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THAINE'S THEOREM IN FUNCTION FIELD

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Abstract. Let F be a finite real abelian extension of a global function field k with $G = \mathrm{Gal}(\mathrm{F}/\mathrm{k})$. Assume that F is an extension field of the Hilbert class field $\mathrm{K}_{\mathfrak{e}}$ of k and is contained in a cyclotomic function field $\mathrm{K}_{\mathfrak{n}}$. Let ℓ be any prime number not dividing $ph_{\mathrm{k}}|G|$. In this paper, we show that if $\theta \in \mathbb{Z}[G]$ annihilates the Sylow ℓ-subgroup of $\mathcal{O}_{\mathrm{F}}^{\times}/\mathcal{C}_{\mathrm{F}}$, then $(q-1)\theta$ annihilates the Sylow ℓ-subgroup of $\mathcal{C}l_{\mathrm{F}}$.

1. Introduction

Let F be a totally real abelian number field with $G = \mathrm{Gal}(\mathrm{F}/\mathbb{Q})$. Let $\mathcal{O}_{\mathrm{F}}^{\times}$ be the group of global units of F, $\mathcal{C}_{\mathrm{F}}$ the group of cyclotomic units of F of conductor level and $\mathcal{C}l_{\mathrm{F}}$ the class group of F. Let p be a prime not dividing $[\mathrm{F} : \mathbb{Q}]$. In [4], Thaine has shown the following remarkable result: *If $\theta \in \mathbb{Z}[G]$ annihilates the Sylow p-subgroup of $\mathcal{O}_{\mathrm{F}}^{\times}/\mathcal{C}_{\mathrm{F}}$, then 2θ annihilates the Sylow p-subgroup of $\mathcal{C}l_{\mathrm{F}}$.*

In this paper we consider the analogous problem in function fields. Let k be a global function field over the finite field $\mathbb{F}_q$ with q elements of characteristic p. Fix a place ∞ of k of degree one and a sign function $\mathrm{sgn} : \mathrm{k}_{\infty} \to \mathbb{F}_q$ with $\mathrm{sgn}(0) = 0$, where k_{∞} is the completion of k at ∞. Let $\mathbb{A}$ be the Dedekind subring of k consisting of the functions regular away from ∞. For any finite separable extension F of k, write $\mathcal{O}_{\mathrm{F}}$ for the integral closure of $\mathbb{A}$ in F and $\mathcal{O}_{\mathrm{F}}^{\times}$ be its group of units. Also write $\mathcal{C}l_{\mathrm{F}}$ for the ideal class group of $\mathcal{O}_{\mathrm{F}}$ and $h_{\mathrm{F}} = |\mathcal{C}l_{\mathrm{F}}|$. In this paper we only consider finite abelian extension F of k contained in a cyclotomic function field of $(\mathrm{k}, \infty, \mathrm{sgn})$. Let F be a finite real abelian extension of k, i.e., ∞ splits completely in F, with $G = \mathrm{Gal}(\mathrm{F}/\mathrm{k})$. We assume that F contains the Hilbert class field of (k, ∞). Let $\mathcal{C}_{\mathrm{F}}$ be the group

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of cyclotomic units of F in the sense of [1] (see the definition 2.2). The main result of this paper is

Theorem 1.1. *Let ℓ be a prime not dividing $\ell \nmid ph_{\mathrm{k}}|G|$. If $\theta \in \mathbb{Z}[G]$ annihilates the Sylow ℓ-subgroup of $\mathcal{O}_{\mathrm{F}}^{\times}/\mathcal{C}_{\mathrm{F}}$, then $(q-1)\theta$ annihilates the Sylow ℓ-subgroup of $\mathcal{C}l_{\mathrm{F}}$.*

For any proper integral ideal of $\mathfrak{n}$ of $\mathbb{A}$, we write $\Phi(\mathfrak{n}) = |(\mathbb{A}/\mathfrak{n})^{\times}|$ and denote by $\mathrm{K}_{\mathfrak{n}}$ the cyclotomic function field of $(\mathrm{k}, \infty, \mathrm{sgn})$ of conductor $\mathfrak{n}$. Moreover we write $\mathrm{K}_{\mathfrak{e}}$ for the Hilbert class field of (k, ∞). For details in the theory of sgn-normalized Drinfeld $\mathbb{A}$-module and cyclotomic function field over the global function field k, we refer the readers to [2] or [3].

