JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 4, December 2009

G^f-SPACES FOR MAPS AND POSTNIKOV SYSTEMS

Yeon Soo Yoon*

Abstract. For a map $f : A \to X$, we define and study a concept of G^f-space for a map, which is a generalized one of a G-space. Any G-space is a G^f-space, but the converse does not hold. In fact, S^2 is a G^η-space, but not G-space. We show that X is a G^f-space if and only if $G_n(A, f, X) = \pi_n(X)$ for all n. It is clear that any H^f-space is a G^f-space and any G^f-space is a W^f-space. We can also obtain some results about G^f-spaces in Postnikov systems for spaces, which are generalization of Haslam's results about G-spaces.

1. Introduction

The Goottlieb groups $G_n(X)$ of a space X have been defined by Gottlieb in [3,4]. A space X is called *G-space* if $G_n(X) = \pi_n(X)$ for all n. It is well known [4] that any H-space is a G-space, but the converse does not hold. A space X is called [14] *W-space* if every Whitehead product $[\alpha, \beta] = 0$ in $\pi_{m+n-1}(X)$ for any $\alpha \in \pi_m(X), \beta \in \pi_n(X)$ and any $m, n \geq 1$. It is known [14] that any G-space is a W-space, but the converse does not hold. In [5], Haslam showed that if X is a G-space, then each X_n is G-space and all the k invariants k_X^{n+2} are G-primitive, and if X_{n-1} is a G-space and the k-invariants k_X^{n+1} is G-primitive, then X_n is a G-space, where f_n is an induced map from f.

For a map $f : A \to X$, *the Gottlieb groups $G_n(A, f, X)$ of a map* $f : A \to X$, which are generalizations of Gottlieb groups $G_n(X)$, are defined in [15]. In general, $G_n(X) \subset G_n(A, f, X) \subset \pi_n(X)$ for any map $f : A \to X$. In this paper, for a map $f : A \to X$, we define a *G^f-space X for a map $f : A \to X$* , and show that X is a G^f-space if and only if $G_n(A, f, X) = \pi_n(X)$ for all n. Any G-space is a G^f-space, but the converse does not hold. In fact, S^2 is a G^η-space, but not

Received November 03, 2009; Accepted November 24, 2009.
2000 Mathematics Subject Classification: Primary 55P45, 55P35.
Key words and phrases: G^f-spaces for maps, Postnikov systems.
This work was supported by Hannam University Research Fund, 2009.

G-space. It is well known [7] that if A and X are spaces having the homotopy type of 1-connected countable CW-complexes and $f; A \to X$ is a map, then there exist Postnikov systems $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ for A and X respectively and induced maps $\{f_n : A_n \to X_n\}$ satisfying (1) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & X_n \\ {\scriptstyle k_A^{n+2}}\downarrow & & {\scriptstyle k_X^{n+2}}\downarrow \\ K(\pi_{n+1}(A), n+2) & \xrightarrow{\tilde{f}_\#} & K(\pi_{n+1}(X), n+2) \end{array}$$

, that is, $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$. (2) $f_{n+1} : A_{n+1} \to X_{n+1}$ given by $f_{n+1} = (f_n, P\tilde{f}_\#)$ satisfying commute diagram

$$\begin{array}{ccc} A_{n+1}(= E_{k_A^{n+2}}) & \xrightarrow{f_{n+1}=(f_n, P\tilde{f}_\#)} & X_{n+1}(= E_{k_X^{n+2}}) \\ {\scriptstyle p'_n(=p_{k_A^{n+2}})}\downarrow & & {\scriptstyle p_n(=p_{k_X^{n+2}})}\downarrow \\ A_n & \xrightarrow{f_n} & X_n. \end{array}$$

(3) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i'_n}\downarrow & & {\scriptstyle i_n}\downarrow \\ A_n & \xrightarrow{f_n} & X_n. \end{array}$$

