Korean J. Math. **17** (2009), No. 3, pp. 257–270

ON HENSTOCK INTEGRAL OF FUZZY MAPPINGS IN BANACH SPACES

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Abstract. In this paper we introduce the Henstock integral of fuzzy mappings in Banach spaces as a generalization of the Henstock integral of set-valued mappings and investigate some properties of it.

1. Introduction

Several types of integrals of set-valued mappings were studied by Aumann [1], Di Piazza and Musial [3,4], El Amri and Hess [5], Papageoriou [10] and others. In particular, Di Piazza and Musial [3] introduced the Henstock integral of set-valued mappings whose values are convex compact subsets in Banach spaces and obtained some properties of the integral. Several authors introduced the integrals of fuzzy mappings in terms of the integrals of set-valued mappings. Kaleva [9] introduced the integral of fuzzy mappings in $\mathbb{R}^n$ in terms of the integral of set-valued mappings in $\mathbb{R}^n$. Wu and Gong [2] introduced the Henstock integral of fuzzy mappings in $\mathbb{R}$. Xue, Ha and Ma [12], Xue, Wang and Wu [13] also introduced integrals of fuzzy mappings in Banach spaces in terms of Aumann-Pettis and Aumann-Bochner integrals of set-valued mappings.

In this paper we introduce the Henstock integral of fuzzy mappings in Banach spaces as a generalization of the Henstock integral of set-valued mappings and investigate some properties of it.

Received May 25, 2009. Revised September 12, 2009.

2000 Mathematics Subject Classification: 03E72, 26A39, 26E25, 28B05, 28B20, 46G10.

Key words and phrases: set-valued mapping, fuzzy mapping, Kurzweil-Henstock integral, Henstock integral, Kurzweil-Henstock-Pettis integral.

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2. Preliminaries

Throughout this paper, $\mathcal{L}$ denotes the family of all Lebesgue measurable subsets of $[a,b]$ and X a real separable Banach space with dual X^*. The closed unit ball of X^* is denoted by B_{X^*}. $CL(X)$ denotes the family of all nonempty closed subsets of X, $CB(X)$ the family of all nonempty closed bounded subsets of X, $CWK(X)$ the family of all nonempty convex weakly compact subsets of X.

For $A \subseteq X$ and $x^* \in X^*$, let $s(x^*, A) = \sup\{x^*(x) : x \in A\}$, the support function of A. For $A, B \in CB(X)$, let $h(A,B)$ denote the Hausdorff metric of A and B defined by

$$h(A,B) = \max\left(\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right),$$

where $d(a,B) = \inf_{b\in B} \|a-b\|$ and $d(b,A) = \inf_{a\in A} \|a-b\|$. Especially,

$$h(A,B) = \sup_{\|x^*\|\leq 1} |s(x^*,A) - s(x^*,B)|$$

whenever A, B are convex sets. Note that $(CWK(X), h)$ is a complete metric space. The number $\|A\|$ is defined by $\|A\| = h(A,\{0\}) = \sup_{x\in A} \|x\|$.

Let $u : X \to [0,1]$. We denote $[u]^r = \{x \in X : u(x) \geq r\}$ for $r \in (0,1]$ and $[u]^0 = cl\{x \in X : u(x) > 0\}$. u is called a *generalized fuzzy number* on X if for each $r \in (0,1]$, $[u]^r \in CWK(X)$. Let $\mathbf{F}(X)$ denote the family of all generalized fuzzy numbers on X. The addition and scalar multiplication in $\mathbf{F}(X)$ are defined according to Zadeh's extension principle. For $u, v \in \mathbf{F}(X)$ and $\lambda \in \mathbb{R}$, $[u+v]^r = [u]^r + [v]^r$ and $[\lambda u]^r = \lambda [u]^r$ for each $r \in (0,1]$. Hence $u+v, \lambda u \in \mathbf{F}(X)$. For $u, v \in \mathbf{F}(X)$, we define $u \leq v$ as follows:

