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SOME GEOMETRIC PROPERTY OF BANACH SPACES-PROPERTY (C_k)

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Abstract. In this paper, we define property (C_k) and show that property (C_k) implies property (C_{k+1}). The converse does not hold. Moreover, we prove that property (C_k) implies the Banach-Saks property.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space. We denote the dual of X as X^* and the second dual of X as X^{**} respectively.

By B_X and S_X, we denote the closed unit ball and the unit sphere of X, respectively. For any subset A of X by span$\{A\}$ we denote the set of all linear combinations of vectors of A. $(X, \|\cdot\|)$ is said to be reflexive if the natural embedding maps X onto X^{**}.

$(X, \|\cdot\|)$ is said to be uniformly convex (UC) if for all $\epsilon > 0$, there exists a $\delta(\epsilon) < 1$ such that for $x,\ y \in B_X$ with $\|x - y\| \geq \epsilon$,

$$\left\| \frac{1}{2}(x+y) \right\| \leq \delta(\epsilon).$$

A Banach space is said to have the Banach-Saks property if any bounded sequence in the space admits a subsequence whose arithmetic means converges in norm. In 1930, S. Banach and S. Saks[2] showed that every bounded sequence in $L_p[0,1]$, $1 < p < \infty$, has a subsequence with arithmetic means converging in norm. J. Schreier[7] showed that $C[0,1]$ does not have the Banach-Saks property. T. Nishiura and D. Waterman [6] proved that the Banach-Saks property implies reflexivity

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in Banach spaces (See also [3]) and S. Kakutani [5] showed that Uniform convexity implies the Banach-Saks property. (See also [4])

The natural questions are the followings : For a Banach space X with the Banach-Saks property, is it uniformly convex? And does every reflexive Banach space have the Banach-Saks property? In 1972, A. Baernstein [1] gave an example of a reflexive Banach space which does not have the Banach-Saks property. In 1978, C. J. Seifert[8] showed that the dual of Baernstein space which is not uniformly convex has the Banach-Saks property.

2. Main result

In this section, we give the definition of property (C_k) and prove that property (C_k) implies the Banach-Saks property. Property (C_k) is defined for $k \geq 2$ in an obvious fashion so that a uniform convexity is just property (C_2).

Definition 1. $(X, \|\cdot\|)$ has property (C_k) if it is reflexive and for all $\epsilon > 0$, there exists a $\delta(\epsilon) < 1$ such that for linearly independent k-elements $x_1, x_2, \cdots, x_k$ in B_X with $\|x_i - x_j\| \geq \epsilon$ for $i \neq j$ and $i, j = 1, 2, \cdots, k$,

$$\left\| \frac{1}{k} \sum_{i=1}^{k} x_i \right\| \leq \delta(\epsilon).$$

Property (C_k) implies property (C_{k+1}).

Proposition 2. *If a Banach space X has property (C_k), then it has property (C_{k+1}).*

Proof. The proof is given by contradiction. Suppose that X has no property (C_{k+1}). Then for all $n \in \mathbb{N}$, there exist linearly independent k-elements $x_1^{(n)}, \cdots, x_{k+1}^{(n)}$ in B_X and $\epsilon_0 > 0$ such that $\|x_i - x_j\| \geq \epsilon_0$, where $i \neq j$ and $i,\ j = 1, 2, \cdots, k+1$
and

$$\left\| \frac{1}{k+1} \left(x_1^{(n)} + x_2^{(n)} \cdots + x_{k+1}^{(n)} \right) \right\| > 1 - \frac{1}{n}.$$

Thus,

$$\begin{aligned}\left\|x_1^{(n)}+x_2^{(n)}\cdots+x_k^{(n)}\right\| &\geq \left\|x_1^{(n)}+x_2^{(n)}\cdots+x_{k+1}^{(n)}\right\|-\left\|x_{k+1}^{(n)}\right\| \\ &\geq (k+1)\left(1-\frac{1}{n}\right)-1 \\ &\geq k\left(1-\frac{2}{n}\right)\end{aligned}$$

