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CHARACTERIZATION ON 2-ISOMETRIES IN NON-ARCHIMEDEAN 2-NORMED SPACES

Jaeyoo Choy* and Se-Hyun Ku**

Abstract. Let f be an 2-isometry on a non-Archimedean 2-normed space. In this paper, we prove that the barycenter of triangle is invariant for f up to the translation by $f(0)$, in this case, needless to say, we can imply naturally the Mazur-Ulam theorem in non-Archimedean 2-normed spaces.

1. Introduction

The theory of isometry was begun by S. Mazur and S. Ulam [5]. They have proved that any isometry f from a real normed linear space onto another is affine (i.e., $x \to f(x) - f(0)$ is linear). But, for a complex normed vector spaces, it does not hold, i.e. not necessarily complex-linear. Consider the conjugation on $\mathbb{C}$ as an example. J. A. Baker raised a question if the result holds without the surjective assumption. As a partial answer of the question, he proved that an isometry from a real normed space into a strictly convex real normed space is affine [2]. We call the problems about the affiness of isometries on linear spaces (preserving a certain normed structure), by the Mazur–Ulam problem. Please see further history and various discussion of isometries on linear spaces relating Mazur–Ulam problem in [6, 7] and the references therein.

The motivation of this paper stems on the previous work of H.-Y. Chu [4] studying the Mazur–Ulam problem of 2-isometries on linear 2-normed spaces.

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Correspondence should be addressed to Se-Hyun Ku, shku@cnu.ac.kr.

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A *valuation* is a map $|\cdot|$ from a field $\mathcal{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r||s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r| + |s| \text{ for all } r, s \in \mathcal{K}.$$

We call a field $\mathcal{K}$ a *valued field* if $\mathcal{K}$ carries an valuation. Throughout this paper, we always assume the base field is a valued field, hence call it simply a field. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\} \text{ for all } r, s \in \mathcal{K},$$

then the map $|\cdot|$ is called a *non-Archimedean* or *ultrametric* valuation, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the map $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$.

Let $\mathcal{X}$ be a vector space over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \to [0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:
(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\| \quad (r \in \mathcal{K}, x \in \mathcal{X})$;
(iii) the strong triangle inequality

$$\|x+y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in \mathcal{X}).$$

Then $(\mathcal{X}, \|\cdot\|)$ is called a *non-Archimedean space*.

In general, the classical Mazur–Ulam theorem is not valid in the content of non-Archimedean 2-normed spaces. But under some conditions, Amyary et al. [1] present a Mazur–Ulam type theorem in non-Archimedean 2-normed spaces.

In this paper, we prove that the barycenter of triangle carries the barycenter of corresponding triangle. As a consequence, we show the Mazur–Ulam problem on non-Archimedean 2-normed spaces using the above statement.

2. Main results

DEFINITION 2.1. [1] Let $\mathcal{X}$ be a vector space of dimension greater than 1 over a field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is said to be a *non-Archimedean* 2-*norm* if it satisfies the following conditions:

(i) $\|x,y\|=0$ if and only if x,y are linearly dependent ;
(ii) $\|x,y\|=\|y,x\|$;
(iii) $\|rx,y\|=|r|\|x,y\| \quad (r\in\mathcal{K}, x,y\in X)$;
(iv) the strong triangle inequality

$$\|x,y+z\|\le\max\{\|x,y\|,\|x,z\|\} \quad (x,y,z\in X).$$

Then $(\mathcal{X},\|\cdot,\cdot\|)$ is called a *non-Archimedean 2-normed space*.

We can say that the set $\|\mathcal{X},\mathcal{X}\|:=\{\|x,y\|: x,y\in\mathcal{X}\}$ is same as $\{|r|: r\in\mathcal{K}\}$ once $\|\mathcal{X},\mathcal{X}\|$ is of a positive dimension. Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean 2-normed spaces and let $f:\mathcal{X}\to\mathcal{Y}$ be a function. Then f is called a 2-*isometry* if

$$\|x-z,y-z\|=\|f(x)-f(z),f(y)-f(z)\|$$

for all x,y and z in $\mathcal{X}$.