$$u \leq v \quad \text{if} \quad u(x) \leq v(x) \quad \text{for all } x \in X.$$

For $u, v \in \mathbf{F}(X)$, $u \leq v$ if and only if $[u]^r \subseteq [v]^r$ for each $r \in (0,1]$. Define $d : \mathbf{F}(X) \times \mathbf{F}(X) \to [0, +\infty]$ by the equation

$$d(u,v) = \sup_{r\in(0,1]} h([u]^r, [v]^r).$$

Then d is a metric on $\mathbf{F}(X)$. The norm $\|u\|$ of $u \in \mathbf{F}(X)$ is defined by

$$\|u\| = d(u, \tilde{0}) = \sup_{r\in(0,1]} h([u]^r, \{0\}) = \sup_{r\in(0,1]} \|[u]^r\|.$$

Definition 2.1[6]. A *partition* of $[a,b]$ is a finite collection $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ such that $\{[c_i,d_i] : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a,b]$ covering $[a,b]$ and $t_i \in [c_i,d_i]$ for $i = 1,2,\cdots,n$. A *gauge* on $[a,b]$ is a function $\delta : [a,b] \to (0,\infty)$. A partition $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ is *δ-fine* if $[c_i,d_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$ for $i = 1,2,\cdots,n$.

A function $f : [a,b] \to X$ is said to be *Henstock integrable* on $[a,b]$ if there exists $w \in X$ with the following property: for each $\epsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that

$$\left\| \sum_{i=1}^{n} f(t_i)(d_i - c_i) - w \right\| < \epsilon$$

for each δ-fine partition $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ of $[a,b]$. We write $w = (H)\int_a^b f(t)dt$. A function $f : [a,b] \to X$ is said to be Henstock integrable on a set $E \subseteq [a,b]$ if the function $f\chi_E$ is Henstock integrable on $[a,b]$, where χ_E denotes the characteristic function of E. We write $(H)\int_E f(t)dt = (H)\int_a^b f\chi_E(t)dt$.

In case when X is the real line, the Henstock integrable function $f : [a,b] \to \mathbb{R}$ is said to be *Kurzweil-Henstock integrable* or simply *KH-integrable* on $[a,b]$ and we write $w = (KH)\int_a^b f(t)dt$.

3. Results

A set-valued mapping $F : [a,b] \to CL(X)$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the real-valued function $s(x^*, F(\cdot))$ is measurable. A set-valued mapping $F : [a,b] \to CL(X)$ is said to be *measurable* if $F^{-1}(A) = \{t \in [a,b] : F(t) \cap A \neq \phi\} \in \mathcal{L}$ for every $A \in CL(X)$. Note that if $F : [a,b] \to CL(X)$ is measurable then $F : [a,b] \to CL(X)$ is scalarly measurable.

A set-valued mapping $F : [a,b] \to CL(X)$ is said to be *Kurzweil-Henstock integrably bounded* or simply *KH-integrably bounded* on $[a,b]$ if there exists a KH-integrable real-valued function h defined on $[a,b]$ such that for each $t \in [a,b]$, $\|x\| \leq h(t)$ for all $x \in F(t)$.

$f : [a, b] \to X$ is called a *selection* of $F : [a, b] \to CL(X)$ if $f(t) \in F(t)$ for every $t \in [a, b]$. A set-valued mapping $F : [a, b] \to CL(X)$ is said to be *scalarly integrable* on $[a, b]$ if for every $x^* \in X^*$, $s(x^*, F(\cdot))$ is Lebesgue integrable on $[a, b]$. A set-valued mapping $F : [a, b] \to CL(X)$ is said to be *scalarly Kurzweil-Henstock integrable* or simply *scalarly KH-integrable* on $[a, b]$ if for every $x^* \in X^*$, $s(x^*, F(\cdot))$ is KH-integrable on $[a, b]$.

Definition 3.1[3]. A set-valued mapping $F : [a, b] \to CWK(X)$ is said to be *Henstock integrable* in $CWK(X)$ on $[a, b]$ if there exists $W \in CWK(X)$ with the following property: for each $\epsilon > 0$ there exists a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$h\left(\sum_{i=1}^{n} F(t_i)(d_i - c_i), W\right) < \epsilon$$

for each δ-fine partition $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ of $[a, b]$. We write $W = (H)\int_a^b F(t)dt$. If $\mathcal{C}$ is a subspace of $CWK(X)$, we say that the set-valued mapping $F : [a, b] \to \mathcal{C}$ is Henstock integrable in $\mathcal{C}$ on $[a, b]$ if $W \in \mathcal{C}$.

