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CONSTRUCTIONS OF $(0,1)$–MATRIX WITH PERMANENT k

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Abstract. The purpose of this paper is to show that for each integer k where $1 \geq k \geq 2^{n-1}$, there exists an $n \times n(0,1)$-matrix A with exactly $PerA = k$. Thus we introduce a constructive approch for such matrices. Using the permanent of $(0,1)$-matrix, we decomposed the number $n!$ with an linear combination of the power of 2. That coefficient is an stiring number.

1. Introduction

Mordern mathematicians have a proclivity to invent flippant names for newly introduced mathematical entities and concepts. They delight in talking about mobs, radicals, derogatory matrices, osculating planes, improper ideals, etc. It may appear that the term "permanent" was also invented by a waggist algebraist.

In his famous memoir of 1812, Cauchy developed the theory of determinants as a special type of alternating symmetric functions, which he distinguished from the ordinary symmetric functions by calling the latter "fonctions symetriques permanentes". He also introduced a certain subclass of symmetric functions, which were later named permanents by Muir[6] and which are nowadays known by this name. These functions can be defined by means of matrices and mordern notaion as follows.

Let $A = [a_{ij}]$ be an $m \times n$ matrix over any commutative ring, $m \leq n$. The permanent of A, written $Per(A)$, or simply $PerA$, is defind by

$$Per(A) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{m\sigma(m)}$$

where the summation extends over all one-to-one functions from

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$\{1, \cdots, m\}$ to $\{1, \cdots, n\}$. The sequence $a_{1\sigma(1)}, a_{2\sigma(2)}, \cdots, a_{m\sigma(m)}$ is called a diagonal of A. Thus the permanent of A is the sum of all diagonal products of A. For a example, if

$$A = \begin{bmatrix} 3 & 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 5 \\ -1 & 2 & -2 \end{bmatrix}$$

then $PerA = 9$, $PerB = 44$, $PerC = 18$. The special case $m = n$ is of particular importance. We denote the permanent of a square matrix A by $per(A)$ instead of $Per(A)$. In fact, most writers restrict the designation "permanent" to the case of square matrices in [1], [2], and [3].

We shall require the following simplifying notation. Let $\Gamma_{r,n}$ denote the set of all n^r sequences $\omega = (\omega_1, \cdots, \omega_r)$ of integers, $1 \leq \omega_i \leq n$, $i = 1, \cdots, n$, Let $G_{r,n}$ denote the subset of $\Gamma_{r,n}$ consisting of all nondecreasing sequences,

$$G_{r,n} = \{(\omega_1, \cdots, \omega_r) \ \in \ \Gamma_{r,n} | \ 1 \leq \omega_1 \leq \cdots \leq \omega_r \leq n\}$$

and let $Q_{r,n}$ be the set of increasing sequences,

$$Q_{r,n} = \{(\omega_1, \cdots, \omega_r) \ \in \ \Gamma_{r,n} | \ 1 \leq \omega_1 < \cdots < \omega_r \leq n\}$$

We shall also require the following matrix notation. Let $M_{m,n}(S)$, or simply $M_{m,n}$, denote the set of all $m \times n$ matrices with entries from a set S. If $m = n$, we shall write M_n instead of $M_{n,n}$. Now, let $A = [a_{ij}] \in M_{m,n}$, and let $\alpha \in G_{h,m}$ and $\beta \in G_{k,n}$. Then $A[\alpha|\beta]$ denotes the $h \times k$ matrix whose (i,j)-entry is $a_{\alpha_i \beta_j}$. If it happens that $\alpha \in Q_{h,m}$ and $\beta \in Q_{k,n}$, then $A[\alpha|\beta]$ is a submatrix of A. If $\alpha = \beta$, we simplify the notation to $A[\alpha]$. Again, if $\alpha \in Q_{h,m}$ and $\beta \in Q_{k,n}$, then $A(\alpha|\beta)$ denotes the $(m-h) \times (n-k)$ submatrix of A complementary to $A[\alpha|\beta]$ - that is, the submatrix obtained from A by deleting rows α and columns β. In particular, the $(m-h) \times n$ submatrix obtained from A by deleting rows α is denoted by $A(\alpha|-)$ simlarly, $A(-|\beta)$ denotes the $m \times (n-k)$ submatrix obtained from A by the deleting columns β .

