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$h-$STABILITY IN CERTAIN INTEGRO-DIFFERENTIAL EQUATIONS

Yoon Hoe Goo*, Myeong Hee Ji**, and Dae Hee Ry***

Abstract. In this paper, we investigate $h-$stability for the nonlinear Volterra integro-differential equations and the functional integro-differential equations.

1. Introduction

Pinto [7, 8] introduced $h-$stability which is an important extension of the notion of exponential asymptotic stability. In [7] he studied some properties about asymptotic behavior of solutions of perturbed $h-$systems. Also, he obtained a general variational stability introduced for nonautonomous systems in [6, 7]. Choi and Ryu [3] dealt with $h-$stability of the solutions of ordinary differential equations and Volterra integro-differential equations. In this paper, we investigate $h-$stability of certain integro-differential equations. The paper is organized as follows. In Section 2, we review a definition and present some related properties needed for our purposes. Finally, in Section 3, we investigate $h-$stability for the nonlinear Volterra integro-differential equations and the functional integro-differential equations.

2. Preliminaries

Let $\mathbb{R}^n$ denote the Euclidean n-space. For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x|\leq 1} |Ax|$. Let $\mathbb{R}^+$ be the half line $0 \leq t < \infty$.

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Correspondence should be addressed to Yoon Hoe Goo, yhgoo@hanseo.ac.kr.

We consider the nonlinear integro-differential equation of Volterra type

$$(2.1)\ x(t) = f(t, x(t)) + \int_{t_0}^{t} g(t, s, x(s))ds, x(t_0) = x_0, \quad t_0 \geq 0,$$

where $f \in C[J \times \mathbb{R}^n, \mathbb{R}^n]$, $g \in C[J \times J \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t, 0) = g(t, s, 0)$ and $J = [t_{0,}\infty)$. Let f and g be continuously differentiable with respect to the components of x on $J \times \mathbb{R}^n$ and $J \times J \times \mathbb{R}^n$, respectively. Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on J. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$(2.2) \qquad y'(t) = f_x(t, 0)y(t) + \int_{t_0}^{t} g_x(t, s, 0)y(s)ds$$

and

$$(2.3)\ \ z'(t) = f_x(t, x(t, t_0, x_0))z(t) + \int_{t_0}^{t} g_x(t, s, x(t, t_0, x_0))z(s)ds$$

with $z(t_0) = z_0$. The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.2). Note that $\Phi(t, t_0, 0) = R(t, t_0)$, where $R(t, s)$ is the resolvent matrix solution which satisfies the initial value problem

$$(2.4)\ \ \frac{\partial}{\partial s} R(t, s) + R(t, s)f_x(s, 0) + \int_{s}^{t} R(t, u)g_x(u, s, 0)du = 0$$

with $R(t, t) = I$, the identity matrix for $0 \leq s \leq t < \infty$. We now give the main definition and some related properties that we need in the sequel.

Definition 2.1. The zero solution of (2.1) is called $h-$stable (hS) if there exist $c \geq 1, \delta > 0$, and a positive bounded continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c|x_0|h(t)h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$.

Remark 2.2. If $h(t) = e^{-t}$, then $h-$stability coincides with exponential stability, and if $h(t)$ is constant, then we have uniform Lipschitz stability.

THEOREM 2.3. [6] *Let $f, g \in C[\mathbb{R}^+ \times D, \mathbb{R}^n]$ in (2.1) satisfy the assumptions listed above. Then, if $x(t, t_0, x_0), x(t, t_0, y_0)$ are any two solutions of (2.1) existing on $[t_0, \infty)$, we have*

$$x(t, t_0, x_0) - x(t, t_0, y_0) = \left[\int_0^1 \frac{\partial x}{\partial x_0}(t, t_0, sx_0 + (1-s)y_0)ds\right](x_0 - y_0).$$

THEOREM 2.4. [9] *Assume that there exists an $n \times n$ continuous matrix function $L(t, s)$ on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $L_s(t, s)$ exists, is continuous, and satisfies*

$$B(t, s) + L_s(t, s) + L(t, s)A(s) + \int_s^t L(t, \sigma)B(\sigma, s)d\sigma = 0,$$

where $A(t)$ and $B(t, s)$ are $n \times n$ matrices continuous on $\mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R}^+$, respectively. Consider the linear system

$$x'(t) = A(t)x(t) + \int_{t_0}^t B(t, s)x(s)ds + f(t), x(t_0) = x_0. \tag{2.5}$$

Then (2.5) is equivalent to

$$v'(t) = C(t)v(t) + L(t, t_0)x_0 + H(t), v(t_0) = x_0,$$

where $C(t) = A(t) - L(t, t)$ and $H(t) = f(t) + \int_{t_0}^t L(t, s)f(s)ds$.

LEMMA 2.5. [3] *The zero solution of (2.3) is hS if and only if there exist $c \geq 1, \delta > 0$ and a positive bounded continuous function h defined on $\mathbb{R}^+$ such that*

$$|\Phi(t, t_0, x_0)| \leq ch(t)h(t_0)^{-1} \tag{2.6}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$, where $\Phi(t, t_0, x_0)$ is a fundamental matrix of (2.3).

