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ON SOME CHRACTERIZATIONS OF THE WEIBULL DISTRIBUTION

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Abstract. In this paper, we establish some characterizations which is satisfied by the independence of the upper record values from the Weibull distribution. One characterization of several results is that $X \in WEI(1,\alpha)$, $\alpha > 0$, if and only if $\frac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$, $1 \le m < n$ are independent.

1. Introduction

Let $X_1,\ X_2, \cdots$ be a sequence of independent and identically distributed (i.i.d.) random variables with a cumulative distribution function (cdf) $F(x)$ and a probability density function (pdf) $f(x)$. Suppose $Y_n = \max\{X_1, X_2, \cdots, X_n\}$ for $n \ge 1$. We say X_j is an upper record value of this sequence, if $Y_j > Y_{j-1}$ for $j > 1$. And we suppose that X_1 is a first upper record value. The indices at which the upper record values occur are given by the record times $\{U(n),\ n \ge 1\}$, where $U(n) = \min\{j | j > U(n-1)\,,\ X_j > X_{U(n-1)}\,,\ n \ge 2\}$ with $U(1) = 1$.

A continuous random variable X is said to have the Weibull distribution with the shape parameter $\theta > 0$ and the scale parameter $\alpha > 0$ if it has a cdf $F(x)$ of the form

$$F(x) = \begin{cases} 1 - e^{-\left(\frac{x}{\theta}\right)^{\alpha}},\ x > 0,\ \theta > 0,\ \alpha > 0 \\ 0, \text{ otherwise} \end{cases} \tag{1}$$

A notation that designates that X has the cdf (1) is $X \in WEI(\theta, \alpha)$.

Some characterizations by the independence of the upper record values are known. In [2] and [3], Ahsanullah studied, if $X \in WEI(\theta, \alpha)$,

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$\theta > 0$, $\alpha > 0$, then $\frac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$, $0 < m < n$ are independent. And Ahsanullah proved, if $X \in PAR(\alpha, \beta)$, $\alpha > 0$, $\beta > 0$, then $\frac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$, $0 < m < n$ are independent. In [4], Chang characterized that $X \in PAR(1, \beta)$, $\beta > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n)} - X_{U(n+1)}}$ and $X_{U(n)}$ or $\frac{X_{U(n)} - X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$, $n \geq 1$ are independent and $X \in PAR(1, \beta)$, $\beta > 0$, if and only if $\frac{X_{U(n)} - X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$, $n \geq 1$ are independent.

In this paper, we will give some characterizations of the Weibull distribution with the shape parameter $\theta = 1$ by the independence of the upper record values.

2. Main theorems

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$, $x > 0$. Then $F(x) = 1 - e^{-x^\alpha}$, $x > 0$, $\alpha > 0$, if and only if $\frac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$, $1 \leq m < n$ are independent.*

Proof. If $F(x) = 1 - e^{-x^\alpha}$, $x > 0$, $\alpha > 0$, then the joint pdf $f_{m,n}(x, y)$ of $X_{U(m)}$ and $X_{U(n)}$ is

$$f_{m,n}(x, y) = \frac{\alpha^2 \, x^{m\alpha - 1} (y^\alpha - x^\alpha)^{n-m-1} y^{\alpha - 1} e^{-y^\alpha}}{\Gamma(m)\Gamma(n-m)}$$

for $0 < x < y$, $\alpha > 0$ and $1 \leq m < n$.

Consider the functions $V = \frac{X_{U(m)}}{X_{U(n)}}$ and $W = X_{U(n)}$. It follows that $x_{U(m)} = vw$, $x_{U(n)} = w$ and $|J| = w$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{\alpha^2 \, v^{m\alpha - 1} (1 - v^\alpha)^{n-m-1} w^{n\alpha - 1} e^{-w^\alpha}}{\Gamma(m)\Gamma(n-m)} \tag{2}$$

for $0 < v < 1$, $w > 0$, $\alpha > 0$ and $1 \leq m < n$.

