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REFINED HYERS–ULAM STABILITY FOR JENSEN TYPE MAPPINGS

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Abstract. In 1940 S.M. Ulam proposed the famous Ulam stability problem. In 1941 D.H. Hyers solved the well-known Ulam stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In this paper we improve results for Jensen type mappings and establish new theorems about the Ulam stability of additive and alternative Jensen type mappings.

1. Introduction

In 1940 and in 1964 S.M. Ulam [17] proposed the famous Ulam stability problem: "When is it true that by changing a little the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" For very general functional equations, the concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? If the answer is affirmative, we would say that the equation is stable.

In 1941 D.H. Hyers [7] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In 1951 D.G. Bourgin [2] was the second author to treat the Ulam stability problem for additive mappings. In 1978 P.M. Gruber [6] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. We wish to

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note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [18] used a stability property of the functional equation $f(x-y)+f(x+y)=2f(x)$ to prove a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials. Th.M. Rassias [15] and then P. Găvruta [5] obtained generalized results of Hyers' Theorem which allow the Cauchy difference to be unbounded. In 1987 Z. Gajda and R. Ger [4] showed that one can get analogous stability results for subadditive multifunctions. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [1, 3, 8, 9, 16]. In 2003-2006 J.M. Rassias and M.J. Rassias [12, 13] and J.M. Rassias [14] solved the above Ulam problem for Jensen and Jensen type mappings.

In this paper we improve our upper bounds and thus generalize Ulam stability results controlled by more general mappings, by considering approximately Jensen and Jensen type mappings satisfying conditions much weaker than the D.H. Hyers and J.M. Rassias conditions on approximately Jensen and Jensen type mappings. Besides we establish new theorems about the Ulam stability results for alternative Jensen and Jensen type equations. These stability results can be applied in stochastic analysis, financial and actuarial mathematics, as well as in psychology and sociology.

Throughout this paper, let X be a real normed space and Y a real Banach space in the case of functional inequalities, as well as let X and Y be real linear spaces for functional equations. Besides let us denote by $\mathbb{N}$ the set of all natural numbers and $\mathbb{R}$ the set of all real numbers.

Definition 1.1. A mapping $A : X \to Y$ is called *additive of the first form* if A satisfies the functional equation

$$A(x_1 + x_2) + A(x_1 - x_2) = 2A(x_1) \tag{1.1}$$

for all $x_1, x_2 \in X$ and $A(0) = 0$.

We note that the equation (1.1) is equivalent to *the Jensen equation*

$$2A(\frac{x+y}{2}) = A(x) + A(y) \tag{1.2}$$

for $x = x_1 + x_2, y = x_1 - x_2$.

Definition 1.2. A mapping $A : X \to Y$ is called *additive of the second form* if A satisfies the functional equation

$$A(x_1 + x_2) - A(x_1 - x_2) = 2A(x_2) \tag{1.3}$$

for all $x_1, x_2 \in X$.

We note that the equation (1.3) is equivalent to *the Jensen type equation*

$$2A(\frac{x-y}{2}) = A(x) - A(y) \tag{1.4}$$

for $x = x_1 + x_2, y = x_1 - x_2$. A mapping $A : X \to Y$ is called *Jensen type mapping* if it satisfies the functional equation (1.4).

2. Stability of Jensen type mappings

We will investigate under what conditions it is then possible to find a true additive mapping of the second form near an approximate additive mapping with small error.

THEOREM 2.1. *Assume that there exists a mapping $\varphi : X^2 \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x_1 + x_2) - f(x_1 - x_2) - 2f(x_2)\| \leq \varphi(x_1, x_2) \tag{2.1}$$

and the series

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x_1, 2^i x_2)}{2^i} < \infty \tag{2.2}$$

$$\left(\sum_{i=1}^{\infty} 2^i \varphi(2^{-i} x_1, 2^{-i} x_2) < \infty, \quad \textit{respectively} \right)$$

for all $x_1, x_2 \in X$. Then there exists a unique additive mapping $A : X \to Y$ of the second form which satisfies the inequality

$$\|f(x) + f(0) - A(x)\| \leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i x)}{2^i}$$

$$\left(\|f(x) - A(x)\| \leq \frac{1}{2} \sum_{i=1}^{\infty} 2^i \varphi(2^{-i} x, 2^{-i} x), \quad \textit{respectively} \right)$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Substitution of $x_i = 0 (i = 1, 2)$ in the functional inequality (2.1) yields that $2\|f(0)\| \leq \varphi(0, 0)$. Note that $\varphi(0, 0) = 0 = f(0)$ in the next case of (2.2) by the convergence of the series $\sum_{i=1}^{\infty} 2^i \varphi(0, 0) < \infty$.