2. Preliminary

2.1. Cyclotomic units

Let ρ be a fixed sgn-normalized Drinfeld $\mathbb{A}$-module. For any proper ideal $\mathfrak{n}$ of $\mathbb{A}$, we fix a primitive $\mathfrak{n}$-torsion point $\lambda_{\mathfrak{n}}$ of ρ.

Lemma 2.1. *Let $\mathfrak{n} = \mathfrak{p}\mathfrak{f}$, where $\mathfrak{f} \neq \mathfrak{e}$ and $\mathfrak{p}$ is a prime ideal of $\mathbb{A}$. Then we have*

$$N_{\mathrm{K}_{\mathfrak{n}}/\mathrm{K}_{\mathfrak{f}}}(\lambda_{\mathfrak{n}}) = \begin{cases} \rho_{\mathfrak{p}}(\lambda_{\mathfrak{n}}), & \text{if } \mathfrak{p}|\mathfrak{f}, \\ \rho_{\mathfrak{p}}(\lambda_{\mathfrak{n}})^{1-\sigma_{\mathfrak{p}}^{-1}}, & \text{if } \mathfrak{p} \nmid \mathfrak{f}, \end{cases}$$

where $\sigma_{\mathfrak{p}}$ denotes the Frobenius automorphism of $\mathfrak{p}$ in $\mathrm{K}_{\mathfrak{f}}$.

Let F be a finite real abelian extension of k. Let $\mathfrak{m}$ be the conductor of F, i.e. $\mathrm{K}_{\mathfrak{m}}$ is the smallest cyclotomic function field containing F. For each proper ideal $\mathfrak{f}$ of $\mathbb{A}$, put $\mathrm{F}_{\mathfrak{f}} = \mathrm{F} \cap \mathrm{K}_{\mathfrak{f}}$ and $\lambda_{\mathfrak{f},\mathrm{F}} = N_{\mathrm{K}_{\mathfrak{f}}/\mathrm{F}_{\mathfrak{f}}}(\lambda_{\mathfrak{f}})$.

Definition 2.2. Let $\mathcal{D}_{\mathrm{F}}$ be the G-submodule of $\mathrm{F}^{\times}$ generated by $\mathbb{F}_q^{\times}$ and $\lambda_{\mathfrak{f},\mathrm{F}}$ with all $\mathfrak{f} \neq \mathfrak{e}$. We define $\mathcal{C}_{\mathrm{F}} := \mathcal{D}_{\mathrm{F}} \cap \mathcal{O}_{\mathrm{F}}^{\times}$, called the *group of cyclotomic units of* F.

As a G-module, $\mathcal{D}_{\mathrm{F}}$ is generated by $\mathbb{F}_q^{\times} \cup \{\lambda_{\mathfrak{f},\mathrm{F}} : \mathfrak{e} \neq \mathfrak{f}|\mathfrak{m}\} \cup \{N_{\mathrm{K}_{\mathfrak{p}}/\mathrm{K}_{\mathfrak{e}}}(\lambda_{\mathfrak{p}})\}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all prime ideals of $\mathbb{A}$ such that $\mathfrak{p} \nmid \mathfrak{m}$.

For any prime ideal $\mathfrak{q}$ of $\mathbb{A}$ which splits completely in F, write $\mathrm{F}(\mathfrak{q}) = \mathrm{F} \cdot \mathrm{K}_{\mathfrak{q}}$. Since all prime ideals of $\mathcal{O}_{\mathrm{F}}$ above $\mathfrak{q}$ are totally ramified in $\mathrm{F}(\mathfrak{q})$, G acts on the set of prime ideals of $\mathcal{O}_{\mathrm{F}(\mathfrak{q})}$ above $\mathfrak{q}$.