LEMMA 2.6. [3] *The zero solution of (2.2) is hS if and only if there exist $c \geq 1, \delta > 0$ and a positive bounded continuous function h defined on $\mathbb{R}^+$ such that*

$$|R(t, t_0)| \leq ch(t)h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0| < \delta$, where $R(t, t_0)$ is a solution of (2.4).

THEOREM 2.7. [3] *If the zero solution of (2.1) is hS, then the zero solution of (2.2) is also hS.*

Theorem 2.8. [3] *Assume that the zero solution of (2.2) is hS. Then the zero solution of (2.3) is also hS under the conditions that*

$$(i) \quad |f_x(t,x) - f_x(t,0)| \le a(t),$$

where $a \in C(R^+)$,

$$(ii) \quad |g_x(t,s,x) - g_x(t,s,0)| \le b(t,s),$$

where $b \in C(R^+ \times R^+)$,

$$(iii) \quad \lambda(t,s) = h(s)\int_s^t h(\tau)^{-1} b(\tau,s)d\tau$$

with $\sup_{t_0 \le s \le t}\{a(s) + \lambda(t,s)\} \le K$, *where* $K \ge 1$ *is a constant.*

Lemma 2.9. [2] *Let* $u(t), a(t), b(t)$ *and* $q(t)$ *be real valued nonnegative continuous functions defined on* $\mathbb{R}^+$, *for which the inequality*

$$u(t) \le d + \int_{t_0}^t a(s)u(s)ds + \int_{t_0}^t b(s)\Big(\int_{t_0}^s q(\tau)u(\tau)d\tau\Big)ds, \quad t \ge t_0 \ge 0,$$

holds, where d *is a nonnegative constant. Then*

$$u(t) \le d\ \exp\int_{t_0}^t \Big(a(s) + b(s)\int_{t_0}^s q(\tau)d\tau\Big)ds$$

for $t_0 \le t \le L$, *where* $W(u) = \int_{u_0}^u \frac{dz}{z},\ u > 0, u_0 > 0$, *and* $W^{-1}(u)$ *is the inverse of* $W(u)$ *and*

$$L = \sup\{t \ge t_0 : W(d) + \int_{t_0}^t \Big(a(s) + b(s)\int_{t_0}^s q(\tau)d\tau\Big)ds \in Dom W^{-1}\}.$$

3. Main results

In this section, we investigate $h-$stability for the nonlinear Volterra integro-differential equations and the functional integro-differential equations. The following theorem shows that *hS* of (2.3) implies *hS* of (2.1).

Theorem 3.1. *If the zero solution of (2.3) is hS, then the zero solution of (2.1) is hS.*

Proof. Let $x(t) = x(t,t_0,x_0)$ be any solution of (2.1). Then by Theorem 2.3, we have

$$x(t,t_0,x_0) = [\int_0^1 \Phi(t,t_0,sx_0)ds]x_0.$$

Since the zero solution of (2.3) is hS, by Lemma 2.5, there exist $c \geq 1, \delta > 0$ and a positive bounded continuous function h on $\mathbb{R}^+$ such that

$$|\Phi(t, t_0, x_0)| \leq ch(t)h(t)^{-1}$$

for $|x_0| < \delta$, where $\Phi(t, t_0, x_0)$ is a fundamental matrix of (2.3). From (2.6) we have

$$\begin{aligned} |x(t, t_0, x_0)| &\leq \int_0^1 |\Phi(t, t_0, sx_0)| ds |x_0| \\ &\leq c|x_0| h(t) h(t_0)^{-1}. \end{aligned}$$

This implies that the zero solution of (2.1) is hS and so the proof is complete. □

By Theorems 2.7 and 3.1, we obtain the following corollary.

Corollary 3.2. *If the zero solution of (2.3) is hS, then the zero solution of (2.2) is also hS.*

Under the assumptions of Theorems 2.4 and 2.8, the following theorem shows that the zero solution of (2.1) is hS.

Theorem 3.3. *Suppose that the assumption of Theorem 2.4 and all the assumptions of Theorem 2.8 hold. Then the hS property of the function*

$$p(t, t_0, |x_0|) = |\Psi(t, t_0)||x_0| + \int_{t_0}^t |\Psi(t, s)||L(s, t_0)||x_0| ds \tag{3.1}$$

where $\Psi(t, s)$ is the fundamental matrix solution of $v' = C(t)v$, implies that the zero solution of (2.1) is hS.

