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ON RETARDED INTEGRAL INEQUALITIES OF BIHARI-TYPE

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Abstract. We obtain some retarded integral inequalities of Bihari-type and apply these results to a retarded differential equation of Bernoulli-type.

1. Introduction

Differential and integral inequalities have played a fundamental role in the study of existence, uniqueness, boundedness, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations. This is the case of the attractive Gronwall-Bellman inequality [4]. Bihari's inequality is the most important nonlinear generalization of the Gronwall-Bellman inequality. Over the years, various investigators have discovered many useful integral inequalities in order to achieve a diversity of desired goals [1–4, 6, 8, 9, 11, 13–15].

Lipovan [7] gave a useful nonlinear generalization of the celebrated Gronwall inequality and presented some of its applications. However, the integral inequalities available in the literature do not apply directly in certain general situations. Pachpatte [12] gave explicit bounds on more general retarded integral inequalities which can be used as tools in the qualitative study of certain retarded integrodifferential equations.

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In this paper we obtain some retarded integral inequalities of Bihari-type and apply these results to a retarded differential equation of Bernoulli-type. Furthermore, we give two examples to explain our results.

2. Main results

In this section we consider retarded linear integral inequalities of Gronwall type for scalar functions of one variable. These retarded inequalities can be used in the study of a retarded nonlinear differential equation of Bernoulli-type. We derive an estimate of a retarded linear integral inequality. Let $\mathbb{R}^+ = [0, \infty)$ and $I = [t_0, T)$ be the subsets of $\mathbb{R}$.

THEOREM 2.1. *Let $u(t), a(t)$, and $b(t)$ be nonnegative continuous functions in I and $\alpha(t) \in C^1(I)$ be nondecreasing with $\alpha(t) \leq t$ on I. Suppose that*

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} a(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u(s)ds, \ t, t_0 \in I, \tag{2.1}$$

where c is a nonnegative constant. Then for $t \geq t_0$,

$$\begin{aligned} u(t) \leq\ & c \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} b(s)ds\right) \\ & + \int_{\alpha(t_0)}^{\alpha(t)} a(s) \exp\left(\int_s^{\alpha(t)} b(\tau)d\tau\right) ds, \ t \in I. \end{aligned} \tag{2.2}$$

Proof. Define $v(t)$ by the right hand side of (2.1). Then $u(t) \leq v(t)$ and $v(t_0) = c$. Then we obtain

$$\begin{aligned} v'(t) &\leq b(\alpha(t))v(\alpha(t))\alpha'(t) + a(\alpha(t))\alpha'(t) \\ &\leq b(\alpha(t))v(t)\alpha'(t) + a(\alpha(t))\alpha'(t), \ t \in I \end{aligned} \tag{2.3}$$

by the monotonicity of v and $\alpha(t) \leq t$. The inequality (2.3) implies

$$\begin{aligned} &\frac{d}{ds}\left(v(s) \exp(\int_s^t b(\alpha(\tau))\alpha'(\tau)d\tau)\right) \\ &= \left[v'(s) - b(\alpha(s))v(s)\alpha'(s)\right] \exp\big(\int_s^t b(\alpha(\tau))\alpha'(\tau)d\tau\big) \\ &\leq a(\alpha(s))\alpha'(s) \exp(\int_s^t b(\alpha(\tau))\alpha'(\tau)d\tau), \ t \geq s \geq t_0. \end{aligned}$$

Integration over s from t_0 to t gives

$$\begin{aligned} v(t) &- v(t_0)\exp\left(\int_{t_0}^{t} b(\alpha(\tau))\alpha'(\tau)d\tau\right) \\ &\le \int_{t_0}^{t} a(\alpha(s))\alpha'(s)\exp\left(\int_{s}^{t} b(\alpha(\tau))\alpha'(\tau)d\tau\right) ds \end{aligned}$$

and the change of variables yields

$$\begin{aligned} u(t) &\le v(t) \\ &\le v(t_0)\exp\left(\int_{\alpha(t_0)}^{\alpha(t)} b(\tau)d\tau\right) + \int_{\alpha(t_0)}^{\alpha(t)} a(s)\exp\left(\int_{s}^{\alpha(t)} b(\tau)d\tau\right) ds. \end{aligned}$$

The proof is complete. □

Using the result of Theorem 2.1 we obtain the retarded linear differential inequality.

