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TOPOLOGICAL CLASSIFICATION OF ω-LIMIT SETS OF HOLOMORPHIC FLOWS ON $\mathbb{C}^1$

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Abstract. This paper aims to study local and global structure of holomorphic flows on $\mathbb{C}^1$. At a singular point of a holomorphic flow, we locally sector the flow into parabolic or elliptic types. By the local structure of holomorphic flows, we classify all the possible types of topologies of ω-limit sets.

1. Introduction

Attractors and ω-limit sets, arising from their ubiquitous applications in Dynamical Systems, have played one of essential roles in the field with the useful properties. Specifically, these are used to describe the time behavior for dynamical systems, and to provide the dynamicists with certain notions for localizing the complexity. The aim of this paper is to understand the topological characterization of ω-limit sets ([2][1][3][10]).

For a manifold M with a vector field on it, and a point $q \in M$, we denote the integral curve (with the initial point q) by $Z_q(t)$. Let an open interval (a_q, b_q) (possibly with $a_q = -\infty$ or $b_q = \infty$) be the maximal domain on which $Z_q(t)$ is defined. Let us define the *ω-limit set* of q by

$$\omega(q) = \{x \in M : x = \lim_{n\to\infty} Z_q(t_n) \text{ for some sequence } t_n \to b_q \text{ as } n \to \infty\}.$$

In the paper, we restrict our interest on $\mathbb{C}^1$ and the holomorphic flows on it. Let $f : \mathbb{C}^1 \to \mathbb{C}^1$ be a holomorphic map. Then f defines a holomorphic vector field on $\mathbb{C}^1$, i.e., $z'(t) = f(z(t))$ gives the associated holomorphic flow. Define $\Phi : D \to \mathbb{R}^2$ by $\Phi(t, u) = Z_u(t)$, where

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$D \subset (-\infty, \infty) \times \mathbb{R}^2$ is the maximal open domain on which the ODE has solution. If one identifies $\mathbb{C}^1$ with $\mathbb{R}^2$, then the holomorphic vector field induced by f is a special case of real-analytic flow on $\mathbb{R}^2$ by imposing the Cauchy-Riemann equation condition. As we recall the results of Jiménez López-Llibre in [7] which classifies the ω-limit sets of real-analytic flows on $\mathbb{R}^2$ up to C^0-topology, the topological characterization of the holomorphic flow on $\mathbb{C}^1$ belongs to the list therein (Theorems A and B in [7]). See Theorems 2.1 and 2.3 in §2.

We now state the main theorems. The first theorem is about a local structure of a holomorphic flow at a singular point (= a point where f vanishes), based on which we are able to draw out every possibilities of ω-limit sets from Theorems 2.1 and 2.3.

THEOREM 1.1. *Let $f : \mathbb{C}^1 \to \mathbb{C}^1$ be a non-constant holomorphic function, and Φ be the local flow of $z' = f(z)$. Let u be a singular point of Φ, i.e. $f(u) = 0$. Then, Φ near u satisfies one of the following.*

(a) *Φ has exactly one sector of parabolic type;*
(b) *Φ has only $2(k-1)$ sectors of elliptic type for some $k > 1$;*
(c) *for every $u' \neq u$ near u, the orbit of u' is periodic.*

Further, we suppose u lies on a 1-dimensional ω-limit set. Then, Φ near u satisfies the first two of the above list as follows:

(a') *Φ has exactly one sector of attracting parabolic type;*
(b) *Φ has only $2(k-1)$ sectors of elliptic type for some $k > 1$.*

THEOREM 1.2. *Let $f : \mathbb{C}^1 \to \mathbb{C}^1$ be a non-constant holomorphic function, let Φ be the local flow of $z' = f(z)$, and let $\Omega = \omega(u)$ for some $u \in \mathbb{C}^1$. Then, Ω is homeomorphic to one of the following:*

(a) *the empty set;*
(b) *a single point;*
(c) *a circle;*
(d) *a wall.*

Conversely, there exists a holomorphic (even better, polynomial) function $f : \mathbb{C}^1 \to \mathbb{C}^1$ such that the ω-limit set is homeomorphic to any one among either (a) or (b) or (c) or (d).