The marginal pdf $f_V(v)$ of V is given by

$$f_V(v) = \int_0^\infty f_{V,W}(v, w) \, dw = \frac{\Gamma(n) \, \alpha \, v^{m\alpha - 1} (1 - v^\alpha)^{n-m-1}}{\Gamma(m)\Gamma(n-m)} \tag{3}$$

for $0 < v < 1$, $\alpha > 0$ and $1 \leq m < n$.

Also, the pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^{n-1} f(w)}{\Gamma(n)} = \frac{\alpha\, w^{n\alpha-1} e^{-w^\alpha}}{\Gamma(n)} \tag{4}$$

for $w > 0$, $\alpha > 0$ and $n \geq 1$.

From (2), (3) and (4), we obtain $f_V(v) f_W(w) = f_{V,W}(v, w)$.

Hence $V = \frac{X_{U(m)}}{X_{U(n)}}$ and $W = X_{U(n)}$ are independent for $1 \leq m < n$.

Now we will prove the sufficient condition. The joint pdf $f_{m,n}(x, y)$ of $X_{U(m)}$ and $X_{U(n)}$ is

$$f_{m,n}(x, y) = \frac{(R(x))^{m-1} r(x) (R(y) - R(x))^{n-m-1} f(y)}{\Gamma(m)\Gamma(n-m)}$$

for $0 < x < y$ and $1 \leq m < n$, where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}\big(R(x)\big) = \frac{f(x)}{1-F(x)}$.

Let us use the transformation $V = \frac{X_{U(m)}}{X_{U(n)}}$ and $W = X_{U(n)}$. The Jacobian of the transformation is $|J| = w$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{(R(vw))^{m-1} r(vw) (R(w) - R(vw))^{n-m-1} f(w)\, w}{\Gamma(m)\Gamma(n-m)} \tag{5}$$

for $0 < v < 1$, $w > 0$ and $1 \leq m < n$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{(R(w))^{n-1} f(w)}{\Gamma(n)} \tag{6}$$

for $w > 0$ and $n \geq 1$.

From (5) and (6), since V and $W$are independent, we must have

$$f_V(v) = \frac{\Gamma(n)(R(vw))^{m-1} r(vw)(R(w) - R(vw))^{n-m-1}\, w}{\Gamma(m)\Gamma(n-m)(R(w))^{n-1}}$$

for $0 < v < 1$, $w > 0$ and $1 \leq m < n$.

That is,

$$f_V(v) = \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left(\frac{R(w) - R(vw)}{R(w)}\right)^{n-m-1} \left(\frac{R(vw)}{R(w)}\right)^{m-1} \frac{r(vw)\, w}{R(w)}$$

where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}\big(R(x)\big) = \frac{f(x)}{1-F(x)}$.

To proof the sufficient condition, the pdf $f_V(v)$ is a function of v only, we need the following lemma.

Lemma 2.2. *Let $F(x)$ be an absolutely continuous function and $F(x) > 0$, $x > 0$. Suppose that $\frac{R(w)-R(vw)}{R(w)} = q(v,w)$ and $k(v,w) = (q(v,w))^r (1-q(v,w))^s \frac{\partial}{\partial v}q(v,w)$ for $r,s \geq 0$, $k(v,w) \neq 0$ and $\frac{\partial}{\partial v}q(v,w) \neq 0$ for any positive v and w. If $k(v,w)$ is independent of w, then $q(v,w)$ is a function of v only.*

Proof. Since $k(v,w)$ is independent of w, we can put $k(v,w) = g(v)$. Then we have

$$\begin{aligned} g(v) = k(v,w) &= (q(v,w))^r (1-q(v,w))^s \frac{\partial}{\partial v}q(v,w) \\ &= \Sigma_{i=0}^{s} \binom{s}{i} (-1)^{s-i} (q(v,w))^{r+s-i} \frac{\partial}{\partial v}q(v,w). \end{aligned}$$

Integrating the above equation for v, we get

$$G(v) + c = \Sigma_{i=0}^{s} \binom{s}{i} (-1)^{s-i} \frac{1}{(r+s-i+1)} (q(v,w))^{r+s-i+1}. \tag{7}$$

Here $G(v)+c$ is a function of v only and c is independent of v but may depend on w. Now letting $v \to 1$, we see that $q(v,w) \to 0$ and hence from (7), we have c as independent of w.