COROLLARY 2.2. *Let $b(t)$ and $a(t)$ be continuous functions for $t \ge t_0$ and let $v(t)$ and $\alpha(t)$ be differentiable nonnegative functions for $t \ge t_0$ with $\alpha(t) \le t$ and $\alpha'(t) \ne 0$. Suppose*

$$v'(t) \le a(t) + b(t)v(\alpha(t)), t \in I.$$

Then for $t \ge t_0$,

$$v(t) \le v(t_0)\exp\left(\int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(\tau)d\tau\right) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s)\exp\left(\int_{s}^{\alpha(t)} \hat{b}(\tau)d\tau\right) ds,$$

where $\hat{a}(t) = \frac{a(\alpha^{-1}(t))}{\alpha'(t)}$ and $\hat{b} = \frac{b(\alpha^{-1}(t))}{\alpha'(t)}$.

REMARK 2.3. Let $\alpha(t) = t$ in Theorem 2.1. Then we obtain a comparison result for linear integral inequalities:

$$\begin{aligned} u(t) &\le u(t_0)\exp\left(\int_{t_0}^{t} b(s)ds\right) \\ &+ \int_{t_0}^{t} a(s)\exp\left(\int_{s}^{t} b(\tau)d\tau\right) ds,\ t \in I. \end{aligned} \tag{2.4}$$

Now, we investigate some retarded nonlinear integral inequalities. To this end we need the following lemmas in [12].

LEMMA 2.4. *[12, Theorem 2.1] Let $u(t), a(t) \in C(I, \mathbb{R}^+), b(t,s) \in C(I \times I, \mathbb{R}^+)$ for $t_0 \le s \le t \le T$ and $\alpha(t) \in C^1(I)$ be nondecreasing with*

$\alpha(t) \le t$ on I and c be a positive constant. Suppose that $w \in C((0,\infty))$ is a nondecreasing function. If

$$u(t) \le c + \int_{\alpha(t_0)}^{\alpha(t)} \left(a(s)w(u(s)) + \int_{\alpha(t_0)}^{s} b(s,\sigma)w(u(\sigma))d\sigma \right) ds, \quad (2.5)$$

for $t \in I$, then for $t_0 \le t \le b_1$,

$$u(t) \le W^{-1}\left[W(c) + \int_{\alpha(t_0)}^{\alpha(t)} \left(a(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma)d\sigma \right) ds \right], \quad (2.6)$$

where W^{-1} is the inverse function of $W(u) = \int_{u_0}^{u} \frac{dz}{w(z)}, u > 0, u_0 > 0$, and b_1 is the largest number such that $b_1 \ge t_0$ and for $t_0 \le t \le b_1$

$$\left[W(c) + \int_{\alpha(t_0)}^{\alpha(t)} \left(a(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma)d\sigma \right) ds \right] \in Dom(W^{-1}).$$

REMARK 2.5. If we set $\alpha(t) = t$ and $b(t,s) = 0$ in [12, Theorem 2.1]. then we obtain the well-known Bihari-type inequality.

We extend the result in [14, Lemma 1] to a retarded nonlinear integral inequality.

THEOREM 2.6. *Let $u(t), a(t), b(t) \in C(I,\mathbb{R}^+)$ and $\alpha(t) \in C^1(I)$ be nondecreasing with $\alpha(t) \le t$ on I and c be a positive constant. Let $w_i, i = 1,2$, be continuous nondecreasing functions on $\mathbb{R}^+$ and positive on $(0,\infty)$ such that $w = \frac{w_2}{w_1}$ is nondecreasing on $(0,\infty)$. If*

$$u(t) \le c + \int_{\alpha(t_0)}^{\alpha(t)} a(s)w_1(u(s))ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)w_2(u(s))ds, \quad (2.7)$$

for $t \in I$, then for $t_0 \le t \le b_1$,

$$u(t) \le W_2^{-1}\left[W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds \right],\ t_0 \le t \le b_1,$$

where $W_1(u) = \int_{c}^{u} \frac{dz}{w_1(z)}, u > 0$, $W_2(u) = \int_{u_0}^{u} \frac{dz}{w_2(z)}, u_0 > 0$, and b_1 is the largest number such that $b_1 \ge t_0$ and

$$\begin{aligned} |a|_{b_1} &= \int_{t_0}^{b_1} a(s)ds \le \int_{c}^{\infty} \frac{dz}{w_1(z)}, \\ |b|_{b_1} &= \int_{t_0}^{b_1} b(s)ds \le \int_{c_1}^{\infty} \frac{dz}{w_2(z)}, c_1 = W_1^{-1}(|a|_{b_1}). \end{aligned}$$