From now on, $(\mathcal{X},\|\cdot,\cdot\|)$ denotes a non-Archimedean 2-normed space.

Lemma 2.2. [1] *Let $(\mathcal{X},\|\cdot,\cdot\|)$ be a non-Archimedean 2-normed space. Then,*

$$\|x,y\|=\|x,y+\gamma x\|$$

for all $x,y\in\mathcal{X}$ and all $\gamma\in\mathcal{K}$.

For non-zero vectors x,y in $\mathcal{X}$, let $\mathcal{V}(x,y)$ denote the subspace of $\mathcal{X}$ generated by x and y. A non-Archimedean 2-normed space $\mathcal{X}$ over a filed $\mathcal{K}$ is called *strictly convex* if for all $x,y,z\in\mathcal{X}$,

$$\|x+y,z\|=\max\{\|x,z\|,\|y,z\|\},$$

and the conditions $\|x,z\|=\|y,z\|$ and $z\notin\mathcal{V}(x,y)$ imply $x=y$; see [1].

Lemma 2.3. *Let $\mathcal{X}$ be a strictly convex non-Archimedean 2-normed space over a non-Archimedean field $\mathcal{K}$ satisfying $|2|=|3|=1$. Suppose a triple $x,y,z\in\mathcal{X}$ satisfies $\|x-y,x-z\|\neq 0$. Then $\frac{x+y+z}{3}$ is the unique member u of $\mathcal{X}$ satisfying*

$$\|x-y,x-u\|=\|y-z,y-u\|=\|z-x,z-u\|=\|x-y,x-z\|.$$

Proof. Let $u=\frac{x+y+z}{3}$. By Lemma 2.2, we have

$$\begin{aligned}\|x-y,x-u\| &= \|x-y,x-\frac{x+y+z}{3}\|=\|x-y,\frac{2x-y-z}{3}\|\\ &= \frac{1}{|3|}\|x-y,2x-y-z\|=\|x-y,2x-y-z\|\\ &= \|x-y,x-z\|,\end{aligned}$$

$$\begin{aligned}\|y-z,y-u\| &= \|y-z,y-\frac{x+y+z}{3}\| = \|y-z,\frac{2y-x-z}{3}\| \\ &= \frac{1}{|3|}\|y-z,2y-x-z\| = \|y-z,2y-x-z\| \\ &= \|y-z,y-x\| = \|x-y,x-z\|,\end{aligned}$$

and

$$\begin{aligned}\|z-x,z-u\| &= \|z-x,z-\frac{x+y+z}{3}\| = \|z-x,\frac{2z-x-y}{3}\| \\ &= \frac{1}{|3|}\|z-x,2z-x-y\| = \|z-x,2z-x-y\| \\ &= \|z-x,z-y\| = \|x-y,x-z\|.\end{aligned}$$

For the uniqueness of u, assume that we have another $v \in \mathcal{X}$ satisfying

$$\|x-y,x-v\| = \|y-z,y-v\| = \|z-x,z-v\| = \|x-y,x-z\|.$$

From the hypothesis $|2| = 1$ of this lemma, we have

$$\|x-y,\frac{x-u}{2}\| = \|x-y,\frac{x-v}{2}\| = \|x-y,x-z\|.$$

Therefore, we have

$$\begin{aligned}\|x-y,x-\frac{u+v}{2}\| &\le \max\{\|x-y,\frac{x-u}{2}\|,\|x-y,\frac{x-v}{2}\|\} \\ &= \max\{\|x-y,x-u\|,\|x-y,x-v\|\} \\ &= \|x-y,x-z\|\,. \qquad (2.1)\end{aligned}$$