Note that if $F : [a, b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on $[a, b]$, then $F : [a, b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on every subinterval of $[a, b]$.

Definition 3.2[3]. A set-valued mapping $F : [a, b] \to CWK(X)$ is said to be *Kurzweil-Henstock-Pettis integrable* or simply *KHP-integrable* in $CWK(X)$ on $[a, b]$ if F is scalarly KH-integrable on $[a, b]$ and for each subinterval $[c, d]$ of $[a, b]$ there exists $W_{[c,d]} \in CWK(X)$ such that

$$s(x^*, W_{[c,d]}) = (KH)\int_c^d s(x^*, F(t))dt$$

for every $x^* \in X^*$. We write $W_{[c,d]} = (KHP)\int_c^d F(t)dt$. If $\mathcal{C}$ is a subspace of $CWK(X)$, we say that the set-valued mapping $F : [a, b] \to \mathcal{C}$ is KHP-integrable in $\mathcal{C}$ on $[a, b]$ if $W_{[c,d]} \in \mathcal{C}$ for each subinterval $[c, d]$ of $[a, b]$.

Note that if $F : [a, b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on $[a, b]$ then $F : [a, b] \to CWK(X)$ is KHP-integrable in $CWK(X)$ on $[a, b]$ and the integrals are equal [3,4].

LEMMA 3.3. *If $F : [a, b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on $[a, b]$, then $F : [a, b] \to CWK(X)$ is measurale.*

Proof. If $F : [a, b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on $[a, b]$, then there exists $W \in CWK(X)$ with the following property: for each $\epsilon > 0$ there exists a gauge $\delta : [a, b] \to (0, \infty)$ such that

$$h\left(\sum_{i=1}^{n} F(t_i)(d_i - c_i), W\right) < \epsilon$$

for each δ-fine partition $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ of $[a, b]$. Since $\sum_{i=1}^{n} F(t_i)\ (d_i - c_i)$ and W are convex sets, we have

$$\begin{aligned} h\left(\sum_{i=1}^{n} F(t_i)(d_i - c_i), W\right) &= \sup_{\|x^*\| \le 1} \left| s\left(x^*, \sum_{i=1}^{n} F(t_i)(d_i - c_i)\right) - s(x^*, W)\right| \\ &= \sup_{\|x^*\| \le 1} \left| \sum_{i=1}^{n} s(x^*, F(t_i))(d_i - c_i) - s(x^*, W)\right| . \end{aligned}$$

Hence for each $x^* \in B_{X^*}$

$$\left| \sum_{i=1}^{n} s(x^*, F(t_i))(d_i - c_i) - s(x^*, W)\right| < \epsilon$$

for each δ-fine partition $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ of $[a, b]$. Thus $s(x^*, F(\cdot))$ is KH-integrable on $[a, b]$ for each $x^* \in B_{X^*}$ and so $s(x^*, F(\cdot))$ is measurable for each $x^* \in B_{X^*}$ by [7, Theorem 9.12]. Hence $s(x^*, F(\cdot))$ is measurable for each $x^* \in X^*$. Thus $F : [a, b] \to CWK(X)$ is scalarly measurable. Hence $F : [a, b] \to CWK(X)$ is measurable.

□

LEMMA 3.4. *If $F : [a,b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on $[a,b]$, then $F : [a,b] \to CWK(X)$ is KHP-integrable in $CWK(X)$ on $[a,b]$ and for each subinterval $[c,d]$ of $[a,b]$*

$$(H)\int_c^d F(t)dt = (KHP)\int_c^d F(t)dt.$$

Proof. Let $F : [a,b] \to CWK(X)$ be Henstock integrable in $CWK(X)$ on $[a,b]$ and let $[c,d]$ be a subinterval of $[a,b]$. Then $F : [a,b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on $[c,d]$. Hence there exists $W_{[c,d]} \in CWK(X)$ with the following property: for each $\epsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that