Proof. Let $v(t)$ by the right side of (2.7). Then $v(t_0) = c, u(t) \le v(t)$ and we obtain

$$\begin{aligned} v'(t) &\le [a(\alpha(t))w_1(v(\alpha(t))) + b(\alpha(t))w_2(v(\alpha(t)))]\,\alpha'(t) \\ &\le w_1(v(t))\left[a(\alpha(t)) + \frac{w_2(v(\alpha(t)))}{w_1(v(\alpha(t)))}b(\alpha(t))\right]\alpha'(t), \end{aligned}$$

by the monotonicity of w_1 and $\alpha(t) \le t$. This implies

$$\begin{aligned} \frac{d(W_1(v(t)))}{dt} &= \frac{v'(t)}{w_1(v(t))} \\ &\le [a(\alpha(t)) + w(v(\alpha(t)))b(\alpha(t))]\,\alpha'(t). \end{aligned} \tag{2.8}$$

Integrating (2.8) from t_0 to t and making the change of variables, we have

$$\begin{aligned} W_1(v(t)) &= W_1(v(t_0)) + \int_{t_0}^{t} [a(\alpha(\sigma)) + w(v(\alpha(\sigma)))b(\alpha(\sigma))]\,\alpha'(\sigma)d\sigma \\ &= W_1(c) + \int_{\alpha(t_0)}^{\alpha(t)} [a(s) + b(s)w(v(s))]\,ds \\ &\le \tilde{c} + \int_{\alpha(t_0)}^{\alpha(t)} b(s)w(v(s))ds, \end{aligned}$$

where $\tilde{c} = \int_{\alpha(t_0)}^{b_1} a(s)ds$. Then, putting $z(t) = W_1(v(t))$, we obtain

$$z(t) \le \tilde{c} + \int_{\alpha(t_0)}^{\alpha(t)} b(s)w(W_1^{-1}(z(s)))ds.$$

Setting $b(t,s) = 0$ in [12, Theorem 2.1], we get, for $t \in [t_0, b_1)$

$$z(t) \le \tilde{W}^{-1}\left[\tilde{W}(\tilde{c}) + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds\right],$$

where $\tilde{W}(u) = \int_{\tilde{u_0}}^{u} \frac{ds}{w(W_1^{-1}(s))}$, $\tilde{u_0} = W_1(u_0)$. Since $u \le v = W_1^{-1}(z)$ and $\tilde{W} = W_2 \circ W_1^{-1}$, we have

$$u(t) \le W_2^{-1}\left[W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds\right],\ t_0 \le t \le b_1.$$

The proof is complete. □

We obtain the retarded nonlinear integral inequality of Bernoulli-type by using the consequences of Theorem 2.6.

COROLLARY 2.7. *Let $w_1(u) = u$ and $w_2(u) = u^p$ with $0 < p \neq 1$ constant in Theorem 2.6. Let $u(t), a(t)$, and $b(t)$ be nonnegative continuous functions in I, and $c > 0$ be constant. Suppose that*

$$u(t) \quad \leq \quad c + \int_{\alpha(t_0)}^{\alpha(t)} a(s)u(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u^p(s)ds, \tag{2.9}$$

holds for $t \geq t_0$. Then for $0 \leq t_0 \leq t \leq b_1$,

$$\begin{aligned} u(t) \quad &\leq \quad \begin{cases} W_1^{-1}\left[W_1(c_2) + \int_{\alpha(t_0)}^{\alpha(t)} a(s)ds\right], \ 0 < p < 1 \\ \\ W_2^{-1}\left[W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds)\right], \ 1 < p \end{cases} \qquad (2.10) \\ &= \quad \begin{cases} c_2 \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} a(s)ds\right), \ 0 < p < 1 \\ \\ \left[c_1^{1-p} + (1-p)\int_{\alpha(t_0)}^{\alpha(t)} b(s)ds\right]^{\frac{1}{1-p}}, \ 1 < p, \end{cases} \end{aligned}$$

where

$$\begin{aligned} c_1 \quad &= \quad W_1^{-1}(|a|_{b_1}) = c\exp(|a|_{b_1}), \\ c_2 \quad &= \quad W_2^{-1}(|b|_{b_1}) = \left[c^{1-p} + (1-p)|b|_{b_1}\right]^{\frac{1}{1-p}}. \end{aligned}$$

Proof. It is easy to find that

$$W_1(u) = \ln(\frac{u}{c}), \ W_1^{-1}(u) = c\exp u,$$

$$W_2(u) = \frac{1}{1-p}\left[u^{1-p} - c^{1-p}\right], \ W_2^{-1}(u) = \left[c^{1-p} + (1-p)u\right]^{\frac{1}{1-p}}.$$

We obtain the increasing function w on $(0, \infty)$ which can be applied in Theorem 2.6:

$$w(u) = \begin{cases} \frac{w_1(u)}{w_2(u)} = u^{1-p}, \ 0 < p < 1 \\ \\ \frac{w_2(u)}{w_1(u)} = u^{p-1}, \ 1 < p. \end{cases}$$

Applying this to (2.9) in Theorem 2.6, we obtain the inequality (2.10). □

REMARK 2.8. For the cases $p = 0$ and $p = 1$ we obtain the retarded linear integral inequality (2.1) in Theorem 2.1.