Now substituting $x_1, x_2 := x$ in the functional inequality (2.1), we obtain

$$\|f(2x) - f(0) - 2f(x)\| \leq \varphi(x, x),$$

$$\left\|g(x) - \frac{g(2x)}{2}\right\| \leq \frac{1}{2}\varphi(x, x), \tag{2.3}$$

for all $x \in X$, where $g(x) := f(x) + f(0)$. Therefore from (2.3) with $2^i x$ in place of x $(i = 1, \cdots, n-1)$, one gets

$$\left\|g(x) - \frac{g(2^n x)}{2^n}\right\| \leq \frac{1}{2}\sum_{i=0}^{n-1} \frac{\varphi(2^i x, 2^i x)}{2^i} \tag{2.4}$$

for all $x \in X$ and all $n \in \mathbb{N}$. By (2.4), for any $n > m \geq 0$ we have

$$\begin{aligned}\left\|\frac{g(2^m x)}{2^m} - \frac{g(2^n x)}{2^n}\right\| &\leq \frac{1}{2^m}\left\|g(2^m x) - \frac{g(2^{n-m}2^m x)}{2^{n-m}}\right\| \\ &\leq \frac{1}{2}\sum_{i=0}^{n-m-1} \frac{\varphi(2^{i+m}x, 2^{i+m}x)}{2^{i+m}},\end{aligned}$$

which tends to zero as m tends to infinity. Thus it follows that a sequence $\left\{\frac{g(2^n x)}{2^n}\right\}$ is Cauchy in Y and it thus converges. Therefore we see that a mapping $A : X \to Y$ defined by

$$A(x) := \lim_{n\to\infty} \frac{g(2^n x)}{2^n} = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in X$. It is obvious that $A(0) = 0$ by the definition of A. In addition it is clear from (2.1) that the following inequality

$$\begin{aligned}&\|A(x_1 + x_2) - A(x_1 - x_2) - 2A(x_2)\| \\ &= \lim_{n\to\infty} 2^{-n}\|f(2^n(x_1 + x_2)) - f(2^n(x_1 - x_2)) - 2f(2^n x_2)\| \\ &\leq \lim_{n\to\infty} 2^{-n}\varphi(2^n x_1, 2^n x_2) = 0\end{aligned}$$

holds for all $x_1, x_2 \in X$. Thus taking the limit $n \to \infty$ in (2.4), we find that the mapping A is an additive mapping of the second form satisfying the inequality (2.3) near the approximate mapping $f : X \to Y$ of the equation (1.1).

To prove the afore-mentioned uniqueness, we assume now that there is another additive mapping $\breve{A} : X \to Y$ of the second form which satisfies the equation (1.1) and the inequality (2.3). Then it follows easily that

$$A(x) = 2^{-n}A(2^n x), \quad \breve{A}(x) = 2^{-n}\breve{A}(2^n x)$$

hold for all $x \in X$ and all $n \in \mathbb{N}$. Thus from the last equality and (2.3) one proves that

$$\begin{aligned}
&\|A(x) - \check{A}(x)\| = \frac{1}{2^n}\|A(2^n x) - \check{A}(2^n x)\| \\
&\leq \frac{1}{2^n}\left(\|A(2^n x) - f(2^n x) - f(0)\| + \left\|f(2^n x) + f(0) - \check{A}(2^n x)\right\|\right) \\
&\leq \sum_{i=0}^{\infty} \frac{\varphi(2^{i+n}x, 2^{i+n}x)}{2^{i+n}}
\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore from $n \to \infty$, one establishes

$$A(x) - \check{A}(x) = 0$$

for all $x \in X$, completing the proof of uniqueness.