$$h\left(\sum_{i=1}^n F(t_i)(d_i - c_i), W_{[c,d]}\right) < \epsilon$$

for each δ-fine partition $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ of $[c,d]$. Since $\sum_{i=1}^n F(t_i)\ (d_i - c_i)$ and $W_{[c,d]}$ are convex sets, we have

$$\begin{aligned} & h\left(\sum_{i=1}^n F(t_i)(d_i - c_i), W_{[c,d]}\right) \\ &= \sup_{\|x^*\|\le 1}\left| s\left(x^*, \sum_{i=1}^n F(t_i)(d_i - c_i)\right) - s(x^*, W_{[c,d]})\right| \\ &= \sup_{\|x^*\|\le 1}\left|\sum_{i=1}^n s(x^*, F(t_i))(d_i - c_i) - s(x^*, W_{[c,d]})\right|. \end{aligned}$$

Hence for each $x^* \in B_{X^*}$

$$\left|\sum_{i=1}^n s(x^*, F(t_i))(d_i - c_i) - s(x^*, W_{[c,d]})\right| < \epsilon$$

for each δ-fine partition $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ of $[c,d]$. Thus $s(x^*, F(\cdot))$ is KH-integrable on $[c,d]$ for each $x^* \in B_{X^*}$ and so $s(x^*, F(\cdot))$ is KH-integrable on $[c,d]$ and

$$s(x^*, W_{[c,d]}) = (KH)\int_c^d s(x^*, F(t))dt.$$

for each $x^* \in X^*$. Therefore $F : [a, b] \to CWK(X)$ is KHP-integrable in $CWK(X)$ on $[a, b]$ and for each subinterval $[c, d]$ of $[a, b]$

$$(H)\int_c^d F(t)dt = W_{[c,d]} = (KHP)\int_c^d F(t)dt.$$

□

Lemma 3.5. *Let $F : [a, b] \to CWK(X)$ and $G : [a, b] \to CWK(X)$ be Henstock integrable set-valued mappings. Then*

(1) *if $F(t) \subseteq G(t)$ a.e. on $[a, b]$, then for each subinterval $[c, d]$ of $[a, b]$*

$$(H)\int_c^d F(t)dt \subseteq (H)\int_c^d G(t)dt;$$

(2) *if $F(t) = G(t)$ a.e. on $[a, b]$, then for each subinterval $[c, d]$ of $[a, b]$*

$$(H)\int_c^d F(t)dt = (H)\int_c^d G(t)dt.$$

Proof. (1) Since $F : [a, b] \to CWK(X)$ and $G : [a, b] \to CWK(X)$ are Henstock integrable in $CWK(X)$ on $[a, b]$, by Lemma 3.4 $F : [a, b] \to CWK(X)$ and $G : [a, b] \to CWK(X)$ are KHP-integrable in $CWK(X)$ on $[a, b]$ and for each subinterval $[c, d]$ of $[a, b]$

$$(H)\int_c^d F(t)dt = (KHP)\int_c^d F(t)dt,\ (H)\int_c^d G(t)dt = (KHP)\int_c^d G(t)dt.$$

If $F(t) \subseteq G(t)$ a.e. on $[a, b]$, then for each subinterval $[c, d]$ of $[a, b]$ and $x^* \in X^*$

$$(KH)\int_c^d s(x^*, F(t))dt \le (KH)\int_c^d s(x^*, G(t))dt,$$

$$s\left(x^*, (KHP)\int_c^d F(t)dt\right) \leq s\left(x^*, (KHP)\int_c^d G(t)dt\right).$$

Since $(KHP)\int_c^d F(t)dt, (KHP)\int_c^d G(t)dt \in CWK(X)$, by the separation theorem

$$(KHP)\int_c^d F(t)dt \subseteq (KHP)\int_c^d G(t)dt,$$

$$(H)\int_c^d F(t)dt \subseteq (H)\int_c^d G(t)dt.$$

(2) The proof is similar to (1).