This means that X has no property (C_k), since $x_1^{(n)},\cdots,x_k^{(n)}$ are linearly independent. We get the contradiction. □

The converse of Proposition 2 does not hold. For simplicity, we give an example of X which is (C_3) but not (C_2). Let D be

$$\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\leq 1, |z|\leq\frac{3}{4}\}$$
$$\cap\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\leq\frac{25}{16},\ \ \frac{3}{4}\leq|z|\leq 1\}$$

We define the new space $(\mathbb{R}^3, |||\cdot|||)$ whose norm is determined by Minkowsky functional under the set D. If we are given three linearly independent elements and two of them are located in a line which is parallel to z-axis and on $x^2+y^2=1$, the rest should be located in outside the line. This shows that $(\mathbb{R}^3, |||\cdot|||)$ is C_3. Furthermore if we are given two linearly independent elements, those two can be possibly located in a line which is parallel to z-axis and on $x^2+y^2=1$. This tells us that $(\mathbb{R}^3, |||\cdot|||)$ is not C_2.

Since uniform convexity implies the Banach-Saks property [5] (See also [4]), it is also a natural question whether property (C_k) implies the Banach-Saks property or not. We need the following lemma.

LEMMA 3. *Let X be a Banach space with property (C_k) and $\{x_i\}$ be a weakly null and linearly independent sequence in X with $\|x_i\|\leq\theta^m$, $i=1,2,3,\cdots$, $m=0,1,2,\cdots$, where $\theta=\max\left\{\delta\left(\frac{1}{k}\right),\frac{k^2-k+1}{k^2}\right\}$. Then for a given $i_1\in\mathbb{N}$, there exist $i_2,i_3,\cdots,i_k$ such that $i_1<i_2<$*

$\cdots < i_k$ *and*

$$\left\| \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\| \leq \theta^{m+1}.$$

Proof. If $\|x_{i_1}\| \leq \frac{\theta^m}{k}$, then for any $i_1 < i_2 < \cdots < i_k$, we have

$$\left\| \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\| \leq \frac{\theta^m}{k^2} + \frac{k-1}{k}\theta^m = \left(\frac{k^2-k+1}{k^2} \right) \cdot \theta^m \leq \theta^{m+1}.$$

Suppose that $\|x_{i_i}\| > \frac{\theta^m}{k}$. Then we can select x_{i_2} satisfying $\|x_{i_1} - x_{i_2}\| > \frac{\theta^m}{k}$ and $i_2 > i_1$. If there does not exist such x_{i_2}, we have $\|x_{i_1} - x_n\| \leq \frac{\theta^m}{k}$ for all $n > i_1$. For any $x^* \in B_{X^*}$, since we have assumed $\{x_n\}$ is a weakly null sequence,

$$\begin{aligned} |x^* x_{i_1}| &= \lim_{n\to\infty} |x^* x_{i_1} - x^* x_n| \\ &\leq \limsup_{n\to\infty} \|x_{i_1} - x_n\| \leq \frac{\theta^m}{k}. \end{aligned}$$

This contradicts to $\|x_{i_1}\| > \frac{\theta^m}{k}$. Thus there exists x_{i_2} such that

$$\|x_{i_1} - x_{i_2}\| > \frac{\theta^m}{k}$$

Now by the same argument we can select $x_{i_3},\ x_{i_4}, \cdots, x_k$ such that

$$\|x_{i_s} - x_{i_t}\| > \frac{\theta^m}{k},$$

where $s, t \in \{1, 2, 3, \cdots, k\}$ and $s < t$. Now by the definition of property (C_k) we have

$$\left\| \frac{1}{k} \sum_{j=1}^{k} x_{i_j} \right\| \leq \delta \left(\frac{1}{k} \right) \theta^m \leq \theta^{m+1}$$

This completes our proof. □

We now show that property (C_k) implies the Banach-Saks property with the similar method of Kakutani's [5].

