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STABILITY OF A SYSTEM OF FUNCTIONAL EQUATIONS ON JENSEN-QUADRATIC MAPPINGS

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Abstract. In this paper, we obtain the generalized Hyers-Ulam stability of a functional equation and a system of functional equations on Jensen-quadratic mappings.

1. Introduction

In 1940, Ulam [7] proposed the general Ulam stability problem:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d\big(h(xy), h(x)h(y)\big) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d\big(h(x), H(x)\big) < \varepsilon$ for all $x \in G_1$?

In 1941, this problem was solved by Hyers [4] in the case of Banach space. Thereafter, we call that type the Hyers-Ulam stability.

Throughout this paper, let X be a vector space and Y a Banach space. A mapping $g : X \to Y$ is called a *Jensen* (respectively, *quadratic*) mapping if g satisfies the functional equation $2g\big(\frac{x+y}{2}\big) = g(x) + g(y)$ (respectively, $g(x+y) + g(x-y) = 2g(x) + 2g(y)$).

Definition 1.1. A mapping $f : X \times X \to Y$ is called *Jensen-quadratic* if f satisfies the system of functional equations

$$(1.1) \qquad 2f\Big(\frac{x+y}{2}, z\Big) = f(x,z) + f(y,z),$$
$$f(x, y+z) + f(x, y-z) = 2f(x,y) + 2f(x,z).$$

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For a mapping $g : X \to Y$, consider the functional equation:

$$\begin{aligned} &3[g(x+y)+g(x-y)+6g(x)] \\ &\quad = 2g(2x+y)+2g(2x-y)+g(-x-y)+g(-x+y)+6g(-x). \end{aligned} \tag{1.2}$$

For a mapping $f : X \times X \to Y$, consider the functional equation:

$$\begin{aligned} &f\Big(\frac{x+y}{2}, z+w\Big) + f\Big(\frac{x+y}{2}, z-w\Big) \\ &\qquad = f(x,z)+f(x,w)+f(y,z)+f(y,w). \end{aligned} \tag{1.3}$$

The authors [2] investigated the relation between (1.2) and (1.3). And they proved that the functional equations (1.1) and (1.3) are equivalent. In this paper, we investigate the stability of (1.1) and (1.2). For more detailed definitions of the functional equation and the Hyers-Ulam stability, we refer the reader to [1, 3, 5, 6].

2. Stability of the functional equation (1.2)

Let $\chi : X \times X \to [0,\infty)$ be a function satisfying

$$\tilde{\chi}(x,y) := \sum_{j=0}^{\infty} \frac{2}{8^{j+2}}\big[\chi(2^j x, 0) + 8\chi(2^j x, 2^j y)\big] < \infty \tag{2.1}$$

for all $x, y \in X$.

Theorem 2.1. *Let $g : X \to Y$ be a mapping such that*

$$\begin{aligned} &\big\| 3[g(x+y)+g(x-y)+6g(x)] - 2g(2x+y) - 2g(2x-y) \\ &\qquad - g(-x-y) - g(-x+y) - 6g(-x) \big\| \le \chi(x,y) \end{aligned} \tag{2.2}$$

for all $x, y \in X$. If g satisfies

$$\|g(x)+g(-x)\| \le \chi(x,x) \tag{2.3}$$

for all $x \in X$, then there exists a cubic mapping $C : X \to Y$ such that

$$\|g(x) - C(x)\| \le \tilde{\chi}(x,x) \tag{2.4}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (2.2), we gain

$$\|6g(x) - g(2x) - 2g(-x)\| \le \frac{1}{4}\chi(x,0) \tag{2.5}$$

for all $x \in X$. By (2.3) and (2.5), we get $\|8g(x) - g(2x)\| \le \frac{1}{4}[\chi(x,0) + 8\chi(x,x)]$ for all $x \in X$. Thus we have $\left\|\frac{1}{8^j}g(2^jx) - \frac{1}{8^{j+1}}g(2^{j+1}x)\right\| \le \frac{2}{8^{j+2}}[\chi(2^jx,0) + 8\chi(2^jx,2^jx)]$ for all $x \in X$ and all nonnegative integers j. For given integers $l, m (0 \le l < m)$, we obtain

$$(2.6)\qquad \left\|\frac{1}{8^l}g(2^lx) - \frac{1}{8^m}g(2^mx)\right\| \le \sum_{j=l}^{m-1} \frac{2}{8^{j+2}}\big[\chi(2^jx,0) + 8\chi(2^jx,2^jx)\big]$$

for all $x \in X$. By (2.1), the sequence $\{\frac{1}{8^j}g(2^jx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{8^j}g(2^jx)\}$ converges for all $x \in X$. Define $C : X \to Y$ by $C(x) := \lim_{j\to\infty} \frac{1}{8^j}g(2^jx)$ for all $x \in X$. Setting $l = 0$ and taking $m \to \infty$ in (2.6), one can obtain the inequality (2.4). By (2.2), we see that

$$\begin{aligned}\frac{1}{8^j}\Big\| & 3\big[g(2^jx + 2^jy) + g(2^jx - 2^jy) + 6g(2^jx)\big] - 2g(2^{j+1}x + 2^jy)\\ & - 2g(2^{j+1}x - 2^jy) - g(-2^jx - 2^jy) - g(-2^jx + 2^jy) - 6g(-2^jx)\Big\|\\ & \le \frac{1}{8^j}\chi(2^jx, 2^jy)\end{aligned}$$

for all $x, y \in X$ and all nonnegative integers j. Letting $j \to \infty$ and using (2.1), we conclude that C is cubic. □

