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A CHARACTERIZATION OF GAMMA DISTRIBUTION BY INDEPENDENT PROPERTY

Min-Young Lee* and Eun-Hyuk Lim**

Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed(i.i.d.) sequence of positive random variables with common absolutely continuous distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$ and $E(X^2) < \infty$. The random variables $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^n X_k)^2}$ and $\Sigma_{k=1}^n X_k$ are independent for $1 \leq i < j \leq n$ if and only if $\{X_n, n \geq 1\}$ have gamma distribution.

1. Introduction

The random variable X is said to have a Gamma distribution with shape parameter α and scale parameter β if

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$. The characteristic function of gamma distribution is given by

$$\phi\ (t;\alpha,\beta) = (1 - it\beta)^{-\alpha}.$$

Here α and $\beta > 0$ are two parameters.

Let X and Y be two independent non-degenerate positive random variables. Then Lukacs(1955) proved that X/Y and $X+Y$ are independent if and only if X and Y are gamma distribution with the same scale parameter.

Using the moment, Findeisen(1978) characterized the gamma distribution. Also, Hwang and Hu(1999) proved a characterization of

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Correspondence should be addressed to Min-Young Lee, leemy@dankook.ac.kr

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the gamma distribution by the independence of the sample mean and the sample coefficient of variation. Recently, Lee and Lim(2007) presented characterizations of gamma distribution that the random variables $\Sigma_{k=1}^{n} X_k$ and $\dfrac{\Sigma_{k=1}^{m} X_k}{\Sigma_{k=1}^{n} X_k}$ are independent for $1 \leq m < n$ if and only if $X_1, \cdots, X_n$ have gamma distribution.

In this paper, we obtain the characterization of gamma distribution by independent property of product and sum of random variables.

2. Main result

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. sequence of positive random variables with common absolutely cdf $F(x)$ and pdf $f(x)$ and $E(X^2) < \infty$. The random variables $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$ and $\Sigma_{k=1}^{n} X_k$ are independent for $1 \leq i < j \leq n$ if and only if $\{X_n, n \geq 1\}$ have gamma distribution.*

Proof. Since $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$ is a statistic scale-invariant, $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$ and $\Sigma_{k=1}^{n} X_k$ are independent for gamma variable [see Lukacs and Laha(1963)]. We have to prove the converse.

We denote the characteristic functions of $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$, $\Sigma_{k=1}^{n} X_k$ and $\left(\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}, \Sigma_{k=1}^{n} X_k\right)$ by $\phi_1(t)$,$\phi_2(s)$ and $\phi(t,s)$, respectively. The independence of $\dfrac{X_i \cdot X_j}{(\Sigma_{k=1}^{n} X_k)^2}$ and $\Sigma_{k=1}^{n} X_k$ is equivalent to

$$\phi(t,s) = \phi_1(t) \cdot \phi_2(s). \tag{1}$$

The left hand side of (1) becomes

$$\phi(t,s) = \int_0^\infty \cdots \int_0^\infty \exp\left\{\frac{is(x_i \cdot x_j)}{(\Sigma_{k=1}^{n} x_k)^2} + it(\Sigma_{k=1}^{n} x_k)\right\} dF$$

where $dF = f(x_1) \cdots f(x_n) \quad dx_1 \ \cdots \ dx_n$. Also the right hand side of (1) becomes

$$\phi_1(t) \cdot \phi_2(s) = \int_0^\infty \cdots \int_0^\infty \exp\left\{ \frac{is(x_i \cdot x_j)}{(\Sigma_{k=1}^n x_k)^2} \right\} dF \cdot \int_0^\infty \cdots \int_0^\infty \exp\{it(\Sigma_{k=1}^n x_k)\} dF.$$

Then (1) gives

$$\int_0^\infty \cdots \int_0^\infty \exp\left\{ \frac{is(x_i \cdot x_j)}{(\Sigma_{k=1}^n x_k)^2} + it(\Sigma_{k=1}^n x_k) \right\} dF = \int_0^\infty \cdots \int_0^\infty \exp\left\{ \frac{is(x_i \cdot x_j)}{(\Sigma_{k=1}^n x_k)^2} \right\} dF \cdot \int_0^\infty \cdots \int_0^\infty \exp\{it(\Sigma_{k=1}^n x_k)\} dF. \tag{2}$$

The integrals in (2) exist not only for reals t and s but also for complex values $t = u + iv, s = u^* + iv^*$, where u and u^* are reals, for which $v = Im(t) \geq 0$, $v^* = Im(s) \geq 0$ and they are analytic for all t, s for $v = Im(t) > 0$, $v^* = Im(s) > 0$ [see Lukacs(1955)].

