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RESOLVENT INEQUALITY OF LAPLACIAN IN BESOV SPACES

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Abstract. For $1 \leq p,\ q \leq \infty$ and $s \in \mathbb{R}$, it is proved that there exists a constant $C > 0$ such that for any $f \in B^{s+2}_{p,q}(\mathbb{R}^n)$

$$\|f\|_{B^{s+2}_{p,q}(\mathbb{R}^n)} \leq C\|f - \Delta f\|_{B^s_{p,q}(\mathbb{R}^n)},$$

which tells us that the operator $I - \Delta$ is $B^{s+2}_{p,q}$-coercive on the Besov space $B^s_{p,q}$.

The Besov spaces $B^s_{p,q}$ together with Triebel-Lizorkin spaces $F^s_{p,q}$ cover many well-known spaces of functions and distributions such as Hölder spaces, Sobolev spaces, Hardy spaces, BMO, etc. Since the theory of $B^s_{p,q}$ and $F^s_{p,q}$ was set up at the end of 1980's, many attempts have been made on applications to the theory of partial differential equations and harmonic analysis. One of the differential operators playing important roles in those fields is the Laplace operator Δ displayed by $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ in the flat (Euclidean) domain $\mathbb{R}^n$. In this note, we present an inequality for a resolvent of the Laplacian Δ in Besov spaces $B^s_{p,q}$. This is the main theorem we prove:

Main theorem *For $1 \leq p,\ q \leq \infty$, and $s \in \mathbb{R}$, there exists a constant $C > 0$ such that for any $f \in B^{s+2}_{p,q}(\mathbb{R}^n)$*

$$\|f\|_{B^{s+2}_{p,q}(\mathbb{R}^n)} \leq C\|f - \Delta f\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

The essential tool for the proof is the Littlewood-Paley decomposition, and the main difficulty is to get over the singularity at the origin in the frequency space.

An immediate application of the inequality above is to explain the m-dissipative property of the Laplacian $\Delta : D \to B^s_{p,q}$ with $D \equiv B^{s+2}_{p,q}$. In

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fact, for $s \ge 0$, $1 \le p,\ q \le \infty$, $\lambda > 0$, and any $F \in B^s_{p,q}(\mathbb{R}^n)$, there exists a unique $u \in B^{s+2}_{p,q}(\mathbb{R}^n)$ satisfying the following stationary problem:

$$\lambda u - \Delta u = F.$$

We introduce some basic notations for Besov spaces. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class and $\hat{f}$ represent the Fourier transform of $f \in \mathcal{S}$ defined by

$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) \equiv \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

We take two radial functions $\Phi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\operatorname{supp}\widehat{\Phi} \subset \{\xi : |\xi| \le \frac{5}{6}\}$, $\operatorname{supp}\widehat{\varphi} \subset \{\xi : \frac{3}{5} \le |\xi| \le \frac{5}{3}\}$ and $\widehat{\Phi}(\xi) + \sum_{j=0}^{\infty} \widehat{\varphi}_j(\xi) = 1$ for $\xi \in \mathbb{R}^n$, where $\varphi_j(x) = 2^{jn}\varphi(2^j x)$, that is, $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j}\xi)$. For $f \in \mathcal{S}'$, $\Delta_{-1} f \equiv \widehat{\Phi}(D) f = \Phi * f$ and when $j \ge 0$, $\Delta_j f \equiv \widehat{\varphi}_j(D) f = \varphi_j * f$. For $s \in \mathbb{R}$, and $1 \le p, q \le \infty$, the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are defined by

$$f \in B^s_{p,q}(\mathbb{R}^n) \Leftrightarrow \{\|2^{js}\Delta_j f\|_{L^p}\}_{j\in\mathbb{Z}} \in l^q.$$

The following lemma exhibits the relationship between the role of the scaling factor 2^{js} and the role of the differentiation index s in $B^s_{p,q}$. The proof of the Lemma can be found in [1, p. 16].

Bernstein's Lemma *There exists a constant $C = C(s)$ such that for any $f \in \mathcal{S}(\mathbb{R}^n)$ with* $\operatorname{supp}\widehat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \le r\}$,

$$\|f\|_{L^{p_1}} \le C r^{n\left(\frac{1}{p} - \frac{1}{p_1}\right)} \|f\|_{L^p}, \quad 1 \le p \le p_1 \le \infty,$$
$$\|D^s f\|_{L^p} \le C r^s \|f\|_{L^p}, \quad 1 \le p \le \infty.$$

Also, there exists a constant $\tilde{C}(s)$ such that for any $f \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\widehat{f} \subset \{\xi : 2^{j-1} \le |\xi| \le 2^{j+1}\}$,

$$\tilde{C}^{-1} 2^{js}\|f\|_{L^p} \le \|D^s f\|_{L^p} \le \tilde{C}\, 2^{js}\|f\|_{L^p}, \qquad 1 \le p \le \infty.$$

Notation Throughout this paper, C denotes various real positive constants.