Lemma 1.1. *(a) The permanent function on $m \times n$ matices, , $m \leq n$, is a multilinear function of the rows of each matrix. If $m = n$, it is also a multilinear function of the columns.*
(b) If A is an $m \times n$ matrix, $m \leq n$, and P and Q are permutation matrices of orders m and n, respectivly, then

$$Per(PAQ) = Per(A)$$

(C) If A is an $n \times n$ matrix, then

$$Per(A^T) = Per(A)$$

All these properties are immediate consequences of the definition of permanents. Our next Lemma is an analogue of the Laplace expantion theorem for determinants.

Lemma 1.2. *If A is an $m \times n$ matrix, $2 \leq m \leq n$, and $\alpha \in Q_{r,m}$, then*

$$Per(A) = \sum_{\omega \in Q_{r,m}} Per(A[\alpha|\omega])Per(A(\alpha|\omega)).$$

In particular, for any i, $1 \leq i \leq m$,

$$Per(A) = \sum_{t=1}^{n} a_{it} Per(A(i|t))$$

If $m = n$ and $\beta \in Q_{r,n}$, then also

$$Per(A) = \sum_{\omega \in Q_{r,n}} Per(A[\omega|\beta])Per(A(\omega|\beta)).$$

and for any j, $1 \leq j \leq n$

$$Per(A) = \sum_{t=1}^{n} a_{tj} Per(A(t|j)).$$

The proof of Lemma 1.2 is quite straightforward. In fact, it is must easier than the corresponding theorem for determinants.

2. The permanent of matrices

Matrices all of whose entries are either 0 or 1 -that is,$(0, 1)$-matrices pay an important part in linear algebra, combinatorics, and graph theory. In some of these applications it is at times preferable to consider 1 as the "all" element in a Boolean algebra, or the identity element in a field of two elements. In what follows, however, the symbol 1 will represent the positive integer 1, since we shall be mainly concerned with enumerations of systems of distinct representatives and with related problems in the theory of permanents.

An n-square nonnegative matrix is said to be *partly decomposable* if it contains a $k \times (n-k)$ zero submatrix. In order wards, a matrix A is partly decomposable if there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where B and D are square. If the matrix contains no $k \times (n-k)$ zero submatrix for $k = 1, \cdots, n-1$, it is called $fully\ indecomposable$.

LEMMA 2.1. ([4,5] Frobenius-Konig) *Let A be an n-square matrix. A necessary and sufficient condition for every diagonal of A to contain a zero entry is that A contain an $s \times t$ zero submatrix such that $s + t = n + 1$.*

By the above, we have the following result.

LEMMA 2.2. *Let A be an n-square matrix. Then $PerA = 0$ if and only if A contains an $s \times t$ zero submatrix such that $s + t = n + 1$.*

By Lemma 2.1 and 2.2, we know that if A is a fully indecomposable n-square $(0, 1)$-matrix with row sums $r_1, \cdots, r_n$, then

$$PerA \geq \{max\ \ r_i | 1 \leq i \leq n\}.$$

Equallity hold if and only if at least $n-1$ of the row sums equal 2. Let J_n be an $n \times n$ matrix that entries are all 1. Then by the definition, $PerJ_n = n!$ and by the Lemma 1.1, there exists an $n \times n$ $(0, 1)$-matrix A with $PerA = 0$. Thus it is clear that if A is an $n \times n$ $(0, 1)$-matrix then