Thus we know that the pair of k invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ is a map from f_n to $\tilde{f}_\#$. So, we can consider a concept of G^f-primitive for the map $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$. Then we can obtain the following results which are generalizations of Haslam's results [5]; Let A and X be spaces having the homotopy type of 1-connected countable CW-complexes and $f; A \to X$ a map, and $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ Postnikov systems for A and X respectively. If X is a G^f-space for a map $f : A \to X$, then each X_n is G^{f_n}-space and the all pair of k invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are G^{f_n}-primitive. Moreover, if X_{n-1} is a $G^{f_{n-1}}$-space and the pair of k-invariants $(k_X^{n+1}, k_A^{n+1}) : f_{n-1} \to \tilde{f}_\#$ is $G^{f_{n-1}}$-primitive, then X_n is a G^{f_n}-space, where f_n is an induced map from f.

2. G^f-spaces for maps

Let $f : A \to X$ be a map. A based map $g : B \to X$ is called *f-cyclic* [12] if there is a map $\phi : B \times A \to X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & X \\ j\uparrow & & \nabla\uparrow \\ A \vee B & \xrightarrow{(f\vee g)} & X \vee X \end{array}$$

is homotopy commute, where $j : A \vee B \to A \times B$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. We call such a map ϕ an *associated map* of a f-cyclic map g. Clearly, g is f-cyclic iff f is g-cyclic. In the case $f = 1_X : X \to X$, a map $g : B \to X$ is called cyclic [15]. We denote the set of all homotopy classes of f-cyclic maps from B to X by $G(B; A, f, X)$ which is called the *Gottlieb set for a map* $f : A \to X$. In the case $f = 1_X : X \to X$, we called such a set $G(B; X, 1, X)$ as the *Gottlieb set*, denoted by $G(B; X)$. In particular, $G(S^n; A, f, X)$ will be denoted by $G_n(A, f, X)$ which is called the *Gottlieb Group for a map* $f : A \to X$. Gottlieb [3,4] introduced and studied the *evaluation subgroups* $G_n(X) = G_n(X, 1, X)$ of $\pi_n(X)$.

The next proposition is an immediate consequence from the definition.

Proposition 2.1.
(1) *For any maps* $f : A \to X$, $\theta : C \to A$, $G_n(A, f, X) \subset G_n(C, f\theta, X)$ *for all* n.
(2) $G_n(X) = G_n(X, 1_X, X) \subset G_n(A, f, X) \subset \pi_n(X)$ *for any space* A *and any map* $f : A \to X$.
(3) $G_n(X) = \cap\{G_n(A, f, X) | f : A \to X$ *is a map and* A *is a space*$\}$.
(4) *If* $h : C \to A$ *is a homotopy equivalence, then* $G_n(A, f, X) = G_n(C, fh, X)$.
(5) *For any map* $k : X \to Y$, $k_\#(G_n(A, f, X)) \subset G_n(A, kf, Y)$.
(6) *For any map* $k : X \to Y$, $k_\#(G_n(X)) \subset G_n(X, k, Y)$.

However, it is known [18] that $G_5(S^5 \times S^5) \cong 2\mathbb{Z} \oplus 2\mathbb{Z} \neq G_5(S^5, i_1, S^5 \times S^5) \cong 2\mathbb{Z} \oplus \mathbb{Z} \neq \pi_5(S^5 \times S^5) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Let $Map(A, X)$ be the space of continuous maps from A to X with compact open topology. For a based map $f : A \to X$, let $Map(A, X; f)$ be the path component of $Map(A, X)$ containing f. Let $Map_*(A, X)$ and $Map_*(A, X; f)$ be the spaces of base point preserving maps in $Map(A, X)$ and $Map(A, X; f)$ respectively. Clearly, the evaluation map

$\omega : Map(A, X) \to X$ is a fibration. Moreover, the restriction to path component $\omega_f = \omega_{|Map(A,X;f)} : Map(A, X; f) \to X$ is a fibration with fiber $Map_*(A, X; f)$.

