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G'-SEQUENCE OF A MAP

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ABSTRACT. Pan, Shen and Woo [8] introduced the concept of the G-sequence of a map. We introduce the G'-sequence of a map, which is a dual concept of the G-sequence of a map. We obtain some sufficient conditions for the all sets in the G'-sequence of a map are groups ,and for the exact G'-sequence of a map.

1. Introduction

Gottlieb [1,2] defined and studied the Gottlieb groups $G_n(X)$ of the homotopy groups $\pi_n(X)$. The Gottlieb groups have many applications on fibration theory and fixed point theory. The homotopy sequence of a topological pair plays an important role in computing homotopy groups. In [11], Woo and Lee introduced the G-sequence of a CW-pair and obtained some sufficient conditions for the G-sequence to be exact. In [8], Pan, Shen and Woo extended the concept of G-sequence of a pair to the concept of G-sequence of a map and obtained some results about exactness for G-sequence of a map. On the other hand, Haslam [3] introduced and studied the dual Gottlieb groups $G^n(X;\pi)$ of $H^n(X;\pi)$. In this paper, we introduce the G'-sequence of a map, which is a dual concept of the G-sequence of a map. We show that there is a sequence $\to G^n_R(g) \xrightarrow{J} G^n(Y) \xrightarrow{g^*} G^n(X,g,Y) \xrightarrow{\delta^*} G^{n+1}_R(g) \xrightarrow{J}$ such that the following diagram is commutative

$$\begin{array}{ccccccccc}
\longrightarrow & G^n_R(g) & \xrightarrow{J} & G^n(Y) & \xrightarrow{g^*} & G^n(X,g,Y) & \xrightarrow{\delta^*} & G^{n+1}_R(g) & \xrightarrow{J} \\
 & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & \\
\longrightarrow & H^n(g) & \xrightarrow{J} & H^n(Y) & \xrightarrow{g^*} & H^n(X) & \xrightarrow{\delta^*} & H^{n+1}(g) & \xrightarrow{J}
\end{array}$$

and $g^*J = 0$ and $\delta^* g^* = 0$ and $J\delta^* = 0$. This sequence is called *the G'-sequence of g*. We do not know whether the all sets in the G'-sequence of g are groups. However, we obtain some sufficient conditions for the all sets in the G'-sequence of g are groups. We show that if $g : X \to Y$

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is a cocyclic map and $g : X \to Y$ has a right homotopy inverse, then the the G'-sequence of g is the same as the cohomology exact sequence

$$\to H^n(g) \xrightarrow{J} H^n(Y) \xrightarrow{g^*} H^n(X) \xrightarrow{\delta^*} H^{n+1}(g) \to$$

of g. In fact, we know that $G^n(X, g, Y) \cong G^n(Y) \oplus G_R^{n+1}(g)$ for all n under the same condition. Moreover, if $g : X \to Y$ has a left homotopy inverse, then the all sets in the G'-sequence of g are groups.

2. Preliminaries

Let $g : X \to Y$ be a map. A based map $f : X \to B$ is called *g-cocyclic* [7] if there is a map $\theta : X \to Y \vee B$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \vee B \\ \Delta \downarrow & & j \downarrow \\ X \times X & \xrightarrow{(g \times f)} & Y \times B \end{array}$$

, where $j : Y \vee B \to Y \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. We call such a map θ a *coassociated map* of a g-cocyclic map f.

In the case $g = 1_X : X \to X$, $f : X \to B$ is called *cocyclic* [9]. Clearly any cocyclic map is a g-cocyclic map and also $f : X \to B$ is g-cocyclic iff $g : X \to Y$ is f-cocyclic. The *dual Gottlieb set* $DG(X, g, Y; B)$ *of* $g : X \to Y$ is the set of all homotopy classes of g-cocyclic maps from X to B. In the case $g = 1_X : X \to X$, we called such a set $DG(X, 1_X, X; B)$ the *dual Gottlieb set* denoted $DG(X; B)$, that is, the dual Gottlieb set is exactly same with the dual Gottlieb set of the identity map. Haslam [3] introduced and studied the *coevaluation subgroups* $G^n(X; \pi) = DG(X; K(\pi, n))$ of $H^n(X; \pi)$. In particular, $DG(X, g, Y; K(\mathbb{Z}, n))$ and $DG(X; K(\mathbb{Z}, n))$ are denoted by $G^n(X, g, Y)$ and $G^n(X)$ respectively. The next proposition is an immediate consequence from the definition.