Proposition 2.3. *For any unit $\varepsilon \in \mathcal{C}_{\mathrm{F}}$, there exists $u \in \mathcal{O}_{\mathrm{F}(\mathfrak{q})}^{\times}$ such that $N_{\mathrm{F}(\mathfrak{q})/\mathrm{F}}(u) = 1$ and $u \equiv \varepsilon \bmod \widetilde{\mathfrak{Q}}^{\sigma}$ for all $\sigma \in G$.*

Proof. It suffices to show that for any $\varepsilon \in \mathbb{F}_q^\times \cup \{\lambda_{\mathfrak{f},\mathrm{F}} : \mathfrak{e} \neq \mathfrak{f}|\mathfrak{m}\} \cup \{N_{\mathrm{K}_\mathfrak{p}/\mathrm{K}_\mathfrak{e}}(\lambda_\mathfrak{p})\}_\mathfrak{p}$, where $\mathfrak{p}$ runs over all prime ideals of $\mathbb{A}$ such that $\mathfrak{p} \nmid \mathfrak{m}$ and $\mathfrak{p} \neq \mathfrak{q}$, there exists $u \in \mathcal{O}_{\mathrm{F}(\mathfrak{q})}^\times$ satisfying the required conditions. For $\varepsilon \in \mathbb{F}_q^\times$, it is easy to see that $u = \varepsilon$ satisfies the required ones. For $\varepsilon = \lambda_{\mathfrak{f},\mathrm{F}}$ with $\mathfrak{e} \neq \mathfrak{f}|\mathfrak{m}$, take $u = \prod_{\tau \in \mathrm{Gal}(\mathrm{K}_\mathfrak{f}/\mathrm{F}_\mathfrak{f})}(\lambda_\mathfrak{f}^\tau + \lambda_\mathfrak{q})$. Then $u \in \mathrm{F}_\mathfrak{f}(\mathfrak{q}) \subset \mathrm{F}(\mathfrak{q})$, and so $u \in \mathcal{O}_{\mathrm{F}(\mathfrak{q})}^\times$. Since $N_{\mathrm{F}(\mathfrak{q})/\mathrm{F}}(u) = N_{\mathrm{K}_{\mathfrak{f}\mathfrak{q}}/\mathrm{K}_\mathfrak{f}}(u)$, we have

$$N_{\mathrm{F}(\mathfrak{q})/\mathrm{F}}(u) = \prod_{\tau \in \mathrm{Gal}(\mathrm{K}_\mathfrak{f}/\mathrm{F}_\mathfrak{f})} \left(\frac{\rho_\mathfrak{q}(\lambda_\mathfrak{f}^\tau)}{\lambda_\mathfrak{f}^\tau} \right) = N_{\mathrm{K}_\mathfrak{f}/\mathrm{F}_\mathfrak{f}}(\lambda_\mathfrak{f})^{\sigma_\mathfrak{q}-1} = 1.$$

Since $\lambda_\mathfrak{q}$ is contained in $\widetilde{\mathfrak{Q}}^\sigma$ for all $\sigma \in G$, $u \equiv \prod_{\tau \in \mathrm{Gal}(\mathrm{K}_\mathfrak{f}/\mathrm{F}_\mathfrak{f})} \lambda_\mathfrak{f}^\tau = \lambda_{\mathfrak{f},\mathrm{F}} \bmod \widetilde{\mathfrak{Q}}^\sigma$. For $\varepsilon = N_{\mathrm{K}_\mathfrak{p}/\mathrm{K}_\mathfrak{e}}(\lambda_\mathfrak{p})$ with $\mathfrak{p} \nmid \mathfrak{m}, \mathfrak{p} \neq \mathfrak{q}$, similar as above, we can show that $u = \prod_{\tau \in \mathrm{Gal}(\mathrm{K}_\mathfrak{p}/\mathrm{K}_\mathfrak{e})}(\lambda_\mathfrak{p}^\tau + \lambda_\mathfrak{q})$ satisfies the required ones. □

Fix a generator s of $(\mathbb{A}/\mathfrak{q})^\times$ and let $\tau \in \mathrm{Gal}(\mathrm{F}(\mathfrak{q})/\mathrm{F})$ be the automorphism such that $\tau(\lambda_\mathfrak{q}) = \rho_s(\lambda_\mathfrak{q})$. Let u be a unit in Lemma 2.3 and choose $w \in \mathrm{F}(\mathfrak{q})$ satisfying $w^\tau = uw$. Then we have $(w) = \mathfrak{D} \prod_{\sigma \in G}(\widetilde{\mathfrak{Q}}^{\sigma^{-1}})^{r_\sigma}$, where $\mathfrak{D}$ is the lift of an ideal of $\mathcal{O}_\mathrm{F}$ relatively prime to $\mathfrak{q}$ and $r_\sigma \in \mathbb{Z}$. As in classical case, r_σ is determined uniquely modulo $\Phi(\mathfrak{q})$ by $s^{r_\sigma} \equiv \sigma(\varepsilon) \bmod \mathfrak{Q}$.