If the condition on p in Corollary 2.7 is replaced by $0 < p < 1$, then we obtain more concrete inequality than the inequality in Corollary 2.7 by using the consequence of retarded integral inequality in Theorem 2.1. We extend the result of Theorem 2.1 to the following corollary.

COROLLARY 2.9. *Let* $w_1(u) = u$ *and* $w_2(u) = u^p$ *with* $0 < p < 1$ *in Theorem 2.6. Let* $u(t), a(t)$, *and* $b(t)$ *be nonnegative continuous functions in* I, *and* $c > 0$ *be a constant. Suppose that*

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} a(s)u(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u^p(s)ds \qquad (2.11)$$

holds for $t \in I$. *Then for* $0 \leq t_0 \leq t$,

$$u(t) \leq \left[\left(c\exp(\int_{\alpha(t_0)}^{\alpha(t)} a(s)ds)\right)^q + q\int_{\alpha(t_0)}^{\alpha(t)} b(s)\exp(q\int_s^{\alpha(t)} a(\tau)d\tau)ds\right]^{\frac{1}{q}},$$

where $q = 1 - p$.

Proof. Define $v(t)$ by the right hand side of (2.11). Then $u(t) \leq v(t)$ and $v(t_0) = c$. Then we obtain

$$\begin{aligned} v'(t) &\leq v^p(\alpha(t))\left[b(\alpha(t)) + a(\alpha(t))v^{1-p}(\alpha(t))\right]\alpha'(t) \qquad (2.12) \\ &\leq v^p(t)\left[b(\alpha(t)) + a(\alpha(t))v^{1-p}(\alpha(t))\right]\alpha'(t),\ t \in I \end{aligned}$$

by the monotonicity of v and $\alpha(t) \leq t$, which implies

$$\frac{v'(t)}{v^p(t)} \leq \left[b(\alpha(t)) + a(\alpha(t))v^{1-p}(\alpha(t))\right]\alpha'(t),\ t \in I. \qquad (2.13)$$

By integrating (2.13) from t_0 to t and the change of variables we have

$$\frac{1}{q}v^q(t) \leq \frac{c^q}{q} + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} a(s)v^q(s)ds,$$

where $q = 1 - p$. Letting $z(t) = \frac{1}{q}v^q(t)$, we obtain

$$z(t) \leq \frac{c^q}{q} + \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds + q\int_{\alpha(t_0)}^{\alpha(t)} a(s)z(s)ds,$$

which implies

$$z(t) \leq \frac{c^q}{q}\exp\left(q\int_{\alpha(t_0)}^{\alpha(t)} a(s)ds\right) + \int_{\alpha(t_0)}^{\alpha(t)} b(s)\exp\left(q\int_s^{\alpha(t)} a(\tau)d\tau\right)ds,$$

by Theorem 2.1. Hence we have

$$\begin{aligned} u(t) &\leq v(t) \\ &\leq \left[c^q\exp\left(q\int_{\alpha(t_0)}^{\alpha(t)} a(s)ds\right) + q\int_{\alpha(t_0)}^{\alpha(t)} b(s)\exp\left(q\int_s^{\alpha(t)} a(\tau)d\tau\right)ds\right]^{\frac{1}{q}}, \end{aligned}$$

where $q = 1 - p$. The proof is complete. □

REMARK 2.10. If we set $p = 0$ in Corollary 2.9, then we obtain the same retarded integral inequality as in Theorem 2.1. Also, if we set $\alpha(t) = t$ in Corollary 2.9 we obtain the same result as in [3, Lemma 4.1].