Proposition 2.2. [15] *For the evaluation map* $\omega : Map(A, X; f) \to X, w_\#(\pi_n(Map(A, X; f))) = G_n(A, f, X)$.

Aguade showed [1] that X is a T-space if and only if $e : \Sigma\Omega X \to X$ is cyclic. For a map $f : A \to X$, a space X is called [21] *T^f-space for a map $f : A \to X$* if $e : \Sigma\Omega X \to X$ is f-cyclic. Clearly any T-space is a T^f-space for a map $f : A \to X$, but the converse does not hold.

A based map $g : B \to X$ is called *weakly cyclic* [16] if $g_\#(\pi_n(B)) \subset G_n(X)$ for all n. It is known[16] that any cyclic map is a weakly cyclic map, but the converse does not hold.

We showed [16] that X is a G-space if and only if $e : \Sigma\Omega X \to X$ is weakly cyclic.

Definition 2.3. Let $f : A \to X$ be a based map. A based map $g : B \to X$ is called an weakly f-cyclic if $g_\#(\pi_n(B)) \subset G_n(A, f, X)$ for all n.

The next proposition is an immediate consequence from the definition.

Proposition 2.4.

(1) *If $g : B \to X$ is a weakly cyclic map and $\theta : C \to B$ is an arbitrary map, then $g\theta : C \to X$ is weakly cyclic.*
(2) *For a map $f : A \to X$, any weakly cyclic map $g : B \to X$ is weakly f-cyclic.*
(3) *For a map $f : A \to X$, if $g : B \to X$ is a weakly f-cyclic map and $\theta : C \to B$ is an arbitrary map, then $g\theta : C \to X$ is weakly f-cyclic.*

Definition 2.5. A space X is called a G^f-space for a map $f : A \to X$ if $e : \Sigma\Omega X \to X$ is weakly f-cyclic.

The following theorem says that a G^f-space can be characterized by the Gottlieb groups for a map $f : A \to X$.

Theorem 2.6. *X is a G^f-space for a map $f : A \to X$ if and only if $G_n(A, f, X) = \pi_n(X)$ for all n.*

Proof. Suppose that X is a G^f-space for a map $f : A \to X$. Let $g : S^n \to X$ be any map. Since $g = e\Sigma\tau(g) : \Sigma S^{n-1} \to X$ and $e : \Sigma\Omega X \to X$ is weakly f-cyclic, $g : S^n \to X$ is weakly f-cyclic. On the other hand, suppose that $G_n(A, f, X) = \pi_n(X)$ for all n. Since

$1_X : X \to X$ is weakly f-cyclic, we know that the map $e = 1_X e$ is weakly f-cyclic and X is a G^f-space for a map $f : A \to X$. □

It is introduced [20] that a space is called *H^f-space for a map $f : A \to X$* if there is a map $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion. It is known [20] that X is an H^f-space for a map $f : A \to X$ if and only if $G(B; A, f, X) = [B, X]$ for any space B. Also, it is known [21] that X is an T^f-space for a map $f : A \to X$ if and only if $G(\Sigma B; A, f, X) = [\Sigma B, X]$ for any space B. Thus we have the following corollary.

Corollary 2.7. *Any H^f-space is a T^f-space and any T^f-space is a G^f-space.*

It is known [4] that if X dominates A and X is a G-space, then A is a G-space. This fact can be generalized as the following corollary.