The proof of the last assertion in the theorem is obvious according to the result of [11, 15]. □

We obtain [14, Theorem 3.1] as a corollary for approximately additive mappings of the second form in terms of a product of powers of norms.

Now we improve the bound $\frac{33\delta}{2}$ in [12, Theorem 4] obtained in 2003 for the Ulam stability of additive mappings of the second form on restricted domains.

THEOREM 2.2. *Assume that there exist $d > 0$ and $\delta \geq 0$ for which a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x_1 + x_2) - f(x_1 - x_2) - 2f(x_2)\| \leq \delta \tag{2.5}$$

for all $x_1, x_2 \in X$ with $\|x_1\| + \|x_2\| \geq d$. Then there exists a unique additive mapping $A : X \to Y$ of the second form which satisfies the inequality

$$\|f(x) + f(0) - A(x)\| \leq 7\delta$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Substitution of $x_1 := 0$ and $x_2 := x$ with $\|x\| \geq d$ in the functional inequality (2.5) yields

$$\|f(x) + f(-x)\| \leq \delta$$

for all $x \in X$ with $\|x\| \geq d$. Replacing (x_1, x_2) by $(x, 0)$ in (2.5), one has $2\|f(0)\| \leq \delta$. Now employing the same argument as [12, Theorem

4], we get

$$\begin{aligned}
&f(x_1+x_2)-f(x_1-x_2)-2f(x_2)\\
&=[f(x_1+x_2)-f(x_1-x_2-2t)-2f(x_2+t)]\\
&\quad+[f(x_1+x_2-2t)-f(x_1-x_2)-2f(x_2-t)]\\
&\quad-[f(x_1+x_2-2t)-f(x_1-x_2-2t)-2f(x_2)]\\
&\quad+2[f(t+x_2)-f(t-x_2)-2f(x_2)]\\
&\quad+2[f(t-x_2)+f(-(t-x_2))],
\end{aligned}$$

for all $x_1, x_2 \in X$ with $\|x_1\|+\|x_2\| < d$, where $t \in X$ with $\|t\| = d$. And thus one has

$$\|f(x_1+x_2)-f(x_1-x_2)-2f(x_2)\| \le 7\delta \tag{2.6}$$

for all $x_1, x_2 \in X$. Applying Theorem 2.1 to the functional inequality (2.6), we obtain the desired results. □

3. Stability of alternative additive mappings

DEFINITION 3.1. A mapping $A : X \to Y$ is called *alternative additive of the first form* if A satisfies the functional equation

$$A(x_1+x_2)+A(x_1-x_2)+2A(-x_1)=0 \tag{3.1}$$

for all $x_1, x_2 \in X$.

We note that the equation (3.1) is equivalent to *the alternative Jensen equation*

$$2A(-\frac{x+y}{2})+A(x)+A(y)=0 \tag{3.2}$$

for $x = x_1+x_2, y = x_1-x_2$. A mapping $A : X \to Y$ is called *alternative Jensen mapping* if it satisfies the functional equation (3.2).

DEFINITION 3.2. A mapping $A : X \to Y$ is called *alternative additive of the second form* if A satisfies the functional equation

$$A(x_1+x_2)-A(x_1-x_2)+2A(-x_2)=0 \tag{3.3}$$

for all $x_1, x_2 \in X$.

We note that the equation (3.3) is equivalent to *the alternative Jensen type equation*

$$2A(-\frac{x-y}{2})+A(x)-A(y)=0 \tag{3.4}$$

for $x = x_1 + x_2, y = x_1 - x_2$. A mapping $A : X \to Y$ is called *alternative Jensen type mapping* if it satisfies the functional equation (3.4).

Alternative additive mappings and alternative Jensen mappings together with alternative Jensen type mappings were introduced and investigated through J.M Rassias' publications [12, 13, 14]. We will investigate under what conditions it is then possible to find a true alternative additive mapping of the first form near an approximate alternative additive mapping with small error.