2. Preliminaries

The (C^0-)topological characterization of the ω-limit sets for real-analytic flows on $\mathbb{R}^2$ are done by Jiménez López-Llibre [7]. Their result is as follows.

THEOREM 2.1. *([7] Theorem A) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be real-analytic, let Φ be the local flow of $z' = f(z)$, and let $\Omega = \omega(u)$ for some $u \in \mathbb{R}^2$. Then, Ω is the boundary ∂A of A where A is one of the following:*

(a) *the empty set;*
(b) *a single point;*
(c) *a cactus;*
(d) *the union of a circle C and finitely many pairwise disjoint cactuses, each of them contained in the disk enclosed by C and intersecting C at exactly one point;*
(e) *a union of countably many cacti, half-planes and chains, which are pairwise disjoint except that each cactus intersects either one of the half-planes of one of the chains at exactly one point; moreover, every bounded set of $\mathbb{R}^2$ intersects finitely many of these sets.*

Conversely, for every set $A \subset \mathbb{R}^2$ as in (a)–(e) and $\Omega = \partial A$, there are an analytic function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(\Omega)$ is an ω-limit set for the local flow of $z' = f(z)$.

A *cactus* is a simply connected finite union of homeomorphic closed discs where any two disks have at most one common point. A *half-plane* is a subset A of $\mathbb{R}^2$ such that both A and the complement of the interior of A in $\mathbb{R}^2$, are homeomorphic to $H = \{(x, y) \in \mathbb{R}^2 | x \geq 0\}$. A *chain* is an infinite union $\cup_{i=1}^{\infty} D_i$ of homeomorphic closed discs D_i such that every bounded set of $\mathbb{R}^2$ intersects finitely many D_i and

$$D_i \cap D_j = \begin{cases} \text{a one-point set} & \text{if } |i - j| = 1 \\ \text{the empty set} & \text{if } |i - j| \neq 1 \end{cases} \tag{2.1}$$

REMARK 2.2. Note that in Theorem 1.2, the following cases of Theorem 2.1 are excluded, which are the differences between the real-analytic and holomorphic flows: the boundary of

1. a cactus other than a disc;
2. the union of pairwise disjoint cacti, and a complement of a closed disk B such that $C = \mathrm{Bd}\, B$ encloses all the cacti and each cactus intersects C exactly by one point;
3. a union of countably many cacti, half-planes and chains, which are pairwise disjoint except that each cactus intersects either one of the half-planes or one of the chains at exactly one point, in which moreover, every bounded set of $\mathbb{R}^2$ intersects finitely many of these sets.

If we restrict f in the above theorem as a polynomial map, then we have fewer possibilities of the ω-limts due to finite number of singularities of the flow.

THEOREM 2.3. *([7] Theorem B) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial, let Φ be the local flow of $z' = f(z)$, and let $\Omega = \omega(u)$ for some $u \in \mathbb{R}^2$. Then $\Omega = \partial A$ where A is as in Theorem 2.1, except that in (e) the union is finite and contains no chains.*

Conversely, for every set $A \subset \mathbb{R}^2$ as before and $\Omega = \partial A$, there are a polynomial function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(\Omega)$ is an ω-limit set for the local flow of $z' = f(z)$.

3. Local structure of holomorphic flows and the proof of Theorem 1.1

For the proof of Theorem 1.1, we need a local analysis (of a holomorphic flow) at the singular points. The references we lean on here, are basically [1] and Bendixson-Seidenberg-Dumortier's desingularization [6, 11] (cf. [2, pp.85–86], [7, pp.688–689]). We recollect the facts on the real-analytic flows at a singularity, appearing in [7].

We start with the general fact about the real-analytic flows on $\mathbb{R}^2$ at a singular point.

DEFINITION 3.1. Let $f_h, f_p, f_e : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f_h(x, y) = (x, -y),\ \ f_p(x, y) = (x, y),\ \ f_e(x, y) = (x^2 - 2xy, xy - y^2)$$

respectively. Let us denote by Φ_h, Φ_p, Φ_e the corresponding flows (or the reversed flows) defined on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, which we call them *hyperbolic, parabolic, elliptic flows*, respectively. By *attracting parabolic* flows, we mean the flow induced by $f_p(x, y) = -(x, y)$ while by *repelling parabolic* flows, we mean the flows induced by $f_p(x, y) = (x, y)$.