2.2. Applications of Tchebotarev density theorem

Given ideal class $\underline{\mathfrak{c}}$ of $\mathcal{O}_\mathrm{F}$ and a positive integer N, we define $\mathbf{P}(\underline{\mathfrak{c}}, N)$ as the set of prime ideals $\mathfrak{Q}$ belonging to $\underline{\mathfrak{c}}$ and lying above a prime ideal $\mathfrak{q}$ of $\mathbb{A}$ which splits completely in F and $\Phi(\mathfrak{q}) \equiv 0 \bmod N$. By using the Tchebotarev density theorem ([3, Theorem 9.13 A]), we can show that if N is a positive integer with $p \nmid N$, then $\mathbf{P}(\underline{\mathfrak{c}}, N)$ is an infinite set.

LEMMA 2.4. *For any nonconstant $z \in \mathrm{k}_\infty^\times$ and a positive integer c prime to p, $\mathrm{k}_\infty(\sqrt[c]{z})$ is a totally ramified extension over k_∞.*

Proof. It is an easy consequence of local class field theory. □

PROPOSITION 2.5. *Let N and c be positive integers with $c|N$ and $p \nmid N$. Let x be a nonconstant element of $\mathcal{O}_\mathrm{F}$. Suppose that for all (except possibly a finite set) of the prime ideals $\mathfrak{Q} \in \mathbf{P}(\underline{\mathfrak{c}}, N)$, there exists $y_\mathfrak{Q} \in \mathcal{O}_\mathrm{F}$ such that $x \equiv y_\mathfrak{Q}^c \bmod \mathfrak{Q}$. Then $x = \alpha y^{c/f}$ for some $y \in \mathcal{O}_\mathrm{F}$ and $\alpha \in \mathbb{F}_q^*$, a f-th root of unity, where $f = \gcd(c, q-1)$.*