Proof. Set $A(t) = f_x(t, 0),\ B(t, s) = g_x(t, s, 0)$ and $f(t) \equiv 0$, and we consider the equation (2.2). We see, in view of Theorem 2.4, that it is enough to investigate the equivalent equation

$$v'(t) = C(t)v(t) + L(t, t_0)x_0,\ v(t_0) = x_0 \tag{3.2}$$

with $H(t) = 0$. By the variation of parameters formula, we have

$$v(t, t_0, x_0) = \Psi(t, t_0)x_0 + \int_{t_0}^t \Psi(t, s)L(s, t_0)x_0 ds.$$

Then it follows from (3.1) that

$$|v(t)| \leq p(t, t_0, |x_0|),\ t \geq t_0 \geq 0.$$

Therefore, from the assumption of (3.1) , we obtain

$$|v(t,t_0,x_0)| \le c|x_0|h(t)h^{-1}(t_0),\ t \ge t_0 \ge 0,$$

for some $c \ge 1$ and $|x_0| < \delta$. Hence the zero solution of (3.2) is hS and so is that of (2.2). This, with the assertion of Theorem 2.7, implies that the zero solution of (2.3) is hS. Hence, by Theorem 3.1, the zero solution of (2.1) is hS and so the proof is complete. □

Also, consider the linear integro-differential equation

$$x' = A(t)x + \int_{t_0}^{t} K(t,s)x(s)ds,\ x(t_0) = x_0 \tag{3.3}$$

and a perturbation of (3.3),

$$y' = A(t)y + \int_{t_0}^{t} K(t,s)y(s)ds + g(t,y,Ty),\ y(t_0) = y_0, \tag{3.4}$$

where $A(t), K(t,s)$ are continuous $n \times n$ matrices on $\mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R}^+$, respectively and $f \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n], g \in C[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ and $T : C[\mathbb{R}^+, \mathbb{R}^n] \to C[\mathbb{R}^+, \mathbb{R}^n]$ is a continuous operator. Then the unique solution $y(t)$ of (3.4) through $(t_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ satisfies the integral equation

$$y(t) = R(t,t_0)y_0 + \int_{t_0}^{t} R(t,s)f(s,y(s))ds,\ t \ge t_0 \ge 0,$$

where $R(t,s)$ is the solution of the initial value problem

$$\frac{\partial}{\partial s}R(t,s) + R(t,s)A(s) + \int_{s}^{t} R(t,u)K(u,s)du = 0$$

with $R(t,t) = I$ for $0 \le s < t < \infty$ (see Grossman and Miller [5]). In addition, the unique solution of (3.3) is $x(t) = R(t,t_0)x_0$.

Next, we investigate the property of hS for the functional integro-differential equation (3.4).

Theorem 3.4. *For the perturbed system (3.4), we assume that*

$$|g(t,y,Ty)| \le a(t)|y(t)| + b(t)\int_{t_0}^{t} r(s)|y(s)|ds,$$

where $a, b, c \in C(\mathbb{R}^+)$,

$$M = \int_{t_0}^{t} [a(s) + b(s)\int_{t_0}^{s} r(\tau)d\tau]ds < \infty. \tag{3.5}$$

Then the zero solution of (3.4) is hS whenever the zero solution of (3.3) is hS with increasing function h.

Proof. Let $y(t) = y(t, t_0, y_0)$ be any solution of (3.4) with an initial value $(t_0, y_0), t_0 \geq 0$. By the nonlinear variation of constants formula, we obtain

$$y(t) = R(t, t_0)y_0 + \int_{t_0}^{t} R(t, s)g(s, y(s), Ty(s))ds, \quad t \geq t_0 \geq 0,$$

where $R(t, s)$ is the solution of the initial value problem

$$\frac{\partial}{\partial s}R(t, s) + R(t, s)A(s) + \int_{s}^{t} R(t, u)K(u, s)du = 0$$

with $R(t, t) = I$ for $0 \leq s < t < \infty$. Since the solution $x = 0$ of (3.3) is *hS,* it follows from Lemma 2.6 that

$$\begin{aligned} |y(t)| &= |R(t, t_0)|\,|y_0| + \int_{t_0}^{t} |R(t, s)||g(s, y(s), Ty(s))|ds, \\ &\leq ch(t)h(t_0)^{-1}|y_0| \\ &+ \int_{t_0}^{t} ch(t)h(s)^{-1}\Big(a(s)|y(s)| + b(s)\int_{t_0}^{s} r(\tau)|y(\tau)|d\tau\Big)ds. \end{aligned}$$

Set $u(t) = h(t)^{-1}|y(t)|$. Then, by Lemma 2.9, we obtain

$$|y(t)| \leq c|y_0|h(t)h(t_0)^{-1} \exp\ c\int_{t_0}^{t}\Big[a(s) + b(s)\int_{t_0}^{s} r(\tau)d\tau\Big]ds.$$

In view of (3.5), we have

$$|y(t)| \leq M_1|y_0|h(t)h(t_0)^{-1},\ M_1 = ce^{cM} \geq 1.$$

This completes the proof. □

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Department of Mathematics
Hanseo University
Seosan, Chungnam, 356-706, Republic of Korea
E-mail: yhgoo@hanseo.ac.kr

**
Department of Mathematics
Hanseo University
Seosan, Chungnam, 356-706, Republic of Korea
E-mail: 20060139@hanseo.ac.kr

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Department of Computer Science
Chungwoon University
Hongseong, Chungnam,351-701, Republic of Korea
E-mail: rdh@chungwoon.ac.kr