Corollary 2.8. *Let X be a G^i-space for a map $i : A \to X$.*
(1) *If $i : A \to X$ has a left homotopy inverse $r : X \to A$, then A is a G-space.*
(2) *If $i : A \to X$ has a right homotopy inverse $r : X \to A$, then X is a G-space.*

Proof. (1) It is sufficient to show that $\pi_n(A) \subset G_n(A)$ for all n. Since X is a G^i-space for $i : A \to X$, we know, from Theorem 2.6, that $G_n(A, i, X) = \pi_n(X)$. Thus we have, from Proposition 2.1(5), that $\pi_n(A) = r_\#\pi_n(X) = r_\#(G_n(A, i, X)) \subset G_n(A, ri, A) = G_n(A, 1, A) = G_n(A)$. Thus A is a G-space. (2) We show that $\pi_n(X) \subset G_n(X)$ for all n. By Theorem 2.6 and Proposition 2.1(1), we can obtain that $\pi_n(X) = G_n(A, i, X) \subset G_n(X, ir, X) = G_n(X, 1, X) = G_n(X)$. Thus we know that X is a G-space. □

From Proposition 2.1(2),(3) and Theorem 2.6, we have the following corollary.

Corollary 2.9. *X is a G-space if and only if for any space A and any map $f : A \to X$, X is a G^f-space for a map $f : A \to X$.*

A space X is called [14] *W-space* if every Whitehead product $[\alpha, \beta] = 0$ in $\pi_{m+n-1}(X)$ for any $\alpha \in \pi_m(X), \beta \in \pi_n(X)$ and any $m, n \geq 1$. It is known [14] that any G-space is a W-space, but the converse does not hold.

Definition 2.10. For a map $f : A \to X$, $P_n(A, f, X) = \{\alpha \in \pi_n(X) | \; [f_\#(\beta), \alpha] = 0$ for any map $\beta \in \pi_m(A)\}$ and any $m \geq 1$. A space

X is called a W^f-space for a map $f : A \to X$ if $P_n(A, f, X) = \pi_n(X)$ for all n.

Proposition 2.11. *$G_n(A, f, X) \subset P_n(A, f, X)$ for all n.*

Proof. Let $[h] \in G_n(A, f, X)$. Then there is a map $H : A \times S^n \to X$ such that $Hj \sim \nabla(f \vee h)$, where $j : A \vee S^n \to A \times S^n$ is the inclusion. Let $m \geq 1$ and $\beta = [g] \in \pi_m(A)$. Then consider the map $F = H(g \times 1) : S^m \times S^n \stackrel{(g\times 1)}{\to} A \times S^n \stackrel{H}{\to} X$. Then $Fj' \sim \nabla(fg \vee h)$, where $j' : S^m \vee S^n \to S^m \times S^n$ is the inclusion. Thus we have $[f_\#(\beta), [h]] = 0$ and $[h] \in P_n(A, f, X)$. □

Corollary 2.12. *If X is a G^f-space for a map $f : A \to X$, then X is a W^f-space for $f : A \to X$.*

Consider the natural pairing $\mu : S^3/S^1(= S^2) \times S^3 \to S^3/S^1(= S^2)$. Thus we know that the Hopf map $\eta : S^3 \to S^2$ is cyclic. Thus S^2 is an H^η-space and a T^η-space. Thus we know that S^2 is a W^η-space for $\eta : S^3 \to S^2$. On the other hand, it is known [16] that H-spaces and T-spaces and G-spaces are equivalent in the category of spheres. Thus we know that S^2 is not a G-space. Moreover, it is known [14] that $\eta_\# : \pi_n(S^3) \to \pi_n(S^2)$, $\eta_\#(\beta) = \eta \circ \beta$, is an isomorphism for $n \geq 3$. Thus we have the following example.

Example 2.13.
(1) *S^2 is a G^η-space, but not G-space.*
(2) *For any $x \in \pi_n(S^2)$, $\alpha \in \pi_k(S^2)$ $(n \geq 3,\ k \geq 1)$, $[x, \alpha] = 0$.*

3. G^f-spaces for maps and Postnikov systems

Let $f : A \to X$, $f' : A' \to X'$, $l : A \to A'$, $k : X \to X'$ be maps. Then a pair of maps $(k, l) : (X, A) \to (X', A')$ is called *a map from f to f'* if the following diagram is homotopy commutative;

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle k} \\ A' & \xrightarrow{f'} & X'. \end{array}$$

It will be denoted by $(k, l) : f \to f'$.