Proposition 2.1.
(1) *For any maps* $g : X \to Y$, $h : Y \to Z$, $G^n(X, g, Y) \subset G^n(X, hg, Z)$.
(2) $G^n(X) = G^n(X, 1_X, X) \subset G^n(X, g, Y) \subset G^n(X, *, Y) = H^n(X)$ *for any spaces* X *and any map* $g : X \to Y$.
(3) $G^n(X) = \cap\{G^n(X, g, Y) | g : X \to Y$ *is a map and* Y *is a space*$\}$.
(4) *If* $h : Y \to Z$ *is a homotopy equivalence, then* $G^n(X, g, Y) = G^n(X, hg, Z)$.

(5) *For any map* $k : A \to X$, $k^{\#}(G^n(X, g, Y)) \subset G^n(A, gk, Y)$.
(6) *For any map* $k : A \to X$, $k^{\#}(G^n(X)) \subset G^n(A, k, X)$.
(7) *For any map* $s : B \to C$, $s_{\#}(DG(X, g, Y; B)) \subset DG(X, g, Y; C)$.

It is well known [3] that $G^n(X;\pi)$ is a subgroup of $H^n(X;\pi)$. Moreover, it is also shown [5] that if B is an H-group, then $DG(X, B)$ is a subgroup of $[X, B]$.

COROLLARY 2.2.
(1) *If* $g : X \to Y$ *has a left homotopy inverse, then* $G^n(X, g, Y) = G^n(X)$ *is a subgroup of* $H^n(X)$.
(2) *If* $g : X \to Y$ *is a map such that* $G^n(X, g, Y) \subset g^{\#}(G^n(Y))$, *then* $G^n(X, g, Y) = g^{\#}(G^n(Y))$ *is a subgroup of* $H^n(X)$.

But we do not know whether $DG(X, g, Y; B)$ is a group. However, we have the following theorem.

THEOREM 2.3. [13] *Let* $g : X \to Y$ *be a map and* B *an* H*-group. Then*
(1) *For any* $[\gamma] \in g^{\#}(DG(Y; B))$ *and any* $[\alpha] \in DG(X, g, Y; B)$, $[\gamma]+[\alpha] \in DG(X, g, Y; B)$.
(2) *For any* $[\alpha] \in DG(X, g, Y; B)$, $-[\alpha] \in DG(X, g, Y; B)$.

COROLLARY 2.4. *For any map* $g : X \to Y$, *the group* $g^{\#}(G^n(Y;\pi))$ *acts on* $G^n(X, g, Y;\pi)$.

3. G'-sequence of a map

In [4], he introduced the *category of pairs* whose objects are maps from a pointed space to a pointed space and whose morphism from f to g is a pair of maps (α_1, α_2) such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha_1} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\alpha_2} & Y \end{array}$$

commutes. A homotopy of (α_1, α_2) is just a pair of homotopies $(\alpha_{1t}, \alpha_{2t})$ such that $g\alpha_{1t} = \alpha_{2t}f$. For a map $f : A \to X$, the homotopy group of the map f is defined [4] by $\pi_n(f) = \{[(\alpha_1, \alpha_2)] | (\alpha_1, \alpha_2) : i_n \to f\}$, where $i_n : S^{n-1} \to CS^{n-1}$ is the inclusion, and [ ] denotes the homotopy class. For an element $[(\alpha_1, \alpha_2)] \in \pi_n(f)$, if there exist $F_1 : A \times S^{n-1} \to A$ and

$F_2 : A \times CS^{n-1} \to X$ such that $F_1 j = \nabla(1 \vee \alpha_1)$ and $F_2 j' = \nabla(f \vee \alpha_2)$ and the diagram

$$\begin{array}{ccc} A \times S^{n-1} & \xrightarrow{F_1} & A \\ {\scriptstyle (1\times i_n)}\downarrow & & f\downarrow \\ A \times CS^{n-1} & \xrightarrow{F_2} & X \end{array}$$

commutes, then (F_1, F_2) is called an *affiliated map of* (α_1, α_2).