COROLLARY 2.11. *Let $v(t)$ be a positive differentiable function satisfying the differential inequality*

$$v'(t) \leq a(t)v(\alpha(t)) + b(t)v^p(\alpha(t)), \quad t \in I, \tag{2.14}$$

where $a(t)$ and $b(t)$ are nonnegative continuous functions in I, and $0 < p \neq 1$ is a constant. Then for $0 \leq t_0 \leq t \leq b_1$,

$$\begin{aligned} v(t) &\leq \begin{cases} W_1^{-1}\left[W_1(c_2) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s)ds\right], & 0 < p < 1 \\ \\ W_2^{-1}\left[W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(s)ds)\right], & 1 < p \end{cases} \\ &= \begin{cases} c_2 \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s)ds\right), & 0 < p < 1 \\ \\ \left[c_1^q + q\int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(s)ds\right]^{\frac{1}{q}}, & 1 < p, \end{cases} \end{aligned}$$

where $q = 1 - p$, $\hat{a}(t) = \frac{a(\alpha^{-1}(t))}{\alpha'(t)}$, $\hat{b}(t) = \frac{b(\alpha^{-1}(t))}{\alpha'(t)}, c_1 = W_1^{-1}(|a|_{b_1}) = c\exp(|a|_{b_1})$, and $c_2 = W_2^{-1}(|b|_{b_1}) = [c^q + q|b|_{b_1}]^{\frac{1}{q}}$.

Proof. The integration of (2.14) and the change of variables yield

$$v(t) \leq v(t_0) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s)v(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(s)v^p(s)ds, t \in I,$$

where $\hat{a}(t) = \frac{a(\alpha^{-1}(t))}{\alpha'(t)}$ and $\hat{b}(t) = \frac{b(\alpha^{-1}(t))}{\alpha'(t)}$. By applying this to Corollary 2.7 we obtain the required estimate. □

REMARK 2.12. If we set $\alpha(t) = t$ in Theorem 2.6, then we obtain the same result as in [5, Lemma 2.1].

We illustrate our results by the following example, including a retarded differential equation of Bernoulli-type.

EXAMPLE 2.13. *Consider the retarded differential equation with the given initial condition*

$$x'(t) = a(t)x(t - h(t)) + b(t)x^p(t - h(t)), \; x(t_0) = x_0 \tag{2.15}$$

where $h(t) \in C^1(I)$ is nondecreasing with $t - h(t) \geq 0, h'(t) < 1, h(t_0) = 0$ and $p \neq 1$ is a positive constant. Then the solution $x(t)$ of (2.15) satisfies

$$|x(t)| \leq \begin{cases} c_2 \exp\left(\int_{t_0}^{t-h(t)} \hat{a}(s)ds\right), \ 0 < p < 1 \\ \\ \left[c_1^{1-p} + (1-p)\left(\int_{t_0}^{t-h(t)} \hat{b}(s)ds\right)\right]^{\frac{1}{1-p}}, \ 1 < p, \end{cases} \tag{2.16}$$

where $\hat{a}(t) = \frac{|a(\alpha^{-1}(t))|}{|1-h'(\alpha^{-1}(t))|}, \hat{b}(t) = \frac{|b(\alpha^{-1}(t))|}{|1-h'(\alpha^{-1}(t))|}$ and $\alpha(t) = t - h(t)$.

Proof. We see that the solution $x(t)$ of (2.15) satisfies the integral equation

$$\begin{aligned} x(t) &= x(t_0) + \int_{\alpha(t_0)}^{\alpha(t)} a(\alpha^{-1}(s))x(s)\frac{ds}{1-h'(\alpha^{-1}(s))} \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha^{-1}(s))x^p(s)\frac{ds}{1-h'(\alpha^{-1}(s))}, \end{aligned}$$

where $\alpha^{-1}(t)$ is the inverse function of $\alpha(t) = t - h(t)$, by applying the change of variables. Putting $|x(t)| = u(t)$, we have

$$u(t) \leq u(t_0) + \int_{\alpha(t_0)}^{\alpha(t)} \hat{a}(s)u(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} \hat{b}(s)u^p(s)ds, \tag{2.17}$$

where $\hat{a}(t) = \frac{|a(\alpha^{-1}(t))|}{|1-h'(\alpha^{-1}(t))|}, \hat{b}(t) = \frac{|b(\alpha^{-1}(t))|}{|1-h'(\alpha^{-1}(t))|}$ and $\alpha(t) = t - h(t)$. A suitable application of the inequality given in Corollary 2.7 to (2.17) yields the required estimate in (2.16). □