Differentiating (2) one time with respect to s and then two times respect to t and setting $s = 0$, we get

$$\int_0^\infty \cdots \int_0^\infty x_i \, x_j \exp\{it(\Sigma_{k=1}^n x_k)\} dF = \theta \int_0^\infty \cdots \int_0^\infty (\Sigma_{k=1}^n x_k)^2 \exp\{it(\Sigma_{k=1}^n x_k)\} dF \tag{3}$$

where $\theta = E\left[\frac{X_i \cdot X_j}{(\Sigma_{k=1}^n X_k)^2} \right]$ for $1 \leq i < j \leq n$.

The random variable θ is bounded. Therefore all its moments exist.

Note that

$$\theta = E\left[\frac{X_1 \cdot X_2}{(\Sigma_{k=1}^n X_k)^2} \right] = E\left[\frac{X_1 \cdot X_3}{(\Sigma_{k=1}^n X_k)^2} \right] = \cdots = E\left[\frac{X_{n-1} \cdot X_n}{(\Sigma_{k=1}^n X_k)^2} \right]$$

for i.i.d. random variables $X_1, \cdots, X_n$.

Then we get the following equation by adding of all θ and multiplying 2

$$2 \cdot_n C_2 \cdot \theta = E\left[\frac{2\Sigma\Sigma_{1\le i<j\le n} X_i X_j}{(\Sigma_{k=1}^n X_k)^2}\right]$$

$$(4) \qquad = E\left[\frac{1}{1+\dfrac{\Sigma_{k=1}^n X_k^2}{2\Sigma\Sigma_{1\le i<j\le n} X_i X_j}}\right].$$

Note that, for $x_1 > 0, \cdots, x_n > 0$, $0 < 2\Sigma\Sigma_{1\le i<j\le n} x_i x_j \le (n-1)(\Sigma_{k=1}^n x_k^2)$ and the equality on the right hand side occurs only if $x_n = \cdots = x_n$. By the assumed continuity of $F(x)$, $P(X_1 = \cdots = X_n) = 0$, so $\dfrac{\Sigma_{k=1}^n x_k^2}{2\Sigma\Sigma_{1\le i<j\le n} x_i x_j} > \dfrac{1}{n-1}$, that is, by (4), $0 < \theta < \dfrac{1}{n^2}$.

Let $\varphi(t)$ be the characteristic function of $F(x)$. Then

$$\varphi'(t) = i\int_0^\infty x \exp\{itx\} dF(x)$$

and

$$\varphi''(t) = -\int_0^\infty x^2 \exp\{itx\} dF(x).$$

We can express (3) as a differential equation for the characteristic function $\varphi(t)$ and get

$$(\varphi(t)')^2 \varphi(t)^{n-2} = \theta\{n\varphi''(t)(\varphi(t))^{n-1} + 2 \cdot_n C_2 (\varphi'(t))^2 \varphi(t)^{n-2}\}.$$

That is,

$$\frac{\varphi''(t)}{\varphi'(t)} = \frac{1-2\theta \cdot_n C_2}{n\theta}\frac{\varphi'(t)}{\varphi(t)}, \qquad 0 < \theta < \frac{1}{n^2}.$$

After integrating with the initial conditions $\varphi(0) = 1$, $\varphi'(0) = iE(X)$, we get

$$(5) \qquad \varphi'(t) = iE(X)(\varphi(t))^{\frac{1-2\theta \cdot_n C_2}{n\theta}}, \quad \frac{1-2\theta \cdot_n C_2}{n\theta} > 1.$$

The solution of this differential equation (5) with the above initial conditions is

$$\varphi(t) = \left(1 - \frac{iE(X)}{\alpha}t\right)^{-\alpha}, \qquad \alpha = \frac{n\theta}{1-n^2\theta} > 0.$$

Therefore $F(x)$ is a gamma distribution.

□

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Department of Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
E-mail: leemy@dankook.ac.kr

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Department of Mathematics
Dankook University
Cheonan 330-714, Republic of Korea
E-mail: ehlim@dankook.ac.kr