$$0 \leq PerA \leq n!.$$

THEOREM 2.3. *Let A be an n-square $(0,1)$-matrix. Then there does not exist an $n \times n$ $(0,1)$-matrix A with*

$$\frac{(n-1)}{n} \cdot n! < PerA < n!.$$

Proof. By the Lemma 1.1, we know that for any i, j, the matrix form $J_n - E_{ij}$ has the same permanent as

$$Per(J_n - E_{ij}) = \frac{(n-1)}{n} \cdot n!$$

Where E_{ij} is an $n \times n$ matrix that the(i,j)-th entry is ! and otherwise are all zero. Since $Per(J_n - E_{ij})$ have (n^2-1)'s nonzero entries, therefore there does not exist an $n \times n$ $(0,1)$-matrix A with

$$\frac{(n-1)}{n} \cdot n! < PerA < n!.$$

$\square$

Now, we show that for each integer k with $1 \leq k \leq 2^{n-1}$, there exists an $n \times n$ $(0,1)$-matrix A with exactly $PerA = k$. Thus we introduce a constructive approch for such matrices. Let $A(k)$ denote the matrix with $PerA(k) = k$. Then we can choose the representative $(0,1)$-matrices as followings ;

$$A(1) = \begin{bmatrix} 1 & \vdots & 1 & \cdots & 1 \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ 0 & \vdots & & & \\ \vdots & \vdots & & I_{n-1} & \\ 0 & \vdots & & & \end{bmatrix},$$

and

$$A(2^{n-1}) = \begin{bmatrix} 1 & \vdots & 1 & \cdots & 1 \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ 1 & \vdots & & & \\ \vdots & \vdots & & \Delta_{n-1} & \\ 1 & \vdots & & & \end{bmatrix}$$

where I_{n-1} is the $(n-1)\times(n-1)$ identity matrix and Δ_{n-1} is the the $(n-1)\times(n-1)$ lower-triangular matrix with all 1 entries. Adding the entries from $A(1)$ to $A(2^{n-1})$, we construct the matrix with exactly $PerA=k$ for each integer k with $1\le k\le 2^{n-1}$. By the Lemma 1.2, for each i, $2\le i\le n$,

$$Per(A(1)+E_{i1})=Per(A(1))+1=A(2)$$

and for each $i(3\le i\le n), j(2\le j\le i-1)$

$$Per(A(*)+E_{ij})=Per(A(*))+2^{j-2}=A(2)$$

where

$$A(*)=A(1)+\sum_{p=2}^{i-1}\sum_{q=1}^{p-1}E_{pq}.$$

By the above fact, we have the following theorem.

THEOREM 2.4. *Let A be an n-square $(0,1)$-matrix. Then there exists an $n\times n$ $(0,1)$-matrix A with exactly $PerA=k$ where $1\le k\le 2^{n-1}$.*

For example, let $n=5$ and $Per(A(k))=k$. Then there exists 5×5 $(0,1)$-matrices $A(k)$ where $1\le k\le 2^{5-1}=16$.

$$A(1)=\begin{bmatrix}1&1&1&1&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\qquad A(2)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},$$

$$A(3)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\qquad A(4)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},$$

$$A(5)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&0&0&1&0\\0&0&0&0&1\end{bmatrix},\qquad A(6)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&0&1&0\\0&0&0&0&1\end{bmatrix},$$

$$A(7)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&0&1&1&0\\0&0&0&0&1\end{bmatrix},\qquad A(8)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\0&0&0&0&1\end{bmatrix},$$

$$A(9)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&0&0&0&1\end{bmatrix},\qquad A(10)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&1&0&0&1\end{bmatrix},$$

$$A(11)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&0&1&0&1\end{bmatrix},\qquad A(12)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&0&1\end{bmatrix},$$

$$A(13)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&0&0&1&1\end{bmatrix},\qquad A(14)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&1&0&1&1\end{bmatrix},$$

$$A(15)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&0&1&1&1\end{bmatrix},\qquad A(16)=\begin{bmatrix}1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\\1&1&1&1&0\\1&1&1&1&1\end{bmatrix}$$

REMARK 2.5. Using the permanent of $(0,1)$-matrix, we decomposed the number $n!$ with an linear combination of the power of 2. That coefficient is an *stiring number*. We know that $Per(J_n)=n!$ and there exists the matrix $A(2^{n-1})$ in the above, $n!=2^{n-1}+L$where L is at most an linear combination of the leading power of 2^{n-2}. Adding the entries from $A(2^{n-1})$ to J_n, we give an $n!$-equation.