The relative evaluation subgroup $G_n^R(f)$ [8] is defined by $G_n^R(f) = \{[(\alpha_1, \alpha_2)] \in \pi_n(f) | \exists$ affiliated map (F_1, F_2) of $(\alpha_1, \alpha_2)\}$. Let $\bar{f} : (A^A, 1) \to (X^A, f)$ be a map given by $\bar{f}(g) = fg$. Consider the evaluation maps $\omega : A^A \to A$, and $\omega' : X^A \to X$. Then the map $(\omega, \omega') : \bar{f} \to f$ is called the evaluation maps in the homotopy category of pairs.

There is a commutative diagram

$$\begin{array}{ccccccccccc} \longrightarrow & \pi_n(A^A,1) & \xrightarrow{\bar{f}_\#} & \pi_n(X^A,f) & \xrightarrow{J} & \pi_n(\bar{f}) & \xrightarrow{\partial} & \pi_{n-1}(A^A,1) & \longrightarrow \\ & {\scriptstyle \omega_\#}\downarrow & & {\scriptstyle \omega'_\#}\downarrow & & {\scriptstyle (\omega,\omega')_\#}\downarrow & & {\scriptstyle \omega_\#}\downarrow & \\ \longrightarrow & G_n(A) & \xrightarrow{f_\#} & G_n(A,f,X) & \xrightarrow{J} & G_n^R(f) & \xrightarrow{\partial} & G_{n-1}(A) & \longrightarrow \\ & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & \\ \longrightarrow & \pi_n(A) & \xrightarrow{f_\#} & \pi_n(X) & \xrightarrow{J} & \pi_n(f) & \xrightarrow{\partial} & \pi_{n-1}(A) & \longrightarrow, \end{array}$$

where the top and the bottom rows are also exact and the middle sequence forms a chain complex. This middle sequence is called *the G-sequence of* f.

Now we would like to consider a dual situation of the above concept. For a map $g : X \to Y$, the cohomology group of the map g [4] is defined by $H^n(g) = \{[(\alpha_1, \alpha_2)] | (\alpha_1, \alpha_2) : g \to p_n\}$, where $p_n : PK(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$, $p_n(\eta) = \eta(1)$ is projection and $PK(\mathbb{Z}, n)$ is the space of paths in $K(\mathbb{Z}, n)$ beginning at $*$. In particular, if we take $g = incl : X \hookrightarrow Y$, then $H^n(g) = H^n(Y, X)$. Then there is the cohomology exact sequence of g

$$\to H^n(g) \xrightarrow{J} H^n(Y) \xrightarrow{g^*} H^n(X) \xrightarrow{\delta^*} H^{n+1}(g) \to .$$

Let $[(\alpha_1, \alpha_2)] \in H^n(g)$. If there are maps $\mu_1 : X \to Y \vee PK(\mathbb{Z}, n)$ and $\mu_2 : Y \to Y \vee K(\mathbb{Z}, n)$ such that $j'\mu_1 \sim (g \times \alpha_1)\Delta$ and $j\mu_2 \sim (1 \times \alpha_2)\Delta$, where $j' : Y \vee PK(\mathbb{Z}, n) \to Y \times PK(\mathbb{Z}, n)$, $j : Y \vee K(\mathbb{Z}, n) \to Y \times K(\mathbb{Z}, n)$ are the inclusions and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_1} & Y \vee PK(\mathbb{Z},n) \\ g\downarrow & & {\scriptstyle (1\vee p_n)}\downarrow \\ Y & \xrightarrow{\mu_2} & Y \vee K(\mathbb{Z},n) \end{array}$$

commutes, then (μ_1, μ_2) is called a *coaffiliated map of* (α_1, α_2) with respect to g.

DEFINITION 3.1. $G^n_R(g) = \{[(\alpha_1, \alpha_2)] \in H^n(g) | \exists$ *coaffiliated map* (μ_1, μ_2) *of* (α_1, α_2) *with respect to* $g\}$.

THEOREM 3.2. *There is a sequence* $\to G^n_R(g) \xrightarrow{J} G^n(Y) \xrightarrow{g^*} G^n(X, g, Y) \xrightarrow{\delta^*} G^{n+1}_R(g) \xrightarrow{J}$ *such that the following diagram is commutative*

$$\begin{array}{ccccccccccc} \longrightarrow & G^n_R(g) & \xrightarrow{J} & G^n(Y) & \xrightarrow{g^*} & G^n(X,g,Y) & \xrightarrow{\delta^*} & G^{n+1}_R(g) & \xrightarrow{J} \\ & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & \\ \longrightarrow & H^n(g) & \xrightarrow{J} & H^n(Y) & \xrightarrow{g^*} & H^n(X) & \xrightarrow{\delta^*} & H^{n+1}(g) & \xrightarrow{J} \end{array}$$

and $g^*J = 0$ *and* $\delta^* g^* = 0$ *and* $J\delta^* = 0$.