We say a singular point $u \in \mathbb{R}^2$ of some flow Φ *admits sectors of hyperbolic, parabolic, elliptic types*, if an open neighborhood U of u has finite sectors U_i centered at u with

$$U = \bigcup_{i=1,\cdots,n} U_i$$

such that $\Phi|_{U_i}$ is homeomorphic to one of hyperbolic, parabolic, elliptic flows. The minimal number n in the above is called a *number of sectors*. For Φ with only sectors of parabolic type, the number of sectors may be 1 in the case f_p is defined on $\mathbb{R} \times \mathbb{R}$. For sectors of parabolic type, we discern them into the two classes: the sector of *attracting parabolic* type and the sector of *repelling parabolic* type.

DEFINITION 3.2. In the above definition, we consider the border of sectors, i.e., $\partial U_i \cap U$. They are homeomorphic to the union of two copies of $\mathbb{R}_{\geq 0}$ at 0. We call each homeomorphic image of $\mathbb{R}_{\geq 0}$ by *ray of sectors*.

THEOREM 3.3. *([7, Theorem 4.4, Theorem 4.5]) Let Φ be a local analytic flow induced by $z' = f(z)$ for a nonconstant real-analytic function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Let u be a singular point of Φ, i.e. $f(u) = 0$. If u lies on a 1-dimensional ω-limit set, then there exists a local analytic function h and a local analytic flow Ψ satisfying $\Phi = h\Psi$ such that* ***either***

(a) *u is a smooth point of Ψ;* ***or***
(b) *u is an isolated singular point of Ψ and it admits sectors of hyperbolic, parabolic, elliptic types.*

Proof. One may refer to the proof in [3]. The existence of above h is due to the Weierstrass preparation theorem. See the proof of [7, Theorem 4.4, Theorem 4.5] and the references therein. □

We are ready to prove Theorem 1.1. Note that a holomorphic function on $\mathbb{C}^1$ is a real-analytic function $\mathbb{R}^2$ as soon as we identify $\mathbb{C}^1$ and $\mathbb{R}^2$. Hence, our theorem comes down to special cases in Theorem 3.3.

Proof of Theorem 1.1. Let $f : \mathbb{C}^1 \to \mathbb{C}^1$ be a non-constant holomorphic function, and Φ be the local flow of $z' = f(z)$ as in the statement of the theorem. Let $f(u) = 0$.

We assume $u = 0$ for simplicity. Since f is a holomorphic function, near 0, we can write

$$f(z) = z^k w(z), \text{ where } w(z) = a + \sum_{i>0} a_i z^i, \tag{3.1}$$

is a (non-vanishing) holomorphic function with $a \in \mathbb{C} - \{0\}$. If $|z|$ is sufficiently small, $f(z)$ is sufficiently C^r-close to az^k; the local flows induced by $f(z)$ and az^k are sufficiently C^{r+1}-close near 0. Therefore, the decomposition of the flow by az^k near 0 into the sectors of hyperbolic, parabolic, elliptic types, coincide with Φ near 0. For the equivalent statement of this paragraph, see [4, Theorem 2.5] (cf. [8, 9]).

We classify the local flow induced by az^k near 0. To find the incoming flows and the outgoing flows, we let θ be the argument of the flows in $\mathbb{C}^1$ and then solve the following incoming and outgoing equations:

$$\theta + \pi = \arg a + k\theta + 2n\pi, \tag{3.2}$$

$$\theta = \arg a + k\theta + 2n\pi \tag{3.3}$$

Let $k = 1$. In this case, 0 can be an isolated ω-limit set if and only if $\frac{\pi}{2} < \arg a < \frac{3\pi}{2}$ for $\arg a \in [0, 2\pi)$; the induced flow is of attracting parabolic type. Note that for $\arg a = \frac{\pi}{2}, \frac{3\pi}{2}$, the flows are all periodic and thus 0 cannot be an ω-limit set. On the other hand, if $0 \le \arg a < \frac{\pi}{2}$ or $\frac{3\pi}{2} \le \arg a < 2\pi$ then the flow is of repelling parabolic type.