Proof. Fix $\zeta_c \in \overline{\mathbb{F}}_q$ a primitive c-th root of unity and let $\sqrt[c]{x}$ be a fixed c-th root of x. By Lemma 2.4, $\mathrm{F}(\sqrt[c]{x})/\mathrm{F}$ is a geometric extension and $\widetilde{\mathrm{F}} \cap \mathrm{F}(\sqrt[c]{x}) = \mathrm{F}$, where $\widetilde{\mathrm{F}} = \mathrm{F}(\zeta_c)$. Let L be the Galois closure of $\mathrm{F}(\sqrt[c]{x})$ over F. Clearly, $\mathrm{L} \subseteq \widetilde{\mathrm{F}}(\sqrt[c]{x})$. Let $\mathrm{F}' = \mathbb{F}_{q^d}\mathrm{F} \subseteq \widetilde{\mathrm{F}}$, where $d = [\mathrm{L} : \mathrm{F}(\sqrt[c]{x})]$. Since $\mathrm{F}(\sqrt[c]{x})/\mathrm{F}$ is a geometric extension, $\mathrm{L} = \mathrm{F}'\mathrm{F}(\sqrt[c]{x}) = \mathrm{F}'(\sqrt[c]{x})$, $\widetilde{\mathrm{F}} \cap \mathrm{L} = \mathrm{F}'$ and $\mathrm{F}' \cap \mathrm{F}(\sqrt[c]{x}) = \mathrm{F}$. Let $f = \gcd(c, q^d - 1)$ and $\zeta_f = (\zeta_c)^{c/f}$. Clearly, $\mathrm{F}(\zeta_f) \subseteq \mathrm{F}'$. Let $Q(X)$ be the irreducible polynomial of $\sqrt[c]{x}$ over F. Then $Q(X) = \prod_{j\in J}(X - \zeta_c^j \sqrt[c]{x})$, where J is a subset of $\{1, 2, \ldots, c\}$ and $\zeta_c^j \in \mathbb{F}_{q^d}$ for all $j \in J$. It is easy to see that $\zeta_c^j \in \mathbb{F}_{q^d}$ if and only if j divides c/f. Hence $Q(X) = \prod_{j\in J^*}(X - \zeta_f^j \sqrt[c]{x})$ for some $J^* \subseteq \{1, 2, \ldots, f\}$. Since $\mathrm{L} = \mathrm{F}'\mathrm{F}(\sqrt[c]{x})$ and $\mathrm{F}' \cap \mathrm{F}(\sqrt[c]{x}) = \mathrm{F}$, $Q(X)$ is also the irreducible irreducible polynomial of $\sqrt[c]{x}$ over F'. Since $\mathrm{L} = \mathrm{F}(\sqrt[c]{x}, \zeta_f^j : j \in J^*) \subseteq \mathrm{F}(\sqrt[c]{x}, \zeta_f) \subseteq \mathrm{F}'(\sqrt[c]{x}) = \mathrm{L}$, we have $\mathrm{L} = \mathrm{F}(\sqrt[c]{x}, \zeta_f)$ and $\mathrm{F}' = \mathrm{F}(\zeta_f)$. Note that L/F' is a finite Galois extension and let $\Gamma = \mathrm{Gal}(\mathrm{L}/\mathrm{F}')$. Let $\mathcal{U}_f$ be the group of f-th roots of unity in $\bar{\mathbb{F}}_q$. Since $Q(X) = \prod_{\sigma\in\Gamma}(X - \sigma(\sqrt[c]{x}))$, $(\sqrt[c]{x})^{\sigma-1} \in \mathcal{U}_f$ for all $\sigma \in \Gamma$. Let $\psi : \Gamma \to \mathcal{U}_f$ be the homomorphism defined by $\psi(\sigma) = (\sqrt[c]{x})^{\sigma-1}$. Then $\mathrm{Im}(\psi) = \langle \zeta_{f_0} \rangle$ for some $f_0|f$, and so $Q(X) = \prod_{j=1}^{f_0}(X - \zeta_{f_0}^j \sqrt[c]{x})$. Now we can follow the same argument as in the classical case for the rest of proof. □

For each nonconstant unit $x \in \mathcal{O}_{\mathrm{F}}^\times$, we define the number $\phi(x)$ as the greatest positive integer n such that $x = u^n$ for some $u \in \mathrm{F}$. Clearly, $\phi(\sigma(x)) = \phi(x)$ for all $\sigma \in G$. Fix a triple $(x, \underline{\mathfrak{c}}, N)$, where $x \in \mathcal{C}_{\mathrm{F}}$, $\underline{\mathfrak{c}}$ an ideal class and N a positive integer with $p \nmid N$. For each $\mathfrak{Q} \in \mathbf{P}(\underline{\mathfrak{c}}, N)$, let $s_{\mathfrak{Q}}$ be a fixed generator of $(\mathbb{A}/\mathfrak{q})^\times$, where $\mathfrak{q} = \mathfrak{Q} \cap \mathbb{A}$. Then there exists a nonzero fractional ideal $\mathfrak{D}_{\mathfrak{Q}}$ of $\mathcal{O}_{\mathrm{F}}$ such that $\mathfrak{D}_{\mathfrak{Q}}^N \prod_{\sigma\in G}(\mathfrak{Q}^{\sigma^{-1}})^{r_\sigma(\mathfrak{Q})}$ is a principal ideal, where the integers $r_\sigma(\mathfrak{Q})$ satisfy $s_{\mathfrak{Q}}^{r_\sigma(\mathfrak{Q})} \equiv \sigma(x) \bmod \mathfrak{Q}$. Let $\sigma \in G$ be fixed. We define $g = g(x, \underline{\mathfrak{c}}, N, \sigma)$ as the greatest common divisor of N and of all the $r_\sigma(\mathfrak{Q})$ such that $\mathfrak{Q} \in \mathbf{P}(\underline{\mathfrak{c}}, N)$.

Fix an embedding of F into k_∞. We call $x \in \mathrm{F}$ *positive* if $\mathrm{sgn}(x) = 1$.