Proof. (1)To show that $J(G^n_R(g)) \subset G^n(Y)$, let $[(\alpha_1, \alpha_2)] \in G^n_R(g)$. Then there are maps $\mu_1 : X \to Y \vee PK(\mathbb{Z}, n)$ and $\mu_2 : Y \to Y \vee PK(\mathbb{Z}, n)$ such that $j'\mu_1 \sim (g \times \alpha_1)\Delta$ and $j\mu_2 \sim (1 \times \alpha_2)\Delta$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_1} & Y \vee PK(\mathbb{Z}, n) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (1\vee p_n)} \\ Y & \xrightarrow{\mu_2} & Y \vee K(\mathbb{Z}, n) \end{array}$$

commutes. Since $\alpha_2 : Y \to K(\mathbb{Z}, n)$ is cocyclic, $J([(\alpha_1, \alpha_2)]) = [\alpha_2] \in G^n(Y)$.

(2) It follows from Proposition 2.1(6) that $g^*(G^n(Y)) \subset G^n(X, g, Y)$.

(3) To show that $\delta^*(G^n(X, g, Y)) \subset G^{n+1}_R(g)$, let $[\alpha_1] \in G^n(X, g, Y)$. Then α_1 can be considered as follows;

$$\begin{array}{ccc} X & \xrightarrow{\alpha_1} & PK(\mathbb{Z}, n+1) \\ \downarrow & & \downarrow{\scriptstyle p_{n+1}} \\ * & \longrightarrow & K(\mathbb{Z}, n+1). \end{array}$$

Thus there is a map $\mu_1 : X \to Y \vee PK(\mathbb{Z}, n+1)$ such that $j'\mu_1 \sim (g \times \alpha_1)\Delta$. Define $\mu_2 : Y \to Y \vee PK(\mathbb{Z}, n+1)$ by $\mu_2(y) = (y, *)$. Then $j\mu_2 \sim (1 \times *)\Delta$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_1} & Y \vee PK(\mathbb{Z}, n+1) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (1\vee p_{n+1})} \\ Y & \xrightarrow{\mu_2} & Y \vee K(\mathbb{Z}, n+1) \end{array}$$

commutes. Thus we know that $\delta^*([\alpha_1]) =$

$$\begin{array}{ccccc} X & \xrightarrow{=} & X & \xrightarrow{\alpha_1} & PK(\mathbb{Z}, n+1) \\ {\scriptstyle g}\downarrow & & \downarrow & & {\scriptstyle p_{n+1}}\downarrow \\ Y & \longrightarrow & * & \longrightarrow & K(\mathbb{Z}, n+1). \end{array}$$

$= [(\alpha_1, *)] \in G_R^{n+1}(g)$.

Finally, the fact of $g^*J = 0$ and $\delta^* g^* = 0$ and $J\delta^* = 0$ follows from the exactness of the cohomology exact sequence

$$\to H^n(g) \xrightarrow{J} H^n(Y) \xrightarrow{g^*} H^n(X) \xrightarrow{\delta^*} H^{n+1}(g) \to .$$

□

Therefore we have, by the above theorem, a sequence $\to G_R^n(g) \xrightarrow{J} G^n(Y) \xrightarrow{g^*} G^n(X, g, Y) \xrightarrow{\delta^*} G_R^{n+1}(g) \xrightarrow{J}$ such that $g^*J = 0$ and $\delta^* g^* = 0$ and $J\delta^* = 0$. This sequence is called *the G'-sequence of g*. We still do not know whether the all sets in the G'-sequence of g are groups. However, we obtain some sufficient conditions for the all sets in the G'-sequence of g are groups.

LEMMA 3.3. [12]*$g : X \to Y$ is a cocyclic map if and only if $DG(X, g, Y; B) = [X, B]$ for any space B.*

Thus we know, from the above lemma, that if $g : X \to Y$ is a cocyclic map, then $G^n(X, g, Y) = H^n(X)$ for all n.