THEOREM 2.14. *Let $u(t), a(t) \in C(I, \mathbb{R}^+), b(t,s) \in C(I \times I, \mathbb{R}^+)$ for $t_0 \leq s \leq t \leq T$ and $\alpha(t) \in C^1(I)$ be nondecreasing with $\alpha(t) \leq t$ on I and c be a positive constant. Let $w_i, i = 1, 2$, be continuous nondecreasing functions on $\mathbb{R}^+$ and positive on $(0, \infty)$ such that $w = \frac{w_2}{w_1}$ is nondecreasing on $(0, \infty)$. If*

$$u(t) \leq c + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)w_1(u(s)) + \int_{\alpha(t_0)}^{s} b(s,\sigma)w_2(u(\sigma))d\sigma\right] ds, \tag{2.18}$$

for $t \in I$, then for $t_0 \leq t \leq b_1$,

$$u(t) \leq W_2^{-1}\left[W_2(c_1) + A(t)\right], \tag{2.19}$$

where $A(t) = \int_{\alpha(t_0)}^{\alpha(t)} [\int_{\alpha(t_0)}^{s} b(s,\sigma)d\sigma]ds$, $W_1(u) = \int_c^u \frac{dz}{w_1(z)}, u > 0, W_2(u) = \int_{u_0}^u \frac{dz}{w_2(z)}, u_0 > 0$ *and* b_1 *is the largest number such that* $b_1 \geq t_0$ *and*

$$\begin{aligned} |a|_{b_1} &= \int_{t_0}^{b_1} a(s)ds \leq \int_c^\infty \frac{dz}{w_1(z)}, \\ |A|_{b_1} &= \int_{t_0}^{b_1} A(s)ds \leq \int_{c_1}^\infty \frac{dz}{w_2(z)}, c_1 = W_1^{-1}(|a|_{b_1}). \end{aligned}$$

Proof. Let $v(t)$ by the right side of (2.18). Then $v(t_0) = c, u(t) \leq v(t)$ and we obtain

$$\begin{aligned} v'(t) &\leq \left[a(\alpha(t))w_1(v(\alpha(t))) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t),s)w_2(v(\alpha(s)))ds\right]\alpha'(t) \\ &\leq \left[a(\alpha(t))w_1(v(t)) + w_2(v(t))\int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t),s)ds\right]\alpha'(t) \\ &\leq w_1(v(t))\left[a(\alpha(t)) + \frac{w_2(v(\alpha(t)))}{w_1(v(\alpha(t)))}\int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t),s)ds\right]\alpha'(t), \end{aligned}$$

in view of the monotonicity of w_1 and $\alpha(t) \leq t$. It follows that

$$\begin{aligned} \frac{d(W_1(v(t)))}{dt} &= \frac{v'(t)}{w_1(v(t))} \\ &\leq \left[a(\alpha(t)) + w(v(\alpha(t)))\int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t),s)ds\right]\alpha'(t). \end{aligned} \tag{2.20}$$

Integrating (2.20) from t_0 to t and making the change of variables, we have

$$\begin{aligned} W_1(v(t)) &= W_1(v(t_0)) \\ &\quad + \int_{t_0}^{t} \left[a(\alpha(\sigma)) + w(v(\alpha(\sigma)))\int_{\alpha(t_0)}^{\alpha(\sigma)} b(\alpha(\sigma),\tau)d\tau\right]\alpha'(\sigma)d\sigma \\ &= W_1(c) + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s) + w(v(s))\int_{t_0}^{s} b(s,\tau)d\tau\right]ds \\ &\leq \tilde{c} + \int_{\alpha(t_0)}^{\alpha(t)} \left[\int_{t_0}^{s} b(s,\tau)d\tau\right]w(v(s))ds, \end{aligned}$$

where $\tilde{c} = \int_{\alpha(t_0)}^{b_1} a(s)ds$. Then, putting $z(t) = W_1(v(t))$, we obtain

$$z(t) \leq \tilde{c} + \int_{\alpha(t_0)}^{\alpha(t)} \left[\int_{t_0}^{s} b(s,\tau)d\tau \right] w(W_1^{-1}(z(t)))ds.$$

By Lemma 2.4, we get, for $t \in [t_0, b_1)$

$$z(t) \leq \tilde{W}^{-1} \left[\tilde{W}(\tilde{c}) + \int_{\alpha(t_0)}^{\alpha(t)} \left(\int_{t_0}^{s} b(s,\tau)d\tau \right) ds \right],$$

where $\tilde{W}(u) = \int_{\tilde{u_0}}^{u} \frac{ds}{w(W_1^{-1}(s))}$ and $\tilde{u_0} = W_1(u_0)$. Since $u \leq v = W_1^{-1}(z)$ and $\tilde{W} = W_2 \circ W_1^{-1}$, we have

$$u(t) \leq W_2^{-1} \left[W_2(c_1) + \int_{\alpha(t_0)}^{\alpha(t)} \left(\int_{t_0}^{s} b(s,\tau)d\tau \right) ds \right],\ t_0 \leq t \leq b_1.$$

This completes the proof. □

Remark 2.15. The method of the proof of Theorem 2.14 is similar to that of Lemma 2.1 [14].