Therefore we obtain

$$\begin{aligned} &\frac{\partial}{\partial w}(G(v)+c) = \Sigma_{i=0}^{s} \binom{s}{i} (-1)^{s-i} (q(v,w))^{r+s-i} \frac{\partial}{\partial w}q(v,w) \\ &= \Sigma_{i=0}^{s} \binom{s}{i} (-1)^{s-i} (q(v,w))^{r+s-i} \frac{\partial}{\partial v}q(v,w) \frac{\partial}{\partial w}q(v,w) / \left(\frac{\partial}{\partial v}q(v,w)\right) \\ &= g(v) \left(\frac{\partial}{\partial v}q(v,w)\right)^{-1} \frac{\partial}{\partial w}q(v,w) = 0. \end{aligned}$$

Since $g(v) = k(v,w) \neq 0$ and $\frac{\partial}{\partial v}q(v,w) \neq 0$, we get

$$\frac{\partial}{\partial w}q(v,w) = 0.$$

Hence $q(v,w)$ is a function of v only.

This completes the proof of lemma. □

We now turn to the proof of Theorem 2.1. By Lemma 2.2, we have

$$\frac{R(w)-R(vw)}{R(w)} = c_1(v),$$

where $c_1(v)$ is a function of v only. Thus

$$R(vw) = c(v)R(w), \tag{8}$$

where $c(v)$ is a function of v only.

By the functional equations (see, [1]), the only nonconstant continuous solution of (8) with the boundary condition $R(0) = 0$ is

$$R(x) = x^{\alpha}$$

for $x > 0$ and $\alpha > 0$. Thus we have

$$F(x) = 1 - e^{-x^{\alpha}}$$

for $x > 0$ and $\alpha > 0$.

This completes the proof. □

In the result of Theorem 2.1, let $\alpha = 1$. Then $F(x)$ is the standard exponential distribution. Also, let $n = m + 1$. Then the record times of the record values are connected immediately. Hence we obtain Remark 2.1 and Remark 2.2.

Remark 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$, $x > 0$. Then $F(x) = 1 - e^{-x}$, $x > 0$, if and only if $\frac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$, $1 \leq m < n$ are independent.

Remark 2.2. Let $\{X_m, m \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$, $x > 0$. Then $F(x) = 1 - e^{-x^{\alpha}}$, $x > 0$, $\alpha > 0$, if and only if $\frac{X_{U(m)}}{X_{U(m+1)}}$ and $X_{U(m+1)}$, $m \geq 1$ are independent.

In the result of Remark 2.2, let $\frac{X_{U(m)}}{X_{U(m+1)}} = Y$. Then we can write $\frac{X_{U(m+1)} - X_{U(m)}}{X_{U(m+1)}} = 1 - Y$ and $\frac{X_{U(m+1)} + X_{U(m)}}{X_{U(m+1)}} = 1 + Y$. By the property of independence, $\frac{X_{U(m+1)} - X_{U(m)}}{X_{U(m+1)}}$ and $X_{U(m+1)}$ are independent for $m \geq 1$. Also, $\frac{X_{U(m+1)} + X_{U(m)}}{X_{U(m+1)}}$ and $X_{U(m+1)}$ are independent for $m \geq 1$. Hence we get Remark 2.3 and Remark 2.4.

Remark 2.3. Let $\{X_m, m \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for $x > 0$. Then $F(x) = 1 - e^{-x^{\alpha}}$ for $x > 0$ and $\alpha > 0$ if and only if $\frac{X_{U(m+1)} - X_{U(m)}}{X_{U(m+1)}}$ and $X_{U(m+1)}$ are independent for $m \geq 1$.

REMARK 2.4. Let $\{X_m, m \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$, $x > 0$. Then $F(x) = 1 - e^{-x^{\alpha}}$, $x > 0$, $\alpha > 0$, if and only if $\frac{X_{U(m+1)} + X_{U(m)}}{X_{U(m+1)}}$ and $X_{U(m+1)}$, $m \geq 1$ are independent.

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