From now on, let $\chi : X \times X \to [0,\infty)$ be a function satisfying

$$(2.7)\qquad \tilde{\chi}(x,y) := \sum_{j=0}^{\infty} \frac{1}{4^{j+2}}\big[\chi(2^jx,0) + 8\chi(2^jx,2^jy)\big] < \infty$$

for all $x, y \in X$.

THEOREM 2.2. *Let $g : X \times X \to Y$ be a mapping such that* (2.2) *for all $x, y \in X$. If g satisfies*

$$(2.8)\qquad \|g(x) - g(-x)\| \le \chi(x,x)$$

for all $x \in X$, then there exists a quadratic mapping $Q : X \to Y$ such that

$$(2.9)\qquad \|g(x) - Q(x)\| \le \tilde{\chi}(x,x)$$

for all $x \in X$.

Proof. By (2.5) and (2.8), we gain $\left\|g(x) - \frac{1}{4}g(2x)\right\| \le \frac{1}{4^2}[\chi(x,0) + 8\chi(x,x)]$ for all $x \in X$. Thus we get $\left\|\frac{1}{4^j}g(2^jx) - \frac{1}{4^{j+1}}g(2^{j+1}x)\right\| \le$

$\frac{1}{4^{j+2}}[\chi(2^jx,0)+8\chi(2^jx,2^jx)]$ for all $x \in X$ and all nonnegative integers j. For given integers $l, m (0 \le l < m)$, we have

$$\left\|\frac{1}{4^l}g(2^lx)-\frac{1}{4^m}g(2^mx)\right\| \le \sum_{j=l}^{m-1}\frac{1}{4^{j+2}}\left[\chi(2^jx,0)+8\chi(2^jx,2^jx)\right] \tag{2.10}$$

for all $x \in X$. By (2.7), the sequence $\{\frac{1}{4^j}g(2^jx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^j}g(2^jx)\}$ converges for all $x \in X$. Define $Q : X \to Y$ by $Q(x) := \lim_{j\to\infty}\frac{1}{4^j}g(2^jx)$ for all $x \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.10), one can obtain the inequality (2.9). By (2.2), we obtain

$$\begin{aligned}\frac{1}{4^j}\Big\| & 3\big[g(2^jx+2^jy)+g(2^jx-2^jy)+6g(2^jx)\big]-2g(2^{j+1}x+2^jy)\\ & -2g(2^{j+1}x-2^jy)-g(-2^jx-2^jy)-g(-2^jx+2^jy)-6g(-2^jx)\Big\|\\ & \le \frac{1}{4^j}\chi(2^jx,2^jy)\end{aligned}$$

for all $x, y \in X$ and all nonnegative integers j. Letting $j \to \infty$ and using (2.7), we see that Q is quadratic. □

3. Stability of the system of functional equations (1.1)

Let $\varphi : X \times X \times X \to [0,\infty)$ and $\psi : X \times X \times X \to [0,\infty)$ be two functions satisfying

$$\begin{aligned}\tilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty}\Big(&\frac{1}{3^{j+1}}\left[\varphi(3^jx,-3^jy,z)+\varphi(-3^jx,3^{j+1}y,z)\right]\\ &+\frac{1}{4^j}\varphi(x,y,2^jz)\Big)+\|f(0,z)\| < \infty\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\tilde{\psi}(x,y,z) := \sum_{j=0}^{\infty}&\left[\frac{1}{3^j}\psi(3^jx,y,z)+\frac{1}{4^{j+1}}\psi(x,2^jy,2^jz)\right]\\ &+\frac{1}{3}\|f(x,0)\| < \infty\end{aligned} \tag{3.2}$$

for all $x, y, z \in X$.

THEOREM 3.1. *Let $f : X \times X \to Y$ be a mapping such that*

$$\Big\|2f\Big(\frac{x+y}{2}, z\Big) - f(x,z) - f(y,z)\Big\| \le \varphi(x,y,z), \tag{3.3}$$

$$\big\|f(x,y+z) + f(x,y-z) - 2f(x,y) - 2f(x,z)\big\| \le \psi(x,y,z) \tag{3.4}$$

for all $x, y, z \in X$. Then there exist two Jensen-quadratic mappings $F, G : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \le \tilde{\varphi}(x,x,y), \tag{3.5}$$

$$\|f(x,y) - G(x,y)\| \le \tilde{\psi}(x,y,y) \tag{3.6}$$

for all $x, y \in X$.

