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ON A VARIATIONAL INEQUALITY WITH $\mathcal{C}$-CONCAVITY

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Abstract. In this paper, using the diagonally $\mathcal{C}$-concave condition, we will prove a functional inequality in a topological space, and as an application, we can obtain a new variational inequality.

1. Introduction

In 1966, Hartman and Stampacchia [5] established the well-known existence theorem for variational inequality as follows: If $f : X \to \mathbb{R}^n$ is a continuous mapping on a compact convex subset X of $\mathbb{R}^n$, then there exists $x_0 \in X$ such that $\langle f(x_0), x_0 - x\rangle \geq 0$ for all $x \in X$. This remarkable result has been investigated and generalized in various points of views by Browder [1], Fan [3], Takahashi [8] and others. As we know, the concavities are very essential natures in numerous applications in nonlinear analysis, and in particular, those concepts have been extensively studied and generalized by several authors in the last six decades, e.g., see [1-8]. However, the numerous generalized concavity and their applications can always be useful only in topological vector spaces, i.e., the linear structure and the continuity are always needed. Hence we shall need a minimax inequality which can be suitable for numerous settings in topological spaces to show the existence results.

In this paper, we will introduce the diagonal $\mathcal{C}$-concavity which generalizes the CF-concave and $\mathcal{C}$-concave conditions simultaneously without assuming the linear structure. Next, using this concept, we will prove a

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basic functional inequality in a topological space, and as an application, we can obtain a variational inequality.

2. Preliminaries

Let X be a non-empty convex subset of a vector space E and let $f : X \to \mathbb{R}$. We say that f is *quasiconcave* if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \geq t\}$ is convex; and that f is *quasiconvex* if $-f$ is quasiconcave. And, let E be a topological vector space and E^* be its dual space. Denote the dual pairing on $E^* \times E$ by $\langle w, x\rangle$ for each $w \in E^*, x \in E$. Then it should be noted that for fixed $w \in E^*$, $x \mapsto Re\langle w, x\rangle$ is a continuous linear functional on E.

Denote by $[0,1]^n$ the Cartesian product of n unit intervals $[0,1] \times \cdots \times [0,1]$, and denote the unit simplex in $[0,1]^n$ by Δ_n, i.e., $\Delta_n := \{(\lambda_1, \ldots, \lambda_n) \in [0,1]^n \mid \sum_{i=1}^n \lambda_i = 1\}$, and simply denote $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n$.

Throughout this paper, all topological spaces are assumed to be Hausdorff and for the other standard notations and terminologies, we shall refer to [3,6,7].

In [2], Fan first introduced the concavelike condition in a general space without assuming the linear structure, and next in [4], adding the continuity to concavelike functions, Forgó introduced the CF-concavity, and as remarked, CF-concavity is closely related to concavelike condition. However, the continuity plays a key role in applying the Brouwer fixed point theorem to show the existence theorems in the applications as we can see in [4,6]. In a recent paper [6], the authors introduced the $\mathcal{C}$-concave condition which generalizes the CF-concave and concave conditions simultaneously.

Now we introduce a general concave condition which generalizes $\mathcal{C}$-concave and quasiconcave conditions simultaneously as follows:

DEFINITION 2.1. Let X be a non-empty topological space and Y a non-empty subset of X. Then $f : X \times Y \to \mathbb{R}$ is called *diagonally $\mathcal{C}$-concave on Y* if for every $n \geq 2$, whenever n points $y_1, \ldots, y_n \in Y$ are arbitrarily given, there exists a continuous function $\phi_n : \Delta_n \to Y$ such that for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n$,

$$f\big(\phi_n(\lambda), \phi_n(\lambda)\big) \geq \lambda_1 f(\phi_n(\lambda), y_1) + \cdots + \lambda_n f(\phi_n(\lambda), y_n); \qquad (*)$$

and f is called *diagonally $\mathcal{C}$-convex on Y* if $-f$ is diagonally $\mathcal{C}$-concave on Y.