Given maps $f : A \to X$, $f' : A' \to X'$, let $(k, l) : f \to f'$ be a map from f to f'. Let PX' and PA' be the spaces of paths in X' and A' which begin at $*$ respectively. Let $\epsilon_{X'} : PX' \to X'$ and $\epsilon_{A'} : PA' \to A'$ be the fibrations given by evaluating a path at its end point. Let $p_k : E_k \to X$ be the fibration induced by $k : X \to X'$ from $\epsilon_{X'}$. Let $p_l : E_l \to A$ induced by $l : A \to A'$ from $\epsilon_{A'}$. Then there is a map $(f, Pf') : E_l \to E_k$ such that the following diagram is commutative

$$\begin{array}{ccc} E_l & \xrightarrow{(f,Pf')} & E_k \\ {\scriptstyle p_l}\downarrow & & {\scriptstyle p_k}\downarrow \\ A & \xrightarrow{f'} & X, \end{array}$$

where $E_l = \{(a, \xi) \in A \times PA' | l(a) = \epsilon(\xi)\}$, $E_k = \{(x, \eta) \in X \times PX' | k(x) = \epsilon(\eta)\}$, $(f, Pf')(a, \xi) = (f(a), f' \circ \xi)$, $p_k(x, \eta) = x$, $p_l(a, \xi) = a$.

Definition 3.1. Let X be a G^f-space for a map $f : A \to X$. A map $(k, l) : f \to f'$ is called a G^f-primitive if for each map $g : S^m \to \Sigma\Omega X$,$m$ arbitrary, there is a map $F : A \times S^m \to X$ such that $Fj \sim \nabla(f \vee e \circ g)$, $kF(p_l \times 1) \sim * : E_l \times S^m \to X'$, where $j : A \vee S^m \to A \times S^m$ is the inclusion.

The following lemmas are standard.

Lemma 3.2. *A map $g : B \to X$ can be lifted to a map $B \to E_k$ if and only if $kg \sim *$.*

Lemma 3.3. [5] *Given maps $g_i : A_i \to E_k$, $i = 1, 2$ and $g : A_1 \times A_2 \to E_k$ satisfying $p_k g|_{A_i} \sim p_k g_i$, $i = 1, 2$, then there is a map $h : A_1 \times A_2 \to E_k$ such that $p_k h = p_k g$ and $h|_{A_i} \sim g_i, i = 1, 2$.*

Theorem 3.4. *If X is a G^f-space for a map $f : A \to X$ and $(k, l) : f \to f'$ is a G^f-primitive, then E_k is a $G^{(f,Pf')}$-space a map $(f, Pf') : E_l \to E_k$.*

Proof. Clearly we have the following commutative diagram;

$$\begin{array}{ccc} \Sigma\Omega E_k & \xrightarrow{e_{E_k}} & E_k \\ {\scriptstyle \Sigma\Omega p_k}\downarrow & & {\scriptstyle p_k}\downarrow \\ \Sigma\Omega X & \xrightarrow{e_X} & X. \end{array}$$

Since $(k, l) : f \to f'$ is a G^f-primitive, for each map $g : S^m \to \Sigma\Omega E_k$,$m$ arbitrary, there is a map $F : A \times S^m \to X$ such that $Fj \sim \nabla(f \vee e \circ (\Sigma\Omega p_k \circ g))$, $kF(p_l \times 1) \sim * : E_l \times S^m \to X'$, where $j : A \vee S^m \to A \times S^m$ is the inclusion. From Lemma 3.2, there is a lifting $F' : E_l \times S^m \to E_k$ of $F(p_l \times 1) : E_l \times S^m \to X$, that is, $p_k \circ F' = F(p_l \times 1)$. Then