We obtain the following result [5, Theorem 2.3] as a corollary of Theorem 2.14.

Corollary 2.16. *Let $w_i, i = 1,2,$ be continuous nondecreasing functions on $\mathbb{R}^+$ and positive on $(0,\infty)$ such that $w = \frac{w_2}{w_1}$ is nondecreasing on $(0,\infty)$. Let $u, \lambda_i \in C(\mathbb{R}^+), i = 1,2,3,$ and c a positive constant. If*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)w_1(u(s))ds + \int_{t_0}^{t} \lambda_2(s) \left(\int_{t_0}^{s} \lambda_3(\tau)w_2(u(\tau))d\tau \right) ds,$$

for $t \in I$, then for $0 \leq t_0 \leq t < b_1$,

$$u(t) \leq W_2^{-1} \left[W_2(c_1) + \int_{t_0}^{t} \hat{\lambda}(s)ds \right],$$

where $\hat{\lambda}(t) = \lambda_2(t)\int_{t_0}^{t} \lambda_3(s)ds, W_1(u) = \int_{c}^{u} \frac{dz}{w_1(z)},\ u > 0,\ W_2(u) = \int_{u_0}^{u} \frac{dz}{w_2(z)}, u_0 > 0$ and b_1 is the largest number such that $b_1 \geq t_0$ and

$$\begin{aligned} |\lambda_1|_{b_1} &= \int_{t_0}^{b_1} \lambda_1(s)ds \leq \int_{c}^{\infty} \frac{dz}{w_1(z)}, \\ |\hat{\lambda}|_{b_1} &= \int_{t_0}^{b_1} \hat{\lambda}(s)ds \leq \int_{c_1}^{\infty} \frac{dz}{w_2(z)}, c_1 = W_1^{-1}(|\lambda_1|_{b_1}). \end{aligned}$$

Proof. In Theorem 2.14, if we put $\alpha(t) = t,\ a(t) = \lambda_1(t)$ and $b(t,s) = \lambda_2(t)\lambda_3(s)$, then we obtain the result. □

REMARK 2.17. If $\frac{w_1}{w_2}$ (instead of $\frac{w_2}{w_1}$ in Theorem 2.14) is a nondecreasing function on $(0, \infty)$, then Theorem 2.14 can be applied by changing the roles of w_1 and w_2, and a and b, respectively. In this case, the conclusion is

$$u(t) \le W_1^{-1}\left[W_1(c_2) + A(t)\right], c_2 = W_2^{-1}(|A|_{b_1}).$$

We obtain the integral inequality with differential equation of Bernoulli-type as a corollary of Theorem 2.14.

COROLLARY 2.18. *Let the assumptions be the same as in the Theorem 2.14. If*

$$u(t) \le c + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)u(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma)u^p(\sigma)d\sigma\right] ds, \tag{2.21}$$

where $w_1(u) = u$ and $w_2(u) = u^p$ with a positive constant p, then for $0 \le t_0 \le t \le b_1$,

$$\begin{aligned} u(t) &\le \begin{cases} W_1^{-1}\left[W_1(c_2) + A(t)\right], \ 0 < p \le 1 \\ W_2^{-1}\left[W_2(c_1) + A(t)\right], \ 1 < p \end{cases} \qquad (2.22)\\ &= \begin{cases} c_2 \exp(A(t)), \ 0 < p \le 1 \\ \left[c_1^{1-p} + (1-p)A(t)\right]^{\frac{1}{1-p}}, \ 1 < p. \end{cases} \end{aligned}$$

Proof. Note that the functions of $W_2(u)$ and $W_2^{-1}(u)$ are given by

$$W_2(u) = \begin{cases} \ln \frac{u}{c}, \ p = 1 \\ \frac{1}{1-p}\left[u^{1-p} - c^{1-p}\right], \ 0 < p \ne 1, \end{cases}$$

and

$$W_2^{-1}(u) = \begin{cases} c\exp(u), \ p = 1 \\ \left[(1-p)u + c^{1-p}\right]^{\frac{1}{1-p}}, \ 0 < p \ne 1, \end{cases}$$

respectively. □

COROLLARY 2.19. *[12, Theorem 2.1] Let the assumptions be the same as in the Lemma 2.4. If*

$$u(t) \le c + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)w(u(s)) + \int_{\alpha(t_0)}^{s} b(s,\sigma)w(u(\sigma))d\sigma\right] ds,$$

for $t \in I$, then for $t_0 \le t \le b_1$,

$$u(t) \le W^{-1}\left[W(c) + \int_{\alpha(t_0)}^{\alpha(t)} \left(a(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma)d\sigma\right) ds\right],$$

where W^{-1} is the inverse function of W defined by Lemma 2.4.