Indeed, in Definition 2.1, we can see that the inequality $(*)$ needs to hold only for $x = \phi_n(\lambda)$ but not for all $x \in X$ in the $\mathcal{C}$-concave condition in [6] so that the $\mathcal{C}$-concave condition implies the diagonally $\mathcal{C}$-concave condition. When X is a topological vector space, the concavity clearly implies the diagonal $\mathcal{C}$-concavity by using a continuous function $\phi_n(\lambda_1, \dots, \lambda_n) := \lambda_1 x_1 + \cdots + \lambda_n x_n$, for all $(\lambda_1, \dots, \lambda_n) \in \Delta_n$, whenever $x_1, \dots, x_n \in Y$ are given. Also, we can see that the quasiconvexity implies the diagonally $\mathcal{C}$-convexity.

Here we note that whenever $X = Y$ in Definition 2.1, for removing the ambiguity of notation, we may simply call that the function $y \mapsto f(x,y)$ is *diagonally $\mathcal{C}$-concave on X* (or f is *diagonally $\mathcal{C}$-concave in the second variable*) if the inequality $(*)$ holds.

3. A functional inequality and a variational inequality

We begin with the basic functional inequality in a topological space without assuming the convex condition on X as follows:

THEOREM 3.1. *Let X be a non-empty compact set in a topological space, and let $f : X \times X \to \mathbb{R}$ be a real-valued function satisfying*

(i) for each $y \in X$, the function $x \mapsto f(x,y)$ is upper semicontinuous on X;

(ii) for each $x \in X$, the function $y \mapsto f(x,y)$ is diagonally $\mathcal{C}$-convex on X.

Then there exists $\bar{x} \in X$ such that

$$\inf_{x \in X} f(x,x) \le f(\bar{x}, y) \quad \text{for all } y \in X.$$

Proof. If $\mu := \inf_{x \in X} f(x,x) = -\infty$, then we have done so that we may assume that $\mu > -\infty$. Suppose the contrary, i.e., for each $x \in X$, there exists $y \in X$ such that $f(x,y) < \mu$. Then we let $G_y := \{x \in X \mid f(x,y) < \mu\}$ for each $y \in X$. Since $x \mapsto f(x,y)$ is upper semicontinuous on X, each G_y is open (possibly empty) in X and $X = \bigcup_{y \in X} G_y$. Since X is compact and each G_y is open, there exist a finite number of non-empty open sets $G_{y_1}, \dots, G_{y_n}$ such that $X = \bigcup_{i=1}^n G_{y_i}$, where $\{y_1, \dots, y_n\} \subset X$. Since X is compact, there exists a continuous partition of unity $\{\alpha_1, \dots, \alpha_n\}$ subordinate to the open covering $\{G_{y_1}, \ldots, G_{y_n}\}$ of X, i.e.,

$$0 \le \alpha_i(x) \le 1, \quad \sum_{i=1}^{n} \alpha_i(x) = 1 \text{ for all } x \in X, \ i = 1, \ldots, n;$$

and if $x \notin G_{y_j}$ for some j, then $\alpha_j(x) = 0$.

For any $1 \le i \le n$, if $\alpha_i(x) > 0$, then $f(x, y_i) < \mu$, hence we have

$$\sum_{i=1}^{n} \alpha_i(x) f(x, y_i) < \mu \quad \text{for all } x \in X.$$

For the finite set $\{y_1, \ldots, y_n\} \subset X$, by the assumption (ii), there exists a continuous function $\psi_n : \Delta_n \to X$ such that for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n$,

$$f\big(\psi_n(\lambda), \psi_n(\lambda)\big) \le \lambda_1 f(\psi_n(\lambda), y_1) + \cdots + \lambda_n f(\psi_n(\lambda), y_n). \qquad (\dagger)$$

Next, we consider a continuous mapping $\Psi : \Delta_n \to \Delta_n$, defined by

$$\Psi(\lambda) := \Big(\alpha_1(\psi_n(\lambda)), \ldots, \alpha_n(\psi_n(\lambda))\Big) \quad \text{for all } \lambda = (\lambda_1, \ldots, \lambda_n) \in \Delta_n.$$