THEOREM 3.3. *Assume that there exists a mapping $\varphi : X^2 \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x_1 + x_2) + f(x_1 - x_2) + 2f(-x_1)\| \le \varphi(x_1, x_2) \tag{3.5}$$

and the series

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x_1, 2^i x_2)}{2^i} < \infty \tag{3.6}$$

$$\left(\sum_{i=1}^{\infty} 2^i \varphi(2^{-i} x_1, 2^{-i} x_2) < \infty, \quad \textit{respectively} \right)$$

for all $x_1, x_2 \in X$. Then there exists a unique alternative additive mapping $A : X \to Y$ of the first form which satisfies the inequality

$$\|f(x) - f(0) - A(x)\| \le \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} \Big[\varphi(2^i x, 2^i x) + \varphi(2^i x, 0) \Big] \tag{3.7}$$

$$\left(\|f(x) - A(x)\| \le \frac{1}{2} \sum_{i=1}^{\infty} 2^i \Big[\varphi(2^{-i} x, 2^{-i} x) + \varphi(2^{-i} x, 0) \Big] \right)$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Replacing (x_1, x_2) by $(x, 0)$ and $(0, 0)$ separately, one has an approximate odd condition $2\|f(x) + f(-x)\| \le \varphi(x, 0)$ and $4\|f(0)\| \le \varphi(0, 0)$. Thus we remark that $\varphi(0, 0) = 0 = f(0)$ by the convergence in the next case of (3.6). Substituting $x_1, x_2 := x$ in the functional inequality (3.5), we obtain $\|f(2x) + f(0) + 2f(-x)\| \le \varphi(x, x)$, and so

$$\|f(2x) + f(0) - 2f(x)\| \le \varphi(x, x) + \varphi(x, 0),$$

$$\left\| g(x) - \frac{g(2x)}{2} \right\| \le \frac{1}{2} \Big[\varphi(x, x) + \varphi(x, 0) \Big], \tag{3.8}$$

for all $x \in X$, where $g(x) := f(x) - f(0)$. Therefore from (3.8) with $2^i x$ in place of x $(i = 1, \cdots, n-1)$, one gets

$$\left\| g(x) - \frac{g(2^n x)}{2^n} \right\| \le \frac{1}{2} \sum_{i=0}^{n-1} \frac{1}{2^i} \Big[\varphi(2^i x, 2^i x) + \varphi(2^i x, 0) \Big] \tag{3.9}$$

$$\left(\| f(x) - 2^n f(2^{-n} x) \| \le \frac{1}{2} \sum_{i=1}^{n} 2^i \Big[\varphi(2^{-i} x, 2^{-i} x) + \varphi(2^{-i} x, 0) \Big] \right)$$

for all $x \in X$ and all $n \in \mathbb{N}$.

From the inequality (3.9), we can prove the rest of the proof by the similar way to the proofs of our corresponding Theorem 2.1. □

By virtue of Theorem 3.3, we obtain [13, Theorem 2.1] as a corollary for approximately alternative additive mappings of the first form.

THEOREM 3.4. *Assume that there exists a mapping* $\varphi : X^2 \to [0, \infty)$ *for which a mapping* $f : X \to Y$ *satisfies the inequality*

$$\| f(x_1 + x_2) - f(x_1 - x_2) + 2f(-x_2) \| \le \varphi(x_1, x_2) \tag{3.10}$$

and the series

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x_1, 2^i x_2)}{2^i} < \infty \tag{3.11}$$

$$\left(\sum_{i=1}^{\infty} 2^i \varphi(2^{-i} x_1, 2^{-i} x_2) < \infty, \quad \textit{respectively} \right)$$

for all $x_1, x_2 \in X$. *Then there exists a unique alternative additive mapping* $A : X \to Y$ *of the second form which satisfies the inequality*

$$\| f(x) + f(0) - A(x) \| \le \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} \Big[\varphi(2^i x, 2^i x) + 2\varphi(0, 2^i x) \Big]$$

$$\left(\| f(x) - A(x) \| \le \frac{1}{2} \sum_{i=1}^{\infty} 2^i \Big[\varphi(2^{-i} x, 2^{-i} x) + 2\varphi(0, 2^{-i} x) \Big] \right)$$

for all $x \in X$. *If, moreover,* f *is measurable or* $f(tx)$ *is continuous in* t *for each fixed* $x \in X$ *then* $A(tx) = tA(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