□

THEOREM 3.6. *If $F : [a,b] \to CWK(X)$ and $G : [a,b] \to CWK(X)$ are KH-integrably bounded and Henstock integrable in $CWK(X)$ on $[a,b]$, then $h(F,G)$ is KH-integrable on $[a,b]$ and*

$$h\left((H)\int_a^b F(t)dt, (H)\int_a^b G(t)dt\right) \leq (KH)\int_a^b h(F(t),G(t))dt.$$

Proof. If $F : [a,b] \to CWK(X)$ and $G : [a,b] \to CWK(X)$ are Henstock integrable in $CWK(X)$ on $[a,b]$, then by Lemma 3.4 $F : [a,b] \to CWK(X)$ and $G : [a,b] \to CWK(X)$ are KHP-integrable in $CWK(X)$ on $[a,b]$ and

$$(H)\int_a^b F(t)dt = (KHP)\int_a^b F(t)dt,\ (H)\int_a^b G(t)dt = (KHP)\int_a^b G(t)dt.$$

By [11, Lemma 3.5] $h(F, G)$ is KH-integrable on $[a, b]$ and

$$
\begin{aligned}
& h\left((H)\int_a^b F(t)dt, (H)\int_a^b G(t)dt\right) \\
& = h\left((KHP)\int_a^b F(t)dt, (KHP)\int_a^b G(t)dt\right) \\
& \le (KH)\int_a^b h(F(t), G(t))dt.
\end{aligned}
$$

□

A mapping $\tilde{F} : [a, b] \to \mathbf{F}(X)$ is called a *fuzzy mapping* in a Banach space X. In this case $\tilde{F}^r : [a, b] \to CWK(X)$ defined by $\tilde{F}^r(t) = [\tilde{F}(t)]^r$ is a set-valued mapping for each $r \in (0, 1]$. A fuzzy mapping $\tilde{F} : [a, b] \to \mathbf{F}(X)$ is said to be *measurable* (resp., *scalarly measurable*) if $\tilde{F}^r : [a, b] \to CWK(X)$ is measurable (resp., scalarly measurable) for each $r \in (0, 1]$. A fuzzy mapping $\tilde{F} : [a, b] \to \mathbf{F}(X)$ is said to be *KH-integrably bounded* on $[a, b]$ if $\tilde{F}^r : [a, b] \to CWK(X)$ is KH-integrably bounded on $[a, b]$ for each $r \in (0, 1]$.

Definition 3.7. A fuzzy mapping $\tilde{F} : [a, b] \to \mathbf{F}(X)$ is said to be *Henstock integrable* on $[a, b]$ if there exists $u \in \mathbf{F}(X)$ such that $[u]^r = (H)\int_a^b \tilde{F}^r(t)dt$ for each $r \in (0, 1]$. In this case, $u = (H)\int_a^b \tilde{F}(t)dt$ is called the *Henstock integral* of $\tilde{F}$ over $[a, b]$.

Definition 3.8. A fuzzy mapping $\tilde{F} : [a, b] \to \mathbf{F}(X)$ is said to be *Kurzweil-Henstock-Pettis integrable* or simply *KHP-integrable* on $[a, b]$ if for each subinterval $[c, d]$ of $[a, b]$ there exists $u_{[c,d]} \in \mathbf{F}(X)$ such that $[u_{[c,d]}]^r = (KHP)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0, 1]$. In this case, $u_{[c,d]} = (KHP)\int_c^d \tilde{F}(t)dt$ is called the *Kurzweil-Henstock-Pettis integral* of $\tilde{F}$ over $[c, d]$.

THEOREM 3.9. *If $\tilde{F} : [a,b] \to \mathbf{F}(X)$ is Henstock integrable on $[a,b]$, then $\tilde{F} : [a,b] \to \mathbf{F}(X)$ is Henstock integrable on every subinterval of $[a,b]$.*