Theorem 2.6. *Given $x \in \mathcal{C}_{\mathrm{F}} \backslash \mathbb{F}_q^\times$, $\underline{\mathfrak{c}}$ an ideal class of $\mathcal{O}_{\mathrm{F}}$, N a positive integer with $p \nmid N$ and $\sigma \in G$, let $g = g(x, \underline{\mathfrak{c}}, N, \sigma)$. If $\gcd(N, q-1) = 1$ or $\sigma(x)$ is a positive, then we have $\gcd(N, \phi(x))|g|\gcd(N, f\phi(x))$, where $f = \gcd(g, q-1)$.*

Proof. As in classical case, it can be shown easily that $\gcd(N, \phi(x))$ divides g. For any $\mathfrak{Q} \in \mathbf{P}(\underline{\mathfrak{c}}, N)$, there exists $y_{\mathfrak{Q}} \in \mathbb{A}$ such that $\sigma(x) \equiv y_{\mathfrak{Q}}^g \bmod \mathfrak{Q}$. By Proposition 2.5, $\sigma(x) = \alpha y^{g/f}$ for some $y \in \mathcal{O}_{\mathrm{F}}$ and $\alpha \in \mathbb{F}_q^\times$, a f-th root of unity, where $f = \gcd(g, q-1)$. If $\gcd(N, q-1) = 1$, then $f = 1, \alpha = 1$ and hence $\sigma(x) = y^{g/f}$. If $\sigma(x)$ is a positive, then $1 = \alpha \cdot \operatorname{sgn}(y)^{g/f}$. Replacing y by $y/\operatorname{sgn}(y)$, we also have $\sigma(x) = y^{g/f}$. Thus $g|f\phi(x)$, and so $g|(N, f\phi(x))$. □

3. Proof of Theorem 1.1

For any finite abelian group A, we denote by A_ℓ the Sylow ℓ-subgroup of A for any prime ℓ. Let $x \in \mathcal{C}_{\mathrm{F}}, \underline{\mathfrak{c}} \in \mathcal{C}l_{\mathrm{F}}$ and N positive integer with $p \nmid N$ be given. For any $\mathfrak{Q} \in \mathbf{P}(\underline{\mathfrak{c}}, N)$, there is an ideal class $\underline{\mathfrak{d}}_{\mathfrak{Q}} \in \mathcal{C}l_{\mathrm{F}}$ such that

$$\underline{\mathfrak{d}}_{\mathfrak{Q}}^N \prod_{\sigma \in G} \sigma^{-1}(\underline{\mathfrak{c}})^{r_\sigma(\mathfrak{Q})} = 1, \tag{3.1}$$

where the integers $r_\sigma(\mathfrak{Q})$ satisfy $s_{\mathfrak{Q}}^{r_\sigma(\mathfrak{Q})} \equiv \sigma(x) \bmod \mathfrak{Q}$.

Proposition 3.1. *Let ℓ be a prime number such that $\ell \neq p$. Let $x \in \mathcal{C}_{\mathrm{F}}$ and let ℓ^n be an exponent of $(\mathcal{C}l_{\mathrm{F}})_\ell$. If $\underline{\mathfrak{c}} \in (\mathcal{C}l_{\mathrm{F}})_\ell, \mathfrak{Q} \in \mathbf{P}(\underline{\mathfrak{c}}, \ell^n)$ and if $r_\sigma = r_\sigma(\mathfrak{Q}), \sigma \in G$, are integers satisfying $s_{\mathfrak{Q}}^{r_\sigma} \equiv \sigma(x) \bmod \mathfrak{Q}$, then $\varrho = \varrho_{\mathfrak{Q}} = \sum_{\sigma \in G} r_\sigma(\mathfrak{Q})\sigma^{-1}$ annihilates $\underline{\mathfrak{c}}$.*

Proof. Note that the integers r_σ are uniquely determined modulo ℓ^n because $\ell^n|\Phi(\mathfrak{q})$. Since $\underline{\mathfrak{c}}^{\ell^n} = 1$, (3.1) holds with $N = \ell^n$. Since all conjugates of $\underline{\mathfrak{c}}$ belong to $(\mathcal{C}l_{\mathrm{F}})_\ell$, $\underline{\mathfrak{d}}_{\mathfrak{Q}}^{\ell^n} \in (\mathcal{C}l_{\mathrm{F}})_\ell$, and hence $\underline{\mathfrak{d}}_{\mathfrak{Q}} \in (\mathcal{C}l_{\mathrm{F}})_\ell$ and $\underline{\mathfrak{d}}_{\mathfrak{Q}}^{\ell^n} = 1$. Therefore $\underline{\mathfrak{c}}^\varrho = 1$. □

