Weighted Bloch spaces*, Commun. Korean Math. Soc. **18** (2003), no. 3, 469-479.

[5] K. S. Choi, *Notes On the Bergman Projection type operators in* C^n, Commun. Korean Math. Soc. **21** (2006), no. 1, 65-74.

[6] K. S. Choi, *Notes on Carleson Measures on bounded symmetric domain*, Commun. Korean Math. Soc. **22** (2007), no.1, 65-74.

[7] P. L. Druen, *Theory of* H^p *spaces*, Academic Press, New York, 1970.

[8] K. T. Hahn and K. S. Choi, *Weighted Bloch spaces in* $\mathbb{C}^n$, J. Korean Math. Soc. **35** (1998), no.2, 171-189.

[9] L. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York/London, 1978.

[10] S. Krantz, *Function theory of several complex variables, 2nd ed.*, Wadsworth & Brooks/Cole Math. Series, Pacific Grove, CA.

[11] D.H. Luecking, *A Technique for characterizing Carleson measures on Bergman spaces*, Proc. Amer. Math. Soc. **87** (1983), 656-660.

[12] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer Verlag, New York, 1980.

[13] M. Stoll, *Invariant potential theory in the unit ball of* $\mathbb{C}^n$, London mathematical Society Lecture note series 199, 1994.

*

Department of Information Security
Konyang University
Nonsan 320-711, Republic of Korea
E-mail: gtyang@konyang.ac.kr

**

Department of Information Security
Konyang University
Nonsan 320-711, Republic of Korea
E-mail: ksc@konyang.ac.kr