THEOREM 4. *If a Banach space X has property (C_k), then it has the Banach-Saks property.*

Proof. Suppose that X is a Banach space with property (C_k). Let $\{x_n\}$ be a bounded sequence in X. Since X is reflexive, weak compactness and Eberlein-Šmulian theorem give a weakly convergent subsequence $\{x_{n_j}\}$. Thus we may assume a sequence $\{x_n\}$ in B_X is weakly null and show that it has a subsequence whose arithmetic means converge to 0 in norm. If dim span$\{x_n\} < \infty$, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$. Thus arithmetic means of $\{x_{n_i}\}$ converges. Suppose that dim span$\{x_n\} = \infty$. Then $\{x_n\}$ has a linearly independent subsequence. Without lose of generality, we may assume that $\{x_n\}$ is linearly independent. Let $\theta = \max\left\{\delta\left(\frac{1}{k}\right), \frac{k^2-k+1}{k^2}\right\}$. As the first stage, we select a subsequence by Lemma 3, $\{x_{m_n}\}$ from $\{x_n\}$ such that

$$\left\|\frac{x_{m_{k(n-1)+1}} + x_{m_{k(n-1)+2}} + \cdots + x_{m_{kn}}}{k}\right\| \le \theta \qquad \text{for } n = 1, 2, 3, \cdots$$

with $m_1 = 2,\ m_{k(n-1)+1} = m_{k(n-1)} + 1\ (n \ge 2)$. Lemma 3 also make it possible selecting $\{m_i\}$ as a strictly increasing sequence. We reindex this subsequence as

$$x_n^{(1)} = \frac{x_{m_{k(n-1)+1}} + x_{m_{k(n-1)+2}} + \cdots + x_{m_{k}n}}{k} \qquad \text{for } n = 1, 2, 3, \cdots$$

Then we have $\|x_n^{(1)}\| \le \theta,\ n = 1, 2, 3, \cdots$. Moreover $\{x_n^{(1)}\}$ is also weakly null. For the second step, by applying Lemma 3 again, we select a subsequence $\{x^{(1)}_{m_n^{(1)}}\}$ from $\{x_n^{(1)}\}$ such that

$$\left\|\frac{x^{(1)}_{m^{(1)}_{k(n-1)+1}} + x^{(1)}_{m^{(1)}_{k(n-1)+2}} + \cdots + x^{(1)}_{m^{(1)}_{kn}}}{k}\right\| \le \theta^2 \qquad \text{for } n = 1, 2, 3, \cdots$$

with $m_1^{(1)} = 2,\ m^{(1)}_{k(n-1)+1} = m^{(1)}_{k(n-1)} + 1 \quad (n = 2, 3, \cdots)$. Lemma 3 also make it possible selecting m_j as a strict increasing sequence. We reindex this sequence as

$$x_n^{(2)} = \frac{x^{(1)}_{m^{(1)}_{k(n-1)+1}} + x^{(1)}_{m^{(1)}_{k(n-1)+2}} + \cdots + x^{(1)}_{m^{(1)}_{kn}}}{k}, \qquad n = 1, 2, 3, \cdots$$

Then we have $\|x_n^{(2)}\| \le \theta^2,\ n = 1,2,3,\cdots$. Moreover $\{x_n^{(2)}\}$ is also weakly null. Continuing this process, for all $n \in \mathbb{N}$, we get a sequence $\{x_n^{(p)}\}$ such that

i) $\|x_n^{(p)}\| \le \theta^p, \qquad$ for $n \in \mathbb{N}$.

ii) $x_n^{(p)} = \left(x^{(p-1)}_{m^{(p-1)}_{k(n-1)+1}} + x^{(p-1)}_{m^{(p-1)}_{k(n-1)+2}} + \cdots + x^{(p-1)}_{m^{(p-1)}_{kn}} \right) / k$

iii) $1 < m_1^{(p-1)} < m_2^{(p-1)} < \cdots < m_k^{(p-1)} < m_{k+1}^{(p-1)} < \cdots < m_{2k}^{(p-1)} < \cdots$

iv) $\{x_n^{(p)}\}$ is weakly null.