Proof. Replacing (x_1, x_2) by $(0, x)$ and $(x, 0)$ separately, one has an approximate odd condition $\| f(x) + f(-x) \| \le \varphi(0, x)$ and $2\| f(0) \| \le \varphi(x, 0)$ for all $x \in X$. Thus we see that $\varphi(0, 0) = 0 = f(0)$ by the convergence in the next case of (3.11). Substituting $x_1, x_2 := x$ in the functional inequality (3.10), we obtain $\| f(2x) - f(0) + 2f(-x) \| \le \varphi(x, x)$, and so

$$\begin{aligned}
&\|f(2x) - f(0) - 2f(x)\| \\
&\quad \le \|f(2x) - f(0) + 2f(-x)\| + \| - 2f(x) - 2f(-x)\| \\
&\quad \le \varphi(x,x) + 2\varphi(0,x), \\
&\left\| g(x) - \frac{g(2x)}{2} \right\| \le \frac{1}{2}\Big[\varphi(x,x) + 2\varphi(0,x)\Big],
\end{aligned}$$

for all $x \in X$, where $g(x) := f(x) + f(0)$.

The rest of the proof goes through by the similar way to the proof of our corresponding Theorem 3.3. □

4. Stability of alternative Jensen type mappings

In this section, we will investigate the stability of alternative Jensen type functional equations.

THEOREM 4.1. *Assume that there exists a mapping $\varepsilon : X^2 \to [0,\infty)$ for which a mapping $f : X \to Y$ satisfies the approximately Jensen type inequality*

$$\left\| 2f\left(\frac{x_1 - x_2}{2}\right) - f(x_1) + f(x_2) \right\| \le \varepsilon(x_1, x_2) \tag{4.1}$$

and the series

$$\sum_{i=1}^{\infty} \frac{\varepsilon(2^i x_1, 2^i x_2)}{2^i} < \infty \tag{4.2}$$

$$\left(\sum_{i=0}^{\infty} 2^i \varepsilon(2^{-i}x_1, 2^{-i}x_2) < \infty, \quad \textit{respectively} \right)$$

for all $x_1, x_2 \in X$. Then there exists a unique Jensen type mapping $A : X \to Y$ which satisfies the inequality

$$\|f(x) + f(0) - A(x)\| \le \sum_{i=1}^{\infty} \frac{\varepsilon(2^i x, 0)}{2^i}$$

$$\left(\|f(x) - A(x)\| \le \sum_{i=0}^{\infty} 2^i \varepsilon(2^{-i}x, 0), \quad \textit{respectively} \right)$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Replacing (x_1, x_2) by (x, x), one has a condition $2\|f(0)\| \leq \inf_{x\in X} \varepsilon(x, x)$. Thus we remark that $\varepsilon(0, 0) = 0 = f(0)$ by the convergence in the next case of (4.2). Substituting $x_1 := 2x$ and $x_2 := 0$ in the functional inequality (4.1), we obtain $\|2f(x) - f(2x) + f(0)\| \leq \varepsilon(2x, 0)$, and so

$$\left\| g(x) - \frac{g(2x)}{2} \right\| \leq \frac{1}{2}\varepsilon(2x, 0), \tag{4.3}$$

for all $x \in X$, where $g(x) := f(x) + f(0)$.

The rest of the proof goes through by the similar way to the proofs of our corresponding Theorem 2.1. □

THEOREM 4.2. *Assume that there exists a mapping $\varepsilon : X^2 \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the approximately alternative Jensen inequality*

$$\left\| 2f(-\frac{x_1 + x_2}{2}) + f(x_1) + f(x_2) \right\| \leq \varepsilon(x_1, x_2) \tag{4.4}$$

and the series

$$\sum_{i=0}^{\infty} \frac{\varepsilon(2^i x_1, 2^i x_2)}{2^i} < \infty \tag{4.5}$$

$$\left(\sum_{i=1}^{\infty} 2^i \varepsilon(2^{-i} x_1, 2^{-i} x_2) < \infty, \quad \textit{respectively} \right)$$

for all $x_1, x_2 \in X$. Then there exists a unique alternative Jensen mapping $A : X \to Y$ which satisfies the inequality

$$\|f(x) - f(0) - A(x)\| \leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{\Theta(2^i x)}{2^i} \tag{4.6}$$