Proof. If $\tilde{F} : [a,b] \to \mathbf{F}(X)$ is Henstock integrable on $[a,b]$, then there exists $u \in \mathbf{F}(X)$ such that $[u]^r = (H)\int_a^b \tilde{F}^r(t)dt$ for each $r \in (0,1]$. Let $[c,d]$ be a subinterval of $[a,b]$. Since $\tilde{F}^r : [a,b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on $[a,b]$ for each $r \in (0,1]$, $\tilde{F}^r : [a,b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on $[c,d]$ for each $r \in (0,1]$. Thus $M_r = (H)\int_c^d \tilde{F}^r(t)dt \in CWK(X)$ for each $r \in (0,1]$. For $r_1, r_2 \in (0,1]$ with $r_1 < r_2$, $\tilde{F}^{r_1}(t) \supseteq \tilde{F}^{r_2}(t)$ for each $t \in [c,d]$. By Lemma 3.5 $M_{r_1} = (H)\int_c^d \tilde{F}^{r_1}(t)dt \supseteq (H)\int_c^d \tilde{F}^{r_2}(t)dt = M_{r_2}$. Let $r \in (0,1]$ and let $\{r_n\}$ be a sequence in $(0,1]$ such that $r_1 \le r_2 \le r_3 \le \cdots$ and $\lim_{n\to\infty} r_n = r$. Then $\tilde{F}^r(t) = \cap_{n=1}^{\infty} \tilde{F}^{r_n}(t)$ for each $t \in [a,b]$. By [12, Lemma 4.2] $\lim_{n\to\infty} s(x^*, \tilde{F}^{r_n}(t)) = s(x^*, \tilde{F}^r(t))$ for each $t \in [a,b]$ and $x^* \in X^*$. Since $\tilde{F}^r : [a,b] \to CWK(X)$ is Henstock integrable in $CWK(X)$ on $[a,b]$ for each $r \in (0,1]$, by Lemma 3.4 $\tilde{F}^r : [a,b] \to CWK(X)$ is KHP-integrable in $CWK(X)$ on $[a,b]$ and $M_r = (H)\int_c^d \tilde{F}^r(t)dt = (KHP)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0,1]$. Thus $s(x^*, M_r) = (KH)\int_c^d s(x^*, \tilde{F}^r(t))dt$ for each $r \in (0,1]$ and $x^* \in X^*$. Since $s(x^*, \tilde{F}^r(\cdot)) \le s(x^*, \tilde{F}^{r_n}(\cdot)) \le s(x^*, \tilde{F}^{r_1}(\cdot))$ on $[a,b]$ for each $n \in \mathbb{N}$ and $x^* \in X^*$, by the Dominated Convergence Theorem for the Kurzweil-Henstock integral we have

$$\begin{aligned}\lim_{n\to\infty} s(x^*, M_{r_n}) &= \lim_{n\to\infty}(KH)\int_c^d s(x^*, \tilde{F}^{r_n}(t))dt \\ &= (KH)\int_c^d s(x^*, \tilde{F}^r(t))dt = s(x^*, M_r)\end{aligned}$$

for each $x^* \in X^*$. By [12, Lemma 4.2], $M_r = \cap_{n=1}^{\infty} M_{r_n}$. Let $M_0 = X$. By [12, Lemma 4.1] there exists $u_{[c,d]} \in \mathbf{F}(X)$ such that $[u_{[c,d]}]^r =$

$M_r = (H)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0,1]$. Hence $\tilde{F} : [a,b] \to \mathbf{F}(X)$ is Henstock integrable on $[c,d]$.

□

THEOREM 3.10. *If $\tilde{F} : [a,b] \to \mathbf{F}(X)$ is Henstock integrable on $[a,b]$, then $\tilde{F} : [a,b] \to \mathbf{F}(X)$ is KHP-integrable on $[a,b]$ and for each subinterval $[c,d]$ of $[a,b]$*

$$(H)\int_c^d \tilde{F}(t)dt = (KHP)\int_c^d \tilde{F}(t)dt.$$

Proof. If $\tilde{F} : [a,b] \to \mathbf{F}(X)$ is Henstock integrable on $[a,b]$, then by Theorem 3.9 for each subinterval $[c,d]$ of $[a,b]$ $\tilde{F} : [a,b] \to \mathbf{F}(X)$ is Henstock integrable on $[c,d]$. Hence there exists $u_{[c,d]} \in \mathbf{F}(X)$ such that $[u_{[c,d]}]^r = (H)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0,1]$. By Lemma 3.4 $[u_{[c,d]}]^r = (H)\int_c^d \tilde{F}^r(t)dt = (KHP)\int_c^d \tilde{F}^r(t)dt$ for each $r \in (0,1]$. Therefore $\tilde{F} : [a,b] \to \mathbf{F}(X)$ is KHP-integrable on $[a,b]$ and for each subinterval $[c,d]$ of $[a,b]$