Proof. The proof can be accomplished by the same method as in Theorem 2.14. Here we omit the details. □

Consider the integrodifferential equation with the given initial condition

$$x'(t) = F(t, x(t-h(t)), \int_{t_0}^{t} f\big(t,\sigma,x(\sigma-h(\sigma))\big)d\sigma), x(t_0) = x_0, \quad (2.23)$$

where $f \in C(I \times I \times \mathbb{R}, \mathbb{R}), F \in C(I \times \mathbb{R}^2, \mathbb{R})$ and $h \in C^1(I)$ is nondecreasing with $t - h(t) \ge 0, h'(t) < 1, h(t_0) = 0$. The following example give an estimate on the solution of (2.23).

Example 2.20. *Suppose that f and F satisfy*

$$|f(t,s,x)| \le b(t,s)w_2(|x|),\ |F(t,x,z)| \le a(t)w_1(|x|) + |z|, \quad (2.24)$$

where $a(t), b(t,s), w_1, w_2$ are as defined in Theorem 2.14. Let $M = \max_{t\in I} \frac{1}{1-h'(t)}$. If $x(t)$ is any solution of (2.23), then for $t_0 \le t \le b_1$,

$$|x(t)| \le W_2^{-1}\left[W_2(c_1) + \int_{t_0}^{t-h(t)} [\int_{t_0}^{s} b(s,\sigma)d\sigma]ds\right], \quad (2.25)$$

where $W_1(u) = \int_c^u \frac{dz}{w_1(z)}, u > 0, c = |x_0|$, $W_2(u) = \int_{u_0}^u \frac{dz}{w_2(z)}, u_0 > 0$, and b_1 is the largest number such that $b_1 \ge t_0$ and

$$\begin{aligned} |a|_{b_1} &= \int_{t_0}^{b_1} a(s)ds \le \int_c^\infty \frac{dz}{w_1(z)}, \\ |A|_{b_1} &= \int_{t_0}^{b_1} A(s)ds \le \int_{c_1}^\infty \frac{dz}{w_2(z)}, c_1 = W_1^{-1}(|a|_{b_1}). \end{aligned}$$

Proof. Put $t - h(t) = \alpha(t)$. Then $\alpha(t)$ is invertible. The solution $x(t)$ of (2.23) can be written as

$$x(t) = x_0 + \int_{t_0}^{t} F\left(s, x(s-h(s)), \int_{t_0}^{s} f(s,\sigma,x(\sigma-h(\sigma)))d\sigma\right) ds.$$

Using the change of variables and the condition (2.24), we have

$$|x(t)| \leq |x_0| + \Big| \int_{\alpha(t_0)}^{\alpha(t)} F\big(\alpha^{-1}(s), x(s),$$

$$\int_{\alpha(t_0)}^{\alpha(\alpha^{-1}(s))} f(\alpha^{-1}(s), \alpha^{-1}(\sigma), x(\sigma)) \frac{d\sigma}{1 - h'(\alpha^{-1}(\sigma))}\big) \frac{ds}{1 - h'(\alpha^{-1}(s))} \Big|$$

$$\leq |x_0| + \int_{\alpha(t_0)}^{\alpha(t)} \left[\hat{a}(s) w_1(|x(s)|) + \int_{\alpha(t_0)}^{s} \hat{b}(s, \sigma) w_2(|x(\sigma)|) d\sigma \right] ds,$$

where $\hat{a}(t) = Ma(\alpha^{-1}(t))$ and $\hat{b}(t,s) = M^2 b(\alpha^{-1}(t), \alpha^{-1}(s))$. Now, a suitable application of the inequality in Theorem 2.14 yields the required estimate in (2.25). □

REMARK 2.21. If we set $w_1(u) = w_2(u) = u$ in Example 2.20, then we have

$$|x(t)| \leq |x_0| \exp\left(\int_{t_0}^{t-h(t)} \big(\hat{a}(s) + \int_{t_0}^{s} \hat{b}(s, \sigma) d\sigma \big) ds \right),$$

where $\hat{a}(t) = Ma(\alpha^{-1}(t))$ and $\hat{b}(t,s) = M^2 b(\alpha^{-1}(t), \alpha^{-1}(s))$, $\alpha(t) = t - h(t)$. Also, we note that the inequality in Theorem 2.14 can be used to study the uniqueness and the continuous dependence of solutions of integrodifferential equation (2.23) on initial values. See Theorems 4.2 and 4.3 in [12] for details.

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