For any character χ of G, we define the idempotent element e_χ as follows:

$$e_\chi = \frac{1}{|G|} \sum_{\sigma \in G} \operatorname{Tr}(\chi(\sigma))\sigma^{-1} \in \mathbb{Q}_\ell[G],$$

where "Tr" is the trace map from $\mathbb{Q}_\ell(\chi) := \mathbb{Q}_\ell(\chi(\sigma) : \sigma \in G)$ to $\mathbb{Q}_\ell$. Now assume that $|G|$ is prime to ℓ. Then $e_\chi \in \mathbb{Z}_\ell[G]$ for all character χ of G.

Since there is a Minkowski unit in $\mathcal{O}_{\mathrm{F}}^\times$, as in classical case, we have

Proposition 3.2. *Suppose that $\ell \nmid p|G|$. Let χ be a nontrivial character of G and ℓ^n an exponent of $(\mathcal{O}_{\mathrm{F}}^\times/\mathcal{C}_{\mathrm{F}})_\ell$. Let ℓ^{a_χ} be the exact exponent of the χ-part $e_\chi(\mathcal{O}_{\mathrm{F}}^\times/\mathcal{C}_{\mathrm{F}})_\ell$ of $(\mathcal{O}_{\mathrm{F}}^\times/\mathcal{C}_{\mathrm{F}})_\ell$. Then there exists a positive $x \in e_\chi(\mathcal{C}_{\mathrm{F}}/\mathcal{C}_{\mathrm{F}} \cap (\mathcal{O}_{\mathrm{F}}^\times)^{\ell^n})$ such that $\ell^{a_\chi}||\phi(x)$.*

Lemma 3.3. *Suppose $\theta \in e_\chi\mathbb{Z}/\ell^n\mathbb{Z}[G]$ and ℓ^a is the highest power of ℓ dividing θ $(0 \leq a < n)$. Then $\ell^{-a}\theta e_\chi\mathbb{Z}/\ell^n\mathbb{Z}[G] = e_\chi\mathbb{Z}/\ell^n\mathbb{Z}[G]$. In particular, there exists $\theta' \in e_\chi\mathbb{Z}/\ell^n\mathbb{Z}[G]$ such that $\ell^{-a}\theta\theta' = e_\chi$.*

Proof. See the proof of Lemma 15.6 in [5]. □

Now we give the proof of Theorem 1.1. Let ℓ^n be an exponent of $(\mathcal{C}l_{\mathrm{F}})_\ell$ and $(\mathcal{O}_{\mathrm{F}}^{\times}/\mathcal{C}_{\mathrm{F}})_\ell$. Let $\underline{\mathfrak{c}} \in (\mathcal{C}l_{\mathrm{F}})_\ell$. For each $\mathfrak{Q} \in \mathbf{P}(\underline{\mathfrak{c}}, \ell^n)$, choose a generator $s_{\mathfrak{Q}}$ of $(\mathbb{A}/\mathfrak{q})^{\times}$, where $\mathfrak{q} = \mathfrak{Q} \cap \mathbb{A}$, and define

$$\Psi_{\mathfrak{Q}} : \mathcal{C}_{\mathrm{F}}/\mathcal{C}_{\mathrm{F}} \cap (\mathcal{O}_{\mathrm{F}}^{\times})^{\ell^n} \to \mathbb{Z}/\ell^n\mathbb{Z}[G]$$