$$(H)\int_c^d \tilde{F}(t)dt = (KHP)\int_c^d \tilde{F}(t)dt.$$

□

THEOREM 3.11. *Let $\tilde{F} : [a,b] \to \mathbf{F}(X)$ and $\tilde{G} : [a,b] \to \mathbf{F}(X)$ be Henstock integrable fuzzy mappings. Then*

(1) *if $\tilde{F}(t) \leq \tilde{G}(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$*

$$(H)\int_c^d \tilde{F}(t)dt \leq (H)\int_c^d \tilde{G}(t)dt;$$

(2) *if $\tilde{F}(t) = \tilde{G}(t)$ a.e. on $[a,b]$, then for each subinterval $[c,d]$ of $[a,b]$*

$$(H)\int_c^d \tilde{F}(t)dt = (H)\int_c^d \tilde{G}(t)dt.$$

Proof. (1) Since $\tilde{F} : [a,b] \to \mathbf{F}(X)$ and $\tilde{G} : [a,b] \to \mathbf{F}(X)$ are Henstock integrable on $[a,b]$, by Theorem 3.9 for each subinterval $[c,d]$ of $[a,b]$ there exist $u_{[c,d]}, v_{[c,d]} \in \mathbf{F}(X)$ such that $u_{[c,d]} = (H)\int_c^d \tilde{F}(t)dt$, $v_{[c,d]} = (H)\int_c^d \tilde{G}(t)dt$. If $\tilde{F}(t) \le \tilde{G}(t)$ a.e. on $[a,b]$, then by Lemma 3.5 $[u_{[c,d]}]^r = (H)\int_c^d \tilde{F}^r(t)dt \subseteq (H)\int_c^d \tilde{G}^r(t)dt = [v_{[c,d]}]^r$ for each $r \in (0,1]$ and so $(H)\int_c^d \tilde{F}(t)dt = u_{[c,d]} \le v_{[c,d]} = (H)\int_c^d \tilde{G}(t)dt$.

(2) The proof is similar to (1). □

THEOREM 3.12. *If $\tilde{F} : [a,b] \to \mathbf{F}(X)$ and $\tilde{G} : [a,b] \to \mathbf{F}(X)$ are KH-integrably bounded and Henstock integrable on $[a,b]$, then $d(\tilde{F},\tilde{G})$ is KH-integrable on $[a,b]$ and*

$$d\left((H)\int_a^b \tilde{F}(t)dt, (H)\int_a^b \tilde{G}(t)dt\right) \le (KH)\int_a^b d(\tilde{F}(t),\tilde{G}(t))dt.$$

Proof. If $\tilde{F} : [a,b] \to \mathbf{F}(X)$ and $\tilde{G} : [a,b] \to \mathbf{F}(X)$ are Henstock integrable on $[a,b]$, then by Theorem 3.10 $\tilde{F} : [a,b] \to \mathbf{F}(X)$ and $\tilde{G} : [a,b] \to \mathbf{F}(X)$ are KHP-integrable on $[a,b]$ and

$$(H)\int_a^b \tilde{F}(t)dt = (KHP)\int_a^b \tilde{F}(t)dt,\ (H)\int_a^b \tilde{G}(t)dt = (KHP)\int_a^b \tilde{G}(t)dt.$$

By [11, Theorem 3.6] $d(\tilde{F}, \tilde{G})$ is KH-integrable on $[a, b]$ and

$$\begin{aligned} & d\left((H)\int_a^b \tilde{F}(t)dt, (H)\int_a^b \tilde{G}(t)dt\right) \\ & = d\left((KHP)\int_a^b \tilde{F}(t)dt, (KHP)\int_a^b \tilde{G}(t)dt\right) \\ & \leq (KH)\int_a^b d(\tilde{F}(t), \tilde{G}(t))dt. \end{aligned}$$

□

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