LEMMA 3.4. [5] *If $g : X \to Y$ is a cocyclic map and $g : X \to Y$ has a right homotopy inverse, then $1_Y : Y \to Y$ is a cocyclic map, that is, $DG(Y, B) = [Y, B]$ for any space B.*

Thus we know, from the above lemma, that if $g : X \to Y$ is a cocyclic map and $g : X \to Y$ has a right homotopy inverse, then $G^n(Y) = H^n(Y)$ for all n.

LEMMA 3.5. *If $g : X \to Y$ is a cocyclic map and $g : X \to Y$ has a right homotopy inverse, then $G_R^n(g) = H^n(g)$ for all n.*

Proof. We need only show that $H^n(g) \subset G_R^n(g)$. Let $[(\alpha_1, \alpha_2)] \in H^n(g)$. Then we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha_1} & PK(\mathbb{Z}, n) \\ {\scriptstyle g}\downarrow & & {\scriptstyle p_{n+1}}\downarrow \\ Y & \xrightarrow{\alpha_2} & K(\mathbb{Z}, n). \end{array}$$

Since $g : X \to Y$ is a cocyclic, there is a map $\mu : X \to Y \vee X$ such that $j\mu \sim (g \times 1)\Delta$. Let $f : Y \to X$ be a right homotopy inverse of g. Let $\mu_1 = (1 \vee \alpha_1)\mu : X \to Y \vee PK(\mathbb{Z}, n)$ and $\mu_2 = (1 \vee \alpha_2)(1 \vee g)\mu f : Y \to Y \vee K(\mathbb{Z}, n)$. Then we have that $j'\mu_1 \sim (g \times \alpha_1)\Delta$ and $j\mu_2 \sim (1 \times \alpha_2)\Delta$, and the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_1} & Y \vee PK(\mathbb{Z}, n) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (1\vee p_n)} \\ Y & \xrightarrow{\mu_2} & Y \vee K(\mathbb{Z}, n). \end{array}$$

Thus we know that $[(\alpha_1, \alpha_2)] \in G_R^n(g)$. □

Thus we have the following theorem.

THEOREM 3.6. *If $g : X \to Y$ is a cocyclic map and $g : X \to Y$ has a right homotopy inverse, then the the G'-sequence of g is the same as the cohomology exact sequence*

$$\to H^n(g) \xrightarrow{J} H^n(Y) \xrightarrow{g^*} H^n(X) \xrightarrow{\delta^*} H^{n+1}(g) \to$$

of g.

If $g : X \to Y$ has a right homotopy inverse $f : Y \to X$, then $f^*g^* = 1$ and g^* is a monomorphism. Thus we have the following corollary.

COROLLARY 3.7. *If $g : X \to Y$ is a cocyclic map and $g : X \to Y$ has a right homotopy inverse, then $G^n(X, g, Y) \cong G^n(Y) \oplus G_R^{n+1}(g)$ for all n.*

THEOREM 3.8. *If $g : X \to Y$ has a left homotopy inverse, then $G_R^n(g)$ is a subgroup of $H^n(g)$.*

Proof. Let $m' : PK(\mathbb{Z}, n) \times PK(\mathbb{Z}, n) \to PK(\mathbb{Z}, n)$ and $m : K(\mathbb{Z}, n) \times K(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$ be the H-structures on $PK(\mathbb{Z}, n)$ and $K(\mathbb{Z}, n)$ respectively. Let $\nu' : PK(\mathbb{Z}, n) \to PK(\mathbb{Z}, n)$ and $\nu : K(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$ be the inverses on $PK(\mathbb{Z}, n)$ and $K(\mathbb{Z}, n)$ respectively. Let $[(\alpha_1, \alpha_2)]$, $[(\beta_1, \beta_2)] \in G_R^n(g)$. Then there are maps $\mu_1 : X \to Y \vee PK(\mathbb{Z}, n)$ and $\mu_2 : Y \to Y \vee K(\mathbb{Z}, n)$ such that $j'\mu_1 \sim (g \times \alpha_1)\Delta$ and $j\mu_2 \sim (1 \times \alpha_2)\Delta$, where $j' : Y \vee PK(\mathbb{Z}, n) \to Y \times PK(\mathbb{Z}, n)$, $j : Y \vee K(\mathbb{Z}, n) \to Y \times K(\mathbb{Z}, n)$ are the inclusions, and the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_1} & Y \vee PK(\mathbb{Z}, n) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (1\vee p_n)} \\ Y & \xrightarrow{\mu_2} & Y \vee K(\mathbb{Z}, n), \end{array}$$