By the Lemma 1.1 and 1.2, let the matrix $A_{n-1}=A(2^{n-1})+E_{(n-1)n}$. Then

$$\begin{aligned}Per(A(2^{n-1})+E_{(n-1)n})&=Per(A(2^{n-1}))+Per(A(2^{(n-1)-1}))\\&=2^{n-1}+2^{(n-1)-1}\\&=2^{n-1}+2^{n-2}\end{aligned}$$

and let the matrix $A_{n-2}=A_{n-1}+E_{(n-2)(n-1)}+E_{(n-2)n}$. Then,

$$\begin{aligned}&Per(A(2^{n-1})+E_{(n-2)(n-1)}+E_{(n-2)n})\\&=Per(A(2^{n-1}))+Per(A_{(n-1)-1})+Per(A_{(n-1)-1})\\&=Per(A(2^{n-1}))+2Per(A_{n-2})\\&=2^{n-1}+2\cdot(2^{n-2}+2^{n-3})\end{aligned}$$

Continuating the above method to $Per(A_2) = Per(J_n)$, we have the equation

$$n! = \sum_{q=1}^{n-1} a_{nq} 2^q$$

where $a_{n(n-1)} = 1$, $a_{n1} = (n-2) \cdot a_{n-1})1$ and if $q \neq (n-1)$ or 1 then

$$a_{nq} = (n-2) \cdot a_{(n-1)q} + a_{(n-1)(q-1)}$$

Therefore, we have a coefficient table of the power of 2 as followings.

$$\begin{bmatrix} n & \vdots & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 & 2^8 & \cdots \\ \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 3 & \vdots & 1 & 1 & & & & & & & \\ 4 & \vdots & 2 & 3 & 1 & & & & & & \\ 5 & \vdots & 6 & 11 & 6 & 1 & & & & & \\ 6 & \vdots & 24 & 50 & 35 & 10 & 1 & & & & \\ 7 & \vdots & 120 & 274 & 225 & 85 & 15 & 1 & & & \\ 8 & \vdots & 720 & 1764 & 1624 & 731 & 165 & 21 & 1 & & \\ 9 & \vdots & 5040 & 13068 & 13132 & 6741 & 1886 & 312 & 28 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

For example, let $n = 5$,

$$\begin{aligned} Per(A_4) &= Per(A(2^4) + E_{45}) \\ &= Per(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} + E_{45}) \\ &= Per(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}) \\ &= 2^4 + 2^3 = 24 \end{aligned}$$

$$
\begin{aligned}
Per(A_3) &= Per(A_4 + E_{34} + E_{35}) \\
&= Per(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}) \\
&= (2^4 + 2^3) + 2 \cdot (2^3 + 2^2) = 48
\end{aligned}
$$

and

$$
\begin{aligned}
Per(A_2) &= Per(A_3 + E_{23} + E_{24} + E_{25}) \\
&= Per(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}) = Per(J_n) \\
&= ((2^4 + 2^3) + 2 \cdot (2^3 + 2^2)) \\
&\quad + 3 \cdot ((2^3 + 2^2) + 2 \cdot (2^2 + 2^1)) = 120
\end{aligned}
$$

Therefore,

$$5! = 1 \cdot 2^4 + 6 \cdot 2^3 + 11 \cdot 2^2 + 6 \cdot 2^1 = 120$$

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