Similarly we obtain that

$$\|y-z,y-\frac{u+v}{2}\| \le \|x-y,x-z\|, \qquad (2.2)$$

$$\|z-x,z-\frac{u+v}{2}\| \le \|x-y,x-z\|. \qquad (2.3)$$

On the other hand,

$$\begin{aligned}\|x-y,z-\frac{u+v}{2}\| &= \|y-z+z-x,z-\frac{u+v}{2}\| \\ &\le \max\{\|y-z,z-\frac{u+v}{2}\|,\|z-x,z-\frac{u+v}{2}\|\} \\ &= \max\{\|y-z,y-\frac{u+v}{2}\|,\|z-x,z-\frac{u+v}{2}\|\} \\ &\le \|x-y,x-z\|.\end{aligned}$$

If all the inequalities (2.1), (2.2) and (2.3) were strict, then

$$\begin{aligned}\|x-y, x-z\| &\leq \max\{\|x-y, x-\frac{u+v}{2}\|, \|x-y, z-\frac{u+v}{2}\|\} \\ &\lneq \|x-y, x-z\|\end{aligned}$$

which is a direct contradiction. So at least one of equalities in (2.1), (2.2) and (2.3) must hold.

We now have three cases. Firstly we are going to consider the case which holds the equality in (2.1). Thus we have

$$\|x-y, x-\frac{u+v}{2}\| = \max\{\|x-y, \frac{x-u}{2}\|, \|x-y, \frac{x-v}{2}\|\}.$$

By the strict convexity, we obtain that

$$\frac{x-u}{2} = \frac{x-v}{2}.$$

Therefore $u = v$.

In the remaining cases, we have the parallel proof as in the first case. This completes the proof. □

THEOREM 2.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean 2-normed spaces over a non-Archimedean field $\mathcal{K}$ with $|2| = |3| = 1$ such that $\mathcal{Y}$ is strictly convex. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 2-isometry and $g(x) := f(x) - f(0)$. Then*

$$g\left(\frac{x+y+z}{3}\right) = \frac{g(x)+g(y)+g(z)}{3}$$

for all x, y and z in $\mathcal{X}$ with $\|x-y, x-z\| \neq 0$, that is, the barycenter of triangle is g-invariant.

Proof. Let $g(x) = f(x) - f(0)$ as above. Then we can easily show that g is also a 2-isometry and $g(0) = 0$. Let $x, y, z \in \mathcal{X}$ with $\|x-y, x-z\| \neq 0$. Since g is a 2-isometry, we have

$$\begin{aligned}\|g(x)-g(y), g(x)-g\left(\frac{x+y+z}{3}\right)\| &= \|x-y, x-\frac{x+y+z}{3}\| \\ &= \|x-y, \frac{2x-y-z}{3}\| \\ &= \|x-y, 2x-y-z\| \\ &= \|x-y, x-z\| \\ &= \|g(x)-g(y), g(x)-g(z)\|\end{aligned}$$

and similarly we can obtain

$$\begin{aligned} \|g(y)-g(z), g(y)-g\left(\frac{x+y+z}{3}\right)\| &= \|g(x)-g(y), g(x)-g(z)\| \\ \|g(z)-g(x), g(z)-g\left(\frac{x+y+z}{3}\right)\| &= \|g(x)-g(y), g(x)-g(z)\|. \end{aligned}$$

By Lemma 2.3

$$g\left(\frac{x+y+z}{3}\right)=\frac{g(x)+g(y)+g(z)}{3},$$

for all $x, y, z \in \mathcal{X}$. □

REMARK 2.5. Using the above theorem, we can show that g is $\mathcal{K}$-linear [3], that is, f is affine. As a consequence of Theorem 2.4, we can obtain the Mazur–Ulam theorem in non-Archimedean 2-normed spaces, as well; it is noteworthy that our version of the Mazur–Ulam theorem comes from the invariance of the barycenter of a triangle, without the assumption of collinearity.

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Department of Mathematics
Kyungpook National University
Daegu 702-701, Republic of Korea
E-mail: choy@knu.ac.kr

**
Department of Mathematics
Chungnam National University
Daejeon 305–764, Republic of Korea
E-mail: shku@cnu.ac.kr