For $k \ge 2$, we have $k-1$ solutions $\theta_1, \dots, \theta_{k-1}$ of (3.2) and $k-1$ solutions $\vartheta_1, \dots, \vartheta_{k-1}$ of (3.3) in the period $[\frac{\arg a}{k-1}, \frac{\arg a}{k-1} + 2\pi)$; they satisfy the following alternating arrangement:

$$\vartheta_1 < \theta_1 < \vartheta_2 < \theta_2 < \cdots < \vartheta_{k-1} < \theta_{k-1}. \tag{3.4}$$

The rays of sectors are formed by the rays with the arguments ϑ_i, θ_i's, and the between the consecutive rays, it is an elliptic sector.

Further, let us assume that u lies on a 1-dimensional ω-limit set. Since a non-constant holomorphic function has only isolated zeros, $\Phi = \Psi$ and the case (a) in Theorem 3.3 do not occur. Thus, u is an isolated singular point of Φ, and hence it may admit sectors of attracting parabolic or elliptic types. □

4. Proof of Theorem 1.2

The technique in the proof of Theorem 1.2 partially owes some part of the proof of the main theorem of [3] which classifies all the possible topological types of attractors of the real-analytic flows on $\mathbb{R}^2$.

Proof of Theorem 1.2. To prove the first statement of the theorem, we have to exclude the following three cases in Theorem 2.1:

1. a cactus other than a circle;
2. the union of pairwise disjoint cacti, and a complement of a closed disk B such that $C = \mathrm{Bd}\, B$ encloses all the cacti and each cactus intersects C exactly by one point;
3. a union of countably many cacti, half-planes and chains, which are pairwise disjoint except that each cactus intersects either one of the half-planes or one of the chains at exactly one point; moreover, every bounded set of $\mathbb{R}^2$ intersects finitely many of these sets.

Let us observe that in the above cases, there must be a singularity of Φ at the intersection points of two circles or a circle and a wall. Therefore, for the proof, we have to deny the existence of such types of singularities. By Theorem 1.1, at every singularity, Φ factors into sectors of attracting parabolic or elliptic types. For a sector of attracting parabolic type (i.e. $f_p(x, y) = (-x, -y)$), we meet an immediate contradiction, because in this case, the ω-limit set of u_0 is one point $(0, 0)$. The second case, elliptic type, cannot occur because otherwise, every point in the interior

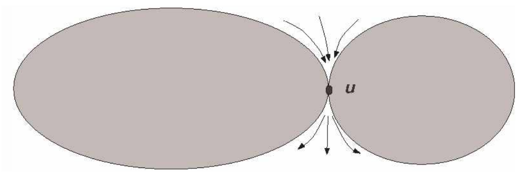

Fig 1 parabolic types of two orientations

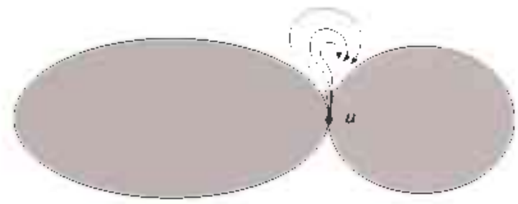

Fig 2. an elliptic type.

of the elliptic sector converges to u (as was indicated in Fig 1 and 2). This implies we cannot have a 1-dimensional ω-limit set containing u.

For the second statement of the theorem, we exhibit a polynomial function f such that the induced local flow has an ω-limit set homeomorphic to either a single point or a circle or a wall, as well as the empty ω-limit set. Let $f(z) = -z$. Then every ω-limit set is the origin. If $f(z) = \sqrt{-1}z$, then every ω-limit set other than the origin, is a circle with the center origin. If $f(z) = (\sqrt{-1} + 1)(z - 1)(z + 1)$, then the line $\Re z = 0$ is an ω-limit set. For $f(z) = z$, the ω-limit sets of points other than the origin are empty.

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