1. The Proof of the theorem

Let $1 < p < \infty$. We first notice that for any test function h in the Schwartz class $\mathcal{S}$, we have

$$(p-1)(|h(x)|^{p-2}\,\nabla h(x)) = \nabla(|h(x)|^{p-2} h(x))$$

for almost every $x \in \mathbb{R}^n$. Hence for $h \in \mathcal{S}$, we observe that

$$\begin{aligned} \frac{4(p-1)}{p^2} \int_{\mathbb{R}^n} |\nabla h^{\frac{p}{2}}|^2 \, dx &= (p-1) \int_{\mathbb{R}^n} |h^{\frac{p}{2}-1} \nabla h|^2 \, dx \\ &= \int_{\mathbb{R}^n} (\nabla h) \cdot ((p-1)(|h|^{p-2} \nabla h)) \, dx \\ &= - \int_{\mathbb{R}^n} (\Delta h) |h|^{p-2} \, h \, dx. \end{aligned} \tag{1}$$

Hence by (1) together with Hölder's inequality, we have for $h \in \mathcal{S}$

$$\begin{aligned} \|h\|_{L^p}^p &\le \int_{\mathbb{R}^n} |h|^p \, dx + \frac{4(p-1)}{p^2} \int_{\mathbb{R}^n} |\nabla h^{\frac{p}{2}}|^2 \, dx \\ &= \int_{\mathbb{R}^n} (h - \Delta h) \, |h|^{p-2} \, h \, dx \\ &\le \|h - \Delta h\|_{L^p} \, \|h\|_{L^p}^{p-1}. \end{aligned}$$

We conclude that for $1 < p < \infty$,

$$\|h\|_{L^p} \le \|h - \Delta h\|_{L^p}. \tag{2}$$

Even for the case when $p = \infty$, the estimate (2) is valid: in fact, for $h \in \mathcal{S}$

$$\|h\|_{L^\infty} = \lim_{p \to \infty} \|h\|_{L^p} \le \lim_{p \to \infty} \|h - \Delta h\|_{L^p} = \|h - \Delta h\|_{L^\infty}.$$

For the case when $p = 1$, Fatou's lemma and the estimate (2) imply that

$$\|h\|_{L^1} = \liminf_{p \to 1} \|h\|_{L^p}^p \le \liminf_{p \to 1} \|h - \Delta h\|_{L^p}^p. \tag{3}$$

The integrand $|h(x) - \Delta h(x)|^p$ in the right hand side of (3) is dominated by an integrable function $|h(x) - \Delta h(x)|^2 + |h(x) - \Delta h(x)|$, and so the Lebesgue's dominated convergence theorem implies that the right hand side of (3) is equal to $\|h - \Delta h\|_{L^1}$. This illustrates that (2) holds even for the critical case $p = 1$.

Next, choose $f \in B^{s+2}_{p,q}(\mathbb{R}^n)$, and we have on one hand

$$\| \Delta_{-1} f \|_{L^p} \le \|\Delta_{-1}(f - \Delta f) \|_{L^p},$$

by applying the estimate (2) to $\Delta_{-1} f$. For $j \ge 0$ and $1 < p < \infty$, by Bernstein's lemma, we see that

$$\|\Delta_j f\|_{L^p}^p = \int_{\mathbb{R}^n} |(\Delta_j f)^{\frac{p}{2}}|^2 \, dx \le C \, 2^{-2j} \int_{\mathbb{R}^n} |\nabla (\Delta_j f)^{\frac{p}{2}}|^2 \, dx.$$

Then by the identity (1) and the same arguments used in the above, we have the following estimate

$$
\begin{aligned}
2^{2j}\|\Delta_j f\|_{L^p}^p &\le C\int_{\mathbb{R}^n} |\nabla(\Delta_j f)^{\frac{p}{2}}|^2\,dx \\
&= -C\int_{\mathbb{R}^n} \Delta(\Delta_j f)\,(\Delta_j f|\Delta_j f|^{p-2})\,dx \\
&\le C\int_{\mathbb{R}^n} (\Delta_j f - \Delta(\Delta_j f))\,(\Delta_j f|\Delta_j f|^{p-2})\,dx \\
&= C\|\Delta_j(f-\Delta f)\|_{L^p}\;\|\Delta_j f\|_{L^p}^{p-1}.
\end{aligned}
$$

Therefore, we have

$$
2^{2j}\,\|\Delta_j f\|_{L^p} \;\le\; C\|\Delta_j(f-\Delta f)\|_{L^p}. \tag{4}
$$

for $1 < p < \infty$. The same argument used in the above delivers the estimate (4) for the critical cases $p = 1$ and $p = \infty$. Collecting terms together, we get

$$
\sum_{j=0}^{\infty} 2^{j(s+2)q}\|\Delta_j f\|_{L^p}^q \;\le\; C\sum_{j=0}^{\infty} 2^{jsq}\|\Delta_j(f-\Delta f)\|_{L^p}^q.
$$

In all, we conclude that

$$
\begin{aligned}
\|f\|_{B^{s+2}_{p,q}} &= \left(\|\,2^{-(s+2)}\,\Delta_{-1}f\,\|_{L^p}^q + \sum_{j=0}^{\infty}\|\,2^{j(s+2)}\,\Delta_j f\,\|_{L^p}^q \right)^{1/q} \\
&\le C\left(\sum_{j=-1}^{\infty}\|2^{js}\,\Delta_j(f\,-\,\Delta f)\,\|_{L^p}^q\right)^{1/q} \\
&= C\,\|\,f\,-\,\Delta f\,\|_{B^s_{p,q}}.
\end{aligned}
$$

This completes the proof. □

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