$$\left(\|f(x) - A(x)\| \leq \frac{1}{2} \sum_{i=1}^{\infty} 2^i \Theta(2^{-i} x), \quad \textit{respectively} \right)$$

for all $x \in X$, where $\Theta(x) := \varepsilon(x, x) + \min\{\varepsilon(2x, 0), \varepsilon(0, 2x)\}, x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. Replacing (x_1, x_2) by (x, x) and $(0, 0)$ separately, one has an approximate odd condition $2\|f(x) + f(-x)\| \leq \varepsilon(x, x)$ and $4\|f(0)\| \leq \varepsilon(0, 0)$ for all $x \in X$. Thus we see that $\varepsilon(0, 0) = 0 = f(0)$ by the convergence in the next case of (4.5). Substituting (x_1, x_2) by $(2x, 0)$ and

$(0, 2x)$ separately in the functional inequality (3.10), we obtain

$$\|2f(-x) + f(2x) + f(0)\| \leq \min\{\varepsilon(2x, 0), \varepsilon(0, 2x)\},$$

and so

$$\|f(2x) + f(0) - 2f(x)\| \leq \Theta(x) = \varepsilon(x, x) + \min\{\varepsilon(2x, 0), \varepsilon(0, 2x)\},$$
$$\left\|g(x) - \frac{g(2x)}{2}\right\| \leq \frac{1}{2}\Theta(x),$$

for all $x \in X$, where $g(x) := f(x) - f(0)$.

From the inequality, we can prove the rest of the proof by the similar way to the proofs of our corresponding Theorem 2.1. □

By virtue of Theorem 4.2, we obtain [13, Theorem 5.1] as a corollary for approximately alternative Jensen mappings.

THEOREM 4.3. *Assume that there exists a mapping $\varepsilon : X^2 \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the approximately alternative Jensen type inequality*

$$\left\|2f(-\frac{x_1 - x_2}{2}) + f(x_1) - f(x_2)\right\| \leq \varepsilon(x_1, x_2) \tag{4.7}$$

and the series

$$\sum_{i=1}^{\infty} \frac{\varepsilon(2^i x_1, 2^i x_2)}{2^i} < \infty \tag{4.8}$$
$$\left(\sum_{i=0}^{\infty} 2^i \varepsilon(2^{-i} x_1, 2^{-i} x_2) < \infty, \quad \textit{respectively}\right)$$

for all $x_1, x_2 \in X$. Then there exists a unique alternative Jensen type mapping $A : X \to Y$ which satisfies the inequality

$$\|f(x) + f(0) - A(x)\| \leq \sum_{i=1}^{\infty} \frac{\varepsilon(0, 2^i x)}{2^i} \tag{4.9}$$
$$\left(\|f(x) - A(x)\| \leq \sum_{i=0}^{\infty} 2^i \varepsilon(0, 2^{-i} x), \quad \textit{respectively}\right)$$

for all $x \in X$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in X$ then $A(tx) = tA(x)$ for all $x \in X$ and $t \in \mathbb{R}$.

Proof. The proof is similar to that of Theorem 4.1. □

5. Stability of Jensen type mappings in Banach modules

In the last part of this paper, let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{M}_1$ and ${}_B\mathbb{M}_2$ be left Banach B-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively.

As an application, we are going to prove the generalized Hyers-Ulam stability problem of Jensen type equations in Banach modules over a unital Banach algebra.

THEOREM 5.1. *Assume that there exists a mapping $\varphi : {}_B\mathbb{M}_1^2 \to [0,\infty)$ for which a mapping $f : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ satisfies the inequality*

$$\|f(ux_1+ux_2)-f(ux_1-ux_2)-2uf(x_2)\| \leq \varphi(x_1,x_2) \tag{5.1}$$

and the series

$$\sum_{i=0}^{\infty} \frac{\varphi(2^i x_1, 2^i x_2)}{2^i} < \infty \tag{5.2}$$

$$\left(\sum_{i=1}^{\infty} 2^i \varphi(2^{-i}x_1, 2^{-i}x_2) < \infty, \quad \textit{respectively}\right)$$

for all $x_1, x_2 \in {}_B\mathbb{M}_1$ and all $u \in B(1) := \{u \in B \mid |u| = 1\}$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in {}_B\mathbb{M}_1$, then there exists a unique B-linear mapping $A : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ of the second form which satisfies the inequality

$$\|f(x)+f(0)-A(x)\| \leq \frac{1}{2}\sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i x)}{2^i} \tag{5.3}$$

$$\left(\|f(x)-A(x)\| \leq \frac{1}{2}\sum_{i=1}^{\infty} 2^i \varphi(2^{-i}x, 2^{-i}x), \quad \textit{respectively}\right)$$

for all $x \in {}_B\mathbb{M}_1$.