by $\Psi_{\mathfrak{Q}}(x) = \sum_{\sigma\in G} r_\sigma \sigma^{-1}$, where the integers $r_\sigma = r_\sigma(\mathfrak{Q})$ are uniquely determined modulo ℓ^n and satisfy $s_{\mathfrak{Q}}^{r_\sigma} \equiv \sigma(x) \bmod \mathfrak{Q}$. By Proposition 3.1, $\underline{\mathfrak{c}}^{\Psi_{\mathfrak{Q}}(x)} = 1$ for all $x \in \mathcal{C}_{\mathrm{F}}/\mathcal{C}_{\mathrm{F}} \cap (\mathcal{O}_{\mathrm{F}}^{\times})^{\ell^n}$. For any nontrivial character χ of G, let $\Psi_{\mathfrak{Q}}^{\chi} : e_\chi(\mathcal{C}_{\mathrm{F}}/\mathcal{C}_{\mathrm{F}} \cap (\mathcal{O}_{\mathrm{F}}^{\times})^{\ell^n}) \to e_\chi\mathbb{Z}/\ell^n\mathbb{Z}[G]$ be the restriction of $\Psi_{\mathfrak{Q}}$. Let ℓ^{a_χ} be the exact exponent of $e_\chi(\mathcal{O}_{\mathrm{F}}^{\times}/\mathcal{C}_{\mathrm{F}})_\ell$. There exists a positive $x \in e_\chi(\mathcal{C}_{\mathrm{F}}/\mathcal{C}_{\mathrm{F}} \cap (\mathcal{O}_{\mathrm{F}}^{\times})^{\ell^n})$ satisfying $\ell^{a_\chi}||\phi(x)$. If $\ell^e||(q-1)$, then $g = g(x, \underline{\mathfrak{c}}, \ell^n, 1)$ divides $\ell^{a_\chi+e}$. There exists $\mathfrak{Q} \in \mathbf{P}(\underline{\mathfrak{c}}, \ell^n)$ satisfying $\Psi_{\mathfrak{Q}}^{\chi}(x) \not\equiv 0 \bmod \ell^{a_\chi+e+1}$. For $\mathfrak{Q}$ as above, let a' be the minimal such that $\Psi_{\mathfrak{Q}}^{\chi}(x) \not\equiv 0 \bmod \ell^{a'+1}$, so that $a' \le a_\chi + e$. Then $\ell^{-a'}\Psi_{\mathfrak{Q}}^{\chi}(x)\mathbb{Z}/\ell^n\mathbb{Z}[G] = e_\chi\mathbb{Z}/\ell^n\mathbb{Z}[G]$. Thus $\mathrm{Im}(\Psi_{\mathfrak{Q}}^{\chi}) \supseteq \ell^{a'}e_\chi\mathbb{Z}/\ell^n\mathbb{Z}[G] \supseteq (q-1)\ell^{a_\chi}e_\chi\mathbb{Z}/\ell^n\mathbb{Z}[G]$, and so $(q-1)\ell^{a_\chi}e_\chi$ annihilates $\underline{\mathfrak{c}}$. Since $\underline{\mathfrak{c}} \in (\mathcal{C}l_{\mathrm{F}})_\ell$ is arbitrary, it proves that $(q-1)\ell^{a_\chi}e_\chi$ annihilates $(\mathcal{C}l_{\mathrm{F}})_\ell$. This is also true for the trivial character χ_0 because e_{χ_0} is essentially the norm and $\ell \nmid h_{\mathrm{k}}$.

Now, suppose $\theta \in \mathbb{Z}[G]$ annihilates $(\mathcal{O}_{\mathrm{F}}^{\times}/\mathcal{C}_{\mathrm{F}})_\ell$. For any character χ, θe_χ annihilates $e_\chi(\mathcal{O}_{\mathrm{F}}^{\times}/\mathcal{C}_{\mathrm{F}})_\ell$. Let ℓ^b be the maximal power of ℓ dividing θe_χ. There exists $\theta' \in e_\chi\mathbb{Z}/\ell^n\mathbb{Z}[G]$ such that $\theta e_\chi \theta' = \ell^b e_\chi$ (by Lemma 3.3). Then ℓ^b annihilates $e_\chi(\mathcal{O}_{\mathrm{F}}^{\times}/\mathcal{C}_{\mathrm{F}})_\ell$, and so $b \ge a_\chi$. Thus, $\ell^{a_\chi}|\theta e_\chi$ and $(q-1)\theta e_\chi$ annihilates $(\mathcal{C}l_{\mathrm{F}})_\ell$. Since $(q-1)\theta = \sum_\chi (q-1)\theta e_\chi$, it follows that $(q-1)\theta$ annihilates $(\mathcal{C}l_{\mathrm{F}})_\ell$.

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