Since ψ_n and each α_i are continuous, $\Psi : \Delta_n \to \Delta_n$ is a continuous mapping on the finite dimensional compact convex set Δ_n. Therefore, by the Brouwer fixed point theorem, we can obtain a fixed point $\bar{\lambda} \in \Delta_n$ for Ψ, i.e., $\Psi(\bar{\lambda}) = \bar{\lambda}$. Putting $\bar{x} := \psi_n(\bar{\lambda})$ then $\alpha_i(\bar{x}) = \bar{\lambda}_i$ for each $i \in I = \{1, \ldots, n\}$. Here we note that for each $j \in I$, $\bar{\lambda}_j \ne 0$ means $\alpha_j(\bar{x}) \ne 0$ so that $\bar{x} \in G_{y_j}$. Therefore, if $\bar{\lambda}_j > 0$, then $f(\bar{x}, y_j) < \mu$. From the inequality (†), we have

$$f(\bar{x}, \bar{x}) \ \le \ \alpha_1(\bar{x}) f(\bar{x}, y_1) + \cdots + \alpha_n(\bar{x}) f(\bar{x}, y_n) \ < \ \mu,$$

which is a contradiction since $f(\bar{x}, \bar{x}) \ \ge \ \inf_{x \in X} f(x, x) = \mu$. This completes the proof. □

Remark 3.2. (1) Theorem 3.1 generalizes Lemma 1 in [8] and its related many generalizations in the following aspects:

(i) X need not be convex nor a subset of a vector space;

(ii) the function $y \mapsto f(x, y)$ need not be convex nor $\mathcal{C}$-concave.

(2) Since Theorem 3.1 is a generalization of Lemma 1 of [8] in a topological space without the linear structure, using the diagonal $\mathcal{C}$-concavity, most of existence results can be generalized in topological spaces without the linear structure by following the method in [8].

Since the quasiconvexity implies the diagonal $\mathcal{C}$-convexity, we have the following Fan's minimax inequality as a consequence of Theorem 3.1:

THEOREM 3.3. *Let X be a non-empty compact convex set in a topological vector space, and let $f : X \times X \to \mathbb{R} \cup \{-\infty\}$ be a function satisfying*

(i) for each $y \in X$, the function $x \mapsto f(x,y)$ is upper semicontinuous on X;

(ii) for each $x \in X$, the function $y \mapsto f(x,y)$ is quasiconvex on X.

Then there exists an $\bar{x} \in X$ such that

$$\inf_{x \in X} f(x,x) \leq f(\bar{x}, y) \quad \textit{for all } y \in X;$$

that is, the minimax inequality $\inf_{x\in X} f(x,x) \leq \max_{x\in X} \inf_{y\in X} f(x,y)$ holds.

As an application of Theorem 3.3, we can obtain the following basic variational inequality in a topological vector space:

THEOREM 3.4. *Let X be a non-empty compact convex set in a topological vector space E and $f : X \to E^*$ be a mapping such that for each $y \in X$, the function $x \mapsto Re\langle f(x), x-y\rangle$ is upper semicontinuous on X. Then there exists a point $\bar{x} \in X$ such that*

$$Re\langle f(\bar{x}), \bar{x} - y\rangle \geq 0 \quad \textit{for all } y \in X.$$

Proof. Define a real-valued function $\psi : X \times X \to \mathbb{R}$ by

$$\psi(x,y) := Re\langle f(x), x-y\rangle \quad \text{for each } (x,y) \in X \times X.$$

Then, by the assumption, for each $y \in X$, the function $x \mapsto \psi(x,y)$ is upper semicontinuous on X. For each fixed $x \in X$, since the function $y \mapsto Re\langle f(x), x-y\rangle$ is a linear mapping on X, it is easy to see that the function $y \mapsto \psi(x,y)$ is quasiconvex. In fact, for fixed $x \in X$, it is clear that the set $\{y \in X \mid Re\langle f(x), x-y\rangle \leq t\}$ is convex for any $t \in \mathbb{R}$. Therefore, the whole assumptions of Theorem 3.3 are satisfied so that there exists a point $\bar{x} \in X$ such that

$$0 = \inf_{x\in X} \psi(x,x) \leq \psi(\bar{x}, y) = Re\langle f(\bar{x}), \bar{x} - y\rangle \quad \text{for all } y \in X.$$

This completes the proof. □

Finally, it should be noted that our theorems can be considered as basic results on non-convex general topological spaces so that it is possible to generalize those results in more general settings by adapting general fixed point theorems or using generalized concave conditions or multivalued mappings, e.g., Eilenberg-Montgomery's fixed point theorem or the G-convexity in [7] or using some coercive conditions.

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