Proof. Letting $y = -x$ in (3.3), we gain $\|2f(0,z) - f(x,z) - f(-x,z)\| \le \varphi(x,-x,z)$ for all $x, z \in X$. Replacing x by $-x$ and y by $3x$ in (3.3), we get $\|2f(x,z) - f(-x,z) - f(3x,z)\| \le \varphi(-x,3x,z)$ for all $x, z \in X$. By the above two inequalities, we have $\big\|f(x,z) - \frac{1}{3}f(3x,z)\big\| \le \frac{1}{3}\big[\varphi(x,-x,z) + \varphi(-x,3x,z) + 2\|f(0,z)\|\big]$ for all $x, z \in X$. Thus we obtain

$$\begin{aligned}&\Big\|\frac{1}{3^j}f(3^j x, z) - \frac{1}{3^{j+1}}f(3^{j+1}x, z)\Big\| \\ &\le \frac{1}{3^{j+1}}\big[\varphi(3^j x, -3^j x, z) + \varphi(-3^j x, 3^{j+1}x, z) + 2\|f(0,z)\|\big]\end{aligned}$$

for all $x, z \in X$ and all nonnegative integers j. Replacing z by y, we gain

$$\begin{aligned}&\Big\|\frac{1}{3^j}f(3^j x, y) - \frac{1}{3^{j+1}}f(3^{j+1}x, y)\Big\| \\ &\qquad \le \frac{1}{3^{j+1}}\big[\varphi(3^j x, -3^j x, y) + \varphi(-3^j x, 3^{j+1}x, y) + 2\|f(0,y)\|\big]\end{aligned}$$

for all $x, y \in X$ and all nonnegative integers j. For given integers $l, m (0 \le l < m)$, we get

$$\begin{aligned}&\Big\|\frac{1}{3^l}f(3^l x, y) - \frac{1}{3^m}f(3^m x, y)\Big\| \\ &\le \sum_{j=l}^{m-1}\frac{1}{3^{j+1}}\big[\varphi(3^j x, -3^j x, y) + \varphi(-3^j x, 3^{j+1}x, y) + 2\|f(0,y)\|\big]\end{aligned} \tag{3.7}$$

for all $x, y \in X$. By (3.1), the sequence $\{\frac{1}{3^j}f(3^j x, y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{3^j}f(3^j x, y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by $F(x,y) := \lim_{j\to\infty}\frac{1}{3^j}f(3^j x, y)$ for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (3.7), one can obtain the inequality (3.5). By (3.3) and (3.4), we have $\frac{1}{3^j}\left\|2f\left(\frac{3^jx+3^jy}{2}, z\right) - f(3^jx, z) - f(3^jy, z)\right\| \le \frac{1}{3^j}\varphi(3^jx, 3^jy, z)$ and $\frac{1}{3^j}\|f(3^jx, y+z)+f(3^jx, y-z)-2f(3^jx, y)-2f(3^jx, z)\| \le \frac{1}{3^j}\psi(3^jx, y, z)$ for all $x, y, z \in X$ and all nonnegative integers j. Letting $j \to \infty$ and using (3.1) and (3.2), we obtain that F is Jensen-quadratic. Next, setting $z = y$ in (3.4), we gain $\left\|f(x, y) - \frac{1}{4}f(x, 2y)\right\| \le \frac{1}{4}[\psi(x, y, y) + \|f(x, 0)\|]$ for all $x, y \in X$. Thus we get $\left\|\frac{1}{4^j}f(x, 2^jy)-\frac{1}{4^{j+1}}f(x, 2^{j+1}y)\right\| \le \frac{1}{4^{j+1}}[\,\psi(x, 2^jy, 2^jy) + \|f(x, 0)\|\,]$ for all $x, y \in X$ and all nonnegative integers j. For given integer $l, m (0 \le l < m)$, we have

$$\begin{aligned} &\left\|\frac{1}{4^l}f(x, 2^ly) - \frac{1}{4^m}f(x, 2^my)\right\| \\ &\le \sum_{j=l}^{m-1} \frac{1}{4^{j+1}}\left[\psi(x, 2^jy, 2^jy) + \|f(x, 0)\|\right] \end{aligned} \tag{3.8}$$

for all $x, y \in X$. By (3.2), the sequence $\{\frac{1}{4^j}f(x, 2^jy)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{4^j}f(x, 2^jy)\}$ converges for all $x, y \in X$. Define $G : X \times X \to Y$ by $G(x, y) := \lim_{j\to\infty} \frac{1}{4^j}f(x, 2^jy)$ for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (3.8), one can obtain the inequality (3.6). By (3.3) and (3.4), we obtain $\frac{1}{4^j}\left\|2f\left(\frac{x+y}{2}, 2^jz\right) - f(x, 2^jz) - f(y, 2^jz)\right\| \le \frac{1}{4^j}\varphi(x, y, 2^jz)$ and

$$\begin{aligned} &\frac{1}{4^j}\left\|f(x, 2^jy + 2^jz) + f(x, 2^jy - 2^jz) - 2f(x, 2^jy) - 2f(x, 2^jz)\right\| \\ &\le \frac{1}{4^j}\psi(x, 2^jy, 2^jz) \end{aligned}$$

for all $x, y, z \in X$ and all nonnegative integers j. Letting $j \to \infty$ and using (3.1) and (3.2), we see that G is Jensen-quadratic. □

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