Proof. By Theorem 2.1, it follows from (5.1) with $u = 1$ that there exists a unique additive mapping $A : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ of the second form such that the inequality (5.3) holds true for all $x \in {}_B\mathbb{M}_1$. The mapping A is given by

$$A(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in {}_B\mathbb{M}_1$.

Under the assumption that f is measurable or the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{M}_1$, the mapping A satisfies

$$A(tx) = tA(x)$$

for all $x \in {}_B\mathbb{M}_1$ and all $t \in \mathbb{R}$. Replacing x_1, x_2 by $2^n x_1, 2^n x_2$ in (5.1), respectively, and dividing it by 2^n, we figure out

$$\frac{\|f(2^n(ux_1 + ux_2)) - f(2^n(ux_1 - ux_2)) - 2uf(2^n x_2)\|}{2^n} \leq \frac{\varphi(2^n x_1, 2^n x_n)}{2^n}$$

for all $u \in B(1)$ and all $x_1, x_2 \in {}_B\mathbb{M}_1$. Taking the limit $n \to \infty$, one obtains by condition (5.2) that

$$A(ux_1 + ux_2) - A(ux_1 - ux_2) - 2uA(x_2) = 0 \tag{5.4}$$

for all $x_1, x_2 \in {}_B\mathbb{M}_1$ and all $u \in B(1)$. Substituting $x_1 = 0, x_2 = x$ in the last equality (5.4), we obtain by oddness of A

$$A(ux) - uA(x) = 0,$$

for all $x \in {}_B\mathbb{M}_1$ and all $u \in B(1)$. The last equality is also true for $u = 0$ vacuously. Now for each element $b \in B (b \neq 0)$ we figure out

$$\begin{aligned} A(bx) &= A\left(|b| \cdot \frac{b}{|b|}x\right) = |b| \cdot A\left(\frac{b}{|b|}x\right) \\ &= |b| \cdot \frac{b}{|b|} \cdot A(x) = bA(x) \end{aligned}$$

for all $b \in B (b \neq 0)$ and all $x \in {}_B\mathbb{M}_1$. Thus the mapping A satisfies

$$A(bx) = bA(x)$$

for all $b \in B$ and all $x \in {}_B\mathbb{M}_1$, as desired. This completes the proof of the theorem. $\square$

THEOREM 5.2. *Assume that there exists a mapping $\varphi : {}_B\mathbb{M}_1^2 \to [0, \infty)$ for which a mapping $f : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ satisfies the inequality*

$$\|f(ux_1 + ux_2) + f(ux_1 - ux_2) + 2uf(-x_1)\| \leq \varphi(x_1, x_2) \tag{5.5}$$

and the series (3.6) converges for all $x_1, x_2 \in {}_B\mathbb{M}_1$ and all $u \in B(1)$. If, moreover, f is measurable or $f(tx)$ is continuous in t for each fixed $x \in {}_B\mathbb{M}_1$, then there exists a unique alternative B-linear mapping $A : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ of the first form which satisfies the inequality (3.7) for all $x \in {}_B\mathbb{M}_1$.

In the last part of this paper, let B be a unital C^*-algebra with norm $|\cdot|$ and unitary group $\mathcal{U}(\mathcal{B}) = \{u \in B : uu^* = u^*u = 1\}$, and let ${}_B\mathbb{M}_1$ and ${}_B\mathbb{M}_2$ be left Banach B-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively.