Before we go to the further step, we emphasize that each element $x_n^{(2)}$ is the average k^2-elements of $\{x_n\}$ where these k^2-elements are selected strictly increasingly. Now we write down the first element $x_1^{(p)}$ in the p-th step.

$$x_1^{(1)} = \frac{x_{m_1} + x_{m_2} + \cdots + x_{m_k}}{k} = \frac{x_2 + x_{m_2} + \cdots + x_{m_k}}{k}$$

$$x_1^{(2)} = \frac{x^{(1)}_{m^{(1)}_1} + x^{(1)}_{m^{(1)}_2} + \cdots + x^{(1)}_{m^{(1)}_k}}{k} = \frac{x^{(1)}_2 + x^{(1)}_{m^{(1)}_2} + \cdots + x^{(1)}_{m^{(1)}_k}}{k}$$

$$= \frac{x_{m_{k+1}} + \cdots + x_{m_{2k}} + x_{m_{k(m_2^{(1)}-1)+1}} + \cdots + x_{m_{km_k^{(1)}}}}{k^2}$$

$$\vdots$$

From the construction of $\{x_1^{(p)}\}$, we can find that $x_1^{(p)}$ is representable in the form

$$x_1^{(p)} = \frac{x_{l_1^{(p)}} + x_{l_2^{(p)}} + \cdots + x_{l_{k^p}^{(p)}}}{k^p}, \qquad p = 1,2,3,\cdots$$

with $1 < l_1^{(1)} < l_2^{(1)} < \cdots < l_k^{(1)} < l_1^{(2)} < \cdots < l_k^{(2)} < l_{k+1}^{(2)} < \cdots < l_{k^2}^{(2)} < \cdots$. Furthermore, for $q < p$ and $1 \le j \le k^{p-q}$, the average of the p-th block of k^q-elements of $\left\{ x_{l_i^{(p)}} \right\}_{i=1}^{k^p}$

$$\frac{x_{l^{(p)}_{(j-1)k^q+1}} + \cdots + x_{l^{(p)}_{jk^q}}}{k^q}$$

is an element of the sequence $\{x_n^{(q)}\}$ and as such has norm $\leq \theta^q$. Now let $n_1 = 1,\ n_{\frac{k^p-1}{k-1}+i} = l_i^{(p)}$ $i = 1, 2, 3, \cdots, k^p$ and $p = 1, 2, \cdots$ (that is, $n_1 = 1,\ n_2 = l_1^{(1)},\ n_3 = l_2^{(1)},\ n_4 = l_3^{(1)}, \cdots, n_{k+1} = l_k^{(1)},\ n_{k+2} = l_1^{(2)}, \cdots$). Then $\{x_{n_m}\}$ is the desired subsequence. For given $\epsilon > 0$, determine q such that $\theta^q < \frac{\epsilon}{3}$. With this $q,\ \epsilon$, determine m such that $\frac{k^q}{m} < \frac{\epsilon}{3}$. Then for any $m \leq 1$, let r be such that

$$\frac{k^q-1}{k-1} + (r-1)k^q + 1 \leq m \leq \frac{k^q-1}{k-1} + rk^q$$

Then we have

$$\begin{aligned}
\frac{1}{m}\|x_{n_1} + \cdots + x_{n_m}\| &\leq \frac{1}{m}\left\|x_{n_1} + \cdots + x_{n_{\frac{k^q-1}{k-1}}}\right\| \\
&+ \frac{1}{m}\sum_{i=1}^{r-1}\left\|x_{n_{\frac{k^q-1}{k-1}+(i-1)k^q+1}} + \cdots + x_{n_{\frac{k^q-1}{k-1}+ik^q}}\right\| \\
&+ \frac{1}{m}\left\|x_{n_{\frac{k^q-1}{k-1}+(r-1)k^q+1}} + \cdots + x_{n_m}\right\| \\
&\leq \frac{1}{m}\cdot\left(\frac{k^q-1}{k-1} - 1\right) + \frac{r-1}{m}\cdot k^q\cdot\theta^q + \frac{k^q}{m} \\
&\leq \frac{k^q}{m} + \theta^q + \frac{k^q}{m} < \epsilon.
\end{aligned}$$

It follow that the averages of $\{x_{n_m}\}$ converge to 0 in norm. □

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