and maps $\theta_1 : X \to Y \vee PK(\mathbb{Z}, n)$ and $\theta_2 : Y \to Y \vee K(\mathbb{Z}, n)$ such that $j'\theta_1 \sim (g \times \beta_1)\Delta$ and $j\theta_2 \sim (1 \times \beta_2)\Delta$, where $j' : Y \vee PK(\mathbb{Z}, n) \to Y \times PK(\mathbb{Z}, n)$, $j : Y \vee K(\mathbb{Z}, n) \to Y \times K(\mathbb{Z}, n)$ are the inclusions, and the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta_1} & Y \vee PK(\mathbb{Z}, n) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (1\vee p_n)} \\ Y & \xrightarrow{\theta_2} & Y \vee K(\mathbb{Z}, n). \end{array}$$

Consider the maps $\mu_1' = (1 \vee \nu')\mu_1 : X \to Y \vee PK(\mathbb{Z}, n)$ and $\mu_2' = (1 \vee \nu)\mu_2 : Y \to Y \vee K(\mathbb{Z}, n)$. Then we can easily know that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu_1'} & Y \vee PK(\mathbb{Z}, n) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (1\vee p_n)} \\ Y & \xrightarrow{\mu_2'} & Y \vee K(\mathbb{Z}, n) \end{array}$$

commutes. Thus we know that $-[(\alpha_1, \alpha_2)] = [(\alpha_1\nu', \alpha_2\nu)] \in G^n_R(g)$. Now we show that $[(\alpha_1, \alpha_2)] + [(\beta_1, \beta_2)] = [(m'(\alpha_1 \times \beta_1)\Delta, m(\alpha_2 \times \beta_2)\Delta)] \in G^n_R(g)$. Let $f : Y \to X$ be the left homotopy inverse of $g : X \to Y$. Let $\lambda_1 = (1 \vee m')i'(\mu_1 \vee 1)(f \vee 1)\theta_1 : X \to Y \vee PK(\mathbb{Z}, n)$, $\lambda_2 = (1 \vee m)i(\mu_2 \vee 1)\theta_2 : Y \to Y \vee K(\mathbb{Z}, n)$, where $i' : Y \vee PK(\mathbb{Z}, n) \vee PK(\mathbb{Z}, n) \to Y \vee PK(\mathbb{Z}, n) \times PK(\mathbb{Z}, n)$, $i : Y \vee K(\mathbb{Z}, n) \vee K(\mathbb{Z}, n) \to Y \vee K(\mathbb{Z}, n) \times K(\mathbb{Z}, n)$ are the inclusions. Then we know that $j'\lambda_1 = (1 \times m')((g \times \alpha_1)\Delta \times 1)(f \times 1)(g \times \beta_1)\Delta \sim (g \times m'(\alpha_1 \times \beta_1)\Delta)\Delta \sim (g \times (\alpha_1 + \beta_1))\Delta$ and $j\lambda_2 = (1 \times m)((1 \times \alpha_2)\Delta \times 1)(1 \times \beta_2)\Delta = (1 \times m(\alpha_2 \times \beta_2)\Delta)\Delta = (1 \times (\alpha_2 + \beta_2))\Delta$, where $j' : Y \vee PK(\mathbb{Z}, n) \to Y \times PK(\mathbb{Z}, n)$, $j : Y \vee K(\mathbb{Z}, n) \to Y \times K(\mathbb{Z}, n)$ are the inclusions. Also, we can easily know that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda_1} & Y \vee PK(\mathbb{Z}, n) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (1\vee p_n)} \\ Y & \xrightarrow{\lambda_2} & Y \vee K(\mathbb{Z}, n) \end{array}$$

commutes. Thus we know that $[(\alpha_1, \alpha_2)] + [(\beta_1, \beta_2)] = [(m'(\alpha_1 \times \beta_1)\Delta, m(\alpha_2 \times \beta_2)\Delta)] \in G^n_R(g)$.

$\square$

Thus we know, from Corollary 2.2(1) and the above theorem, the following corollary.

Corollary 3.9. *If $g : X \to Y$ has a left homotopy inverse, then the all sets in the G'-sequence of g are groups.*

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