$p_k \circ F'|_{E_l} \sim F(p_l \times 1)|_{E_l} = f \circ p_l = p_k \circ (f, Pf') : E_l \to X$ and $p_k \circ F'|_{S^m} = F|_{S^m} = e_X \circ \Sigma\Omega p_k \circ g = p_k \circ e_{E_k} \circ g : S^m \to X$. Thus we have, from Lemma 3.3, that there is a map $\bar{F} : E_l \times S^m \to E_k$ such that $p_k \bar{F} = p_k F' = F(p_l \times 1)$ and $\bar{F}|_{E_l} \sim (f, Pf')$, $\bar{F}|_{S^m} \sim e_{E_k} \circ g$. Thus $e_{E_k} : \Sigma\Omega E_k \to E_k$ is weakly (f, Pf')-cyclic. That is E_k is a $G^{(f,Pf')}$-space a map $(f, Pf') : E_l \to E_k$. This proves the theorem. □

In 1951, Postnikov [13] introduced the notion of the Postnikov system as follows; *A Postnikov system for X(or homotopy decomposition of X)* $\{X_n, i_n, p_n\}$ consists of a sequence of spaces and maps satisfying (1) $i_n : X \to X_n$ induces an isomorphism $(i_n)_\# : \pi_i(X) \to \pi_i(X_n)$ for $i \leq n$. (2) $p_n : X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$. (3) $p_n i_n \sim i_{n+1}$. It is well known fact [11] that if X is a 1-connected space having a homotopy type of CW-complex, then there is a Postnikov system $\{X_n, i_n, p_n\}$ for X such that $p_{n+1} : X_{n+1} \to X_n$ is the fibration induced from the path space fibration over $K(\pi_{n+1}(X), n+2)$ by a map $k^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$. It is well known [7] that if A and X are spaces having the homotopy type of 1-connected countable CW-complexes and $f; A \to X$ is a map, then there exist Postnikov systems $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ for A and X respectively and induced maps $\{f_n : A_n \to X_n\}$ satisfying (1) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & X_n \\ {\scriptstyle k_A^{n+2}}\downarrow & & {\scriptstyle k_X^{n+2}}\downarrow \\ K(\pi_{n+1}(A), n+2) & \xrightarrow{\tilde{f}_\#} & K(\pi_{n+1}(X), n+2) \end{array}$$

, that is, $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$. (2) $f_{n+1} : A_{n+1} \to X_{n+1}$ given by $f_{n+1} = (f_n, P\tilde{f}_\#)$ satisfying commute diagram

$$\begin{array}{ccc} A_{n+1}(= E_{k_A^{n+2}}) & \xrightarrow{f_{n+1}=(f_n, P\tilde{f}_\#)} & X_{n+1}(= E_{k_X^{n+2}}) \\ {\scriptstyle p'_n(=p_{k_A^{n+2}})}\downarrow & & {\scriptstyle p_n(=p_{k_X^{n+2}})}\downarrow \\ A_n & \xrightarrow{f_n} & X_n. \end{array}$$

(3) for each n, the following diagram is homotopy commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i'_n}\downarrow & & {\scriptstyle i_n}\downarrow \\ A_n & \xrightarrow{f_n} & X_n. \end{array}$$

THEOREM 3.5. *Let A and X be spaces having the homotopy type of 1-connected countable CW-complexes and $f; A \to X$ a map, and $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ Postnikov systems for A and X respectively.*

(1) *If X is a G^f-space for a map $f : A \to X$, then each X_n is G^{f_n}-space and the all pair of k invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are G^{f_n}-primitive.*

(2) *If X_{n-1} is a $G^{f_{n-1}}$-space and the pair of k-invariants $(k_X^{n+1}, k_A^{n+1}) : f_{n-1} \to \tilde{f}_\#$ is $G^{f_{n-1}}$-primitive, then X_n is a G^{f_n}-space, where f_n is an induced map from f.*

