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k-HYPONORMALITY AND WEIGHTED SHIFTS

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Abstract. In this paper we make a connection between the specific class of weighted shifts and general study of k-hyponormality. We show how the k-hyponormality of an arbitrary operator can be ascertained by examining the k-hyponormality of an associated family of weighted shifts.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([2],[3, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{0.1}$$

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Condition (0.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (0.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (0.1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{0.2}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (0.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (0.1); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([11]).

Recall ([1],[4],[11]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^{k} \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e., ([11])

(0.3)

$$\left\langle M_k(T) \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix} \right\rangle \geq 0 \quad \text{for all } x \in \mathcal{H} \text{ and } \lambda_1, \cdots, \lambda_k \in \mathbb{C}.$$

If $k = 2$ then T is said to be *quadratically hyponormal* and if $k = 3$ then T is said to be *cubically hyponormal.* Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \left\{ \begin{array}{cc} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0 \end{array} \right\}.$$

We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [3, III.8.16]): W_α is subnormal if and only if there exists a probability measure ξ supported in $[0, \|W_\alpha\|^2]$ such that $\gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int t^k \, d\xi(t) \quad (k \geq 1)$. If W_α is subnormal, and if for $h \geq 1$ we let $\mathcal{M}_h := \bigvee \{e_n : n \geq h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h} t^h d\xi(t)$.

The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([5],[6],[7], [8],[9],[10],[11],
[12],[13],[15]).

2. Main Results

Let $T \in \mathcal{L}(\mathcal{H})$ with $||T|| = 1$. For each nonzero vector x in $\mathcal{H}$, define W_x to the weighted shift with weighted sequence $\{\frac{||T^{n+1}x||}{||T^n x||}\}_{n=0}^{\infty}$. A. Lambert([14]) showed the following result.

Theorem 1. *([14]) T is subnormal if and only if W_x is subnormal for each $x \in \mathcal{H}$.*

More generally, T. Trent gave([16]) a criterion for subnormality which involves looking at only one vector of $\mathcal{H}$ at a time. For $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, denote the orbit of T i.e., the closed linear span of $\{T^n x : n = 0, 1, \cdots\}$ by $\mathcal{H}_x(T)$.

Theorem 2. *([16]) T is subnormal if and only if the restriction of T to $\mathcal{H}_x(T)$, $T|_{\mathcal{H}_x(T)}$ is subnormal for each $x \in \mathcal{H}$.*

In this paper we discuss about these analogy for k-hyponormal operator. For weighted shifts, there is no gap between hyponormality and paranormality (i.e., $||T^2x|| \geq ||Tx||^2$ for all unit vector $x \in \mathcal{H}$). So the analogy of Theorem 1 does not work for hyponormality (see Example 4). However, we have:

Theorem 3. *If T is k-hyponormal for $k \geq 1$, then W_x is k-hyponormal for each $x \in \mathcal{H}$.*

Proof. Note that $\frac{||T^{n+1}x||}{||T^n x||} = \frac{||T^{n+1}\frac{x}{||x||}||}{||T^n\frac{x}{||x||}||}$ for each $n \geq 0$. It thus suffices to show that W_x is k-hyponormal for each $x \in \mathcal{H}$ with $||x|| = 1$. Recall that ([6, Theorem 4(d)]) W_α is k-hyponormal if and only if

$$A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \geq 0 \quad \text{for all} \quad n \geq 0,$$

where γ_i is the moments of α. Note that $\gamma_n = ||T^n x||^2$ for W_x with $||x|| = 1$. If T is k-hyponormal, then
(1)

$$B(n;k) := D(n)^* \begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} D(n) \geq 0 \quad \text{for all} \quad n \geq 0,$$

where

$$D(n) := \begin{pmatrix} T^n & 0 & \cdots & 0 \\ 0 & T^{n+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T^{n+k} \end{pmatrix}.$$

In particular, (1) implies $B(n;k)$ is weakly positive, i.e.,
(2)

$$\left\langle B(n;k) \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix} \right\rangle \geq 0 \quad \text{for all } x \in \mathcal{H} \text{ and } \lambda_1, \cdots, \lambda_k \in \mathbb{C}.$$

But, (2) is equivalent to

$$\left\langle \begin{pmatrix} \langle T^{*n}T^n x, x\rangle & \cdots & \langle T^{*(n+k)}T^{n+k}x, x\rangle \\ \langle T^{*(n+1)}T^{n+1}x, x\rangle & \cdots & \langle T^{*(n+k+1)}T^{n+k+1}x, x\rangle \\ \vdots & \ddots & \vdots \\ \langle T^{*(n+k)}T^{n+k}x, x\rangle & \cdots & \langle T^{*(n+2k)}T^{n+2k}x, x\rangle \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix}, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} \right\rangle$$

is positive for all $x \in \mathcal{H}$ and $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$. Thus $A(n;k) \geq 0$ for all $n \geq 0$. Therefore W_x is k-hyponormal for each $x \in \mathcal{H}$ with $||x|| = 1$. □

The converse is not true in general.

EXAMPLE 4. *Let $T \in \mathcal{L}(\mathcal{H})$ be a paranormal nonhyponormal operator. Then W_x is hyponormal for each $x \in \mathcal{H}$ but T is not hyponormal.*

Proof. Recall that T is paranormal if and only if $||T^2x|| \geq ||Tx||^2$ for all unit vector $x \in \mathcal{H}$. Hence replacing x by $\frac{x}{||x||}$, we have that if T is paranormal then $\frac{||T^2x||}{||Tx||} \geq \frac{||Tx||}{||x||}$ for every $x \in \mathcal{H}$. Substituting Tx for x and repeating this process, we get $\frac{||T^{n+2}x||}{||T^{n+1}x||} \geq \frac{||T^{n+1}x||}{||T^n x||}$ for every $x \in \mathcal{H}$. Thus, W_x is hyponormal for each $x \in \mathcal{H}$. □

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