THEOREM 5.3. *Assume that there exists a mapping* $\varepsilon : {}_B\mathbb{M}_1^2 \to [0,\infty)$ *for which a mapping* $f : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ *satisfies the approximately Jensen type inequality*

$$\left\|2f(\frac{ux_1 - ux_2}{2}) - uf(x_1) + uf(x_2)\right\| \le \varepsilon(x_1, x_2) \tag{5.6}$$

and the series

$$\sum_{i=1}^{\infty} \frac{\varepsilon(2^i x_1, 2^i x_2)}{2^i} < \infty \tag{5.7}$$

$$\left(\sum_{i=0}^{\infty} 2^i \varepsilon(2^{-i} x_1, 2^{-i} x_2) < \infty, \quad \textit{respectively}\right)$$

for all $x_1, x_2 \in {}_B\mathbb{M}_1$ *and all* $u \in \mathcal{U}(\mathcal{B})$. *Then there exists a unique Jensen type* B*-linear mapping* $A : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ *which satisfies the inequality*

$$\|f(x) + f(0) - A(x)\| \le \sum_{i=1}^{\infty} \frac{\varepsilon(2^i x, 0)}{2^i} \tag{5.8}$$

$$\left(\|f(x) - A(x)\| \le \sum_{i=0}^{\infty} 2^i \varepsilon(2^{-i} x, 0), \quad \textit{respectively}\right)$$

for all $x \in {}_B\mathbb{M}_1$.

Proof. By Theorem 4.1, it follows from (5.6) with $u = 1 \in \mathcal{U}(\mathcal{B})$ that there exists a unique Jensen type mapping $A : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ such that the inequality (5.8) holds true for all $x \in {}_B\mathbb{M}_1$.

By the condition (5.7), we get

$$\begin{aligned}
&\left\|2A(\frac{ux_1 - ux_2}{2}) - uA(x_1) + uA(x_2)\right\| \\
&= \lim_{n\to\infty} 2^{-n} \left\|2f(\frac{u(2^n(x_1 - x_2)}{2}) - uf(2^n x_1) + uf(2^n x_2)\right\| \\
&\le \lim_{n\to\infty} \le 2^{-n}\varepsilon(2^n x_1, 2^n x_2) = 0
\end{aligned}$$

for all $x_1, x_2 \in {}_B\mathbb{M}_1$ and all $u \in \mathcal{U}(\mathcal{B})$, which yields $A(ux) = uA(x)$ for all $x \in {}_B\mathbb{M}_1$ and all $u \in \mathcal{U}(\mathcal{B})$.

Now let a be a nonzero element in B and K a positive integer greater than $4|a|$. Then we have $|\frac{a}{K}| < \frac{1}{4} < 1 - \frac{2}{3}$. By [10, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(B)$ such that $3\frac{a}{K} = u_1 + u_2 + u_3$.

Thus we calculate that

$$\begin{aligned} A(ax) &= A\left(\frac{K}{3}3\frac{a}{K}x\right) = \left(\frac{K}{3}\right)A(u_1x + u_2x + u_3x) \\ &= \left(\frac{K}{3}\right)(A(u_1x) + A(u_2x) + A(u_3x)) \\ &= \left(\frac{K}{3}\right)(u_1 + u_2 + u_3)A(x) = \left(\frac{K}{3}\right)3\frac{a}{K}A(x) = aA(x) \end{aligned}$$

for all $a \in B$ ($a \neq 0$) and all $x \in {}_B\mathbb{M}_1$. It is clear that $A(0{\cdot}x) = 0 = 0A(x)$ for all $x \in {}_B\mathbb{M}_1$. So the mapping A is B-linear, as desired. □

THEOREM 5.4. *Assume that there exists a mapping $\varepsilon : {}_B\mathbb{M}_1^2 \to [0, \infty)$ for which a mapping $f : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ satisfies the approximately alternative Jensen inequality*

$$\left\| 2f(-\frac{ux_1 + ux_2}{2}) + uf(x_1) + uf(x_2) \right\| \leq \varepsilon(x_1, x_2)$$

and the series (4.5) converges for all $x_1, x_2 \in {}_B\mathbb{M}_1$ and all $u \in \mathcal{U}(\mathcal{B})$. Then there exists a unique alternative Jensen B-linear mapping $A : {}_B\mathbb{M}_1 \to {}_B\mathbb{M}_2$ which satisfies the inequality (4.6) for all $x \in {}_B\mathbb{M}_1$.

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