Proof. Let $g : S^m \to X_n$ be any map. Since $(i_n)_\# : \pi_i(X) \to \pi_i(X_n)$ for $i \leq n$ and $\pi_i(X_n) = 0$ for $i > n$, there is a map $g' : S^m \to X$ such that $i_n g' \sim g$. Since X is a G^f-space for a map $f : A \to X$, there is a map $G : A \times S^m \to X$ such that $Gj \sim \nabla(f \vee g')$, where $j : A \vee S^m \to A \times S^m$ is the inclusion. Let $\{B_n, i''_n, p''_n\}$ be a Postnikov system for S^m. Then $\{A_n \times B_n, i'_n \times i''_n, p'_n \times p''_n\}$ is a Postnikov system for $A \times S^m$. Then we have, by Kahn's result [7,Theorem 2.2], that there are families of maps $f_n : A_n \to X_n$ and $G_n : A_n \times B_n \to X_n$ such that $p_n f_n = f_{n-1} p'_n$ and $i_n f \sim f_n i'_n$, and $p_n G_n = G_{n-1}(p'_n \times p''_n)$ and $i_n G \sim G_n(i'_n \times i''_n)$ for $n = 2, 3, \cdots$ respectively, and $k_X^{n+2} f_n \sim \tilde{f}_\# k_A^{n+2} : A_n \to K(\pi_{n+1}(X), n+2)$ and $k_X^{n+2} G_n \sim \tilde{G}_\#(k_A^{n+2} \times k_{S^m}^{n+2}) : A_n \times B_n \to K(\pi_{n+1}(X), n+2)$, where $k_A^{n+2} : A_n \to K(\pi_{n+1}(A), n+2)$, $k_X^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$ and $k_{S^m}^{n+2} : B_n \to K(\pi_{n+1}(S^m), n+2)$ are k-invariants of A, X and S^m respectively, $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ and $\tilde{G}_\# : K(\pi_{n+1}(A), n+2) \times K(\pi_{n+1}(S^m), n+2) \approx K(\pi_{n+1}(A \times S^m), n+2) \to K(\pi_{n+1}(X), n+2)$ are the induced maps by $f : A \to X$ and $G : A \times S^m \to X$ respectively. Consider the composition $G' : G_n(1 \times i''_n) : A_n \times S^m \to X_n$. Since $G|_A \sim f$, we know, from Kahn's another result [8, Theorem 1.2], that $G'_{|A_n} = G_{n|A_n} = (G|_A)_n \sim f_n$ and it is clear that $G'_{|S^m} = G_{n|B_n} i''_n \sim i_n G|_{S^m} = i_n g' \sim g$. Thus $[g] \in G_m(A_n, f_n, X_n)$ and X_n is a G^{f_n}-space for a map $f_n : A_n \to X_n$. Moreover, to show that $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ is G^{f_n}-primitive, let $g'' : S^m \to \Sigma\Omega X_n$ be any map and m arbitrary. Since $e_{X_n} g'' : S^m \to X_n$, by the above fact, there is a map $G' : A_n \times S^m \to X_n$ such that $G'j \sim \nabla(f_n \vee e_{X_n} g'') : A_n \vee S^m \to X_n$, where $j : A_n \vee S^m \to A_n \times S^m$ is the inclusion. Since $G'(p_{k_A^{n+2}} \times 1) = G'(p'_{n+1} \times 1) = G_n(1 \times i''_n)(p'_{n+1} \times 1) = G_n(p'_{n+1} \times i''_n) \sim G_n(p'_{n+1} \times p''_{n+1})(1 \times i''_{n+1}) \sim p_{n+1}G_{n+1}(1 \times i''_{n+1}) : A_{n+1} \times S^m \to X_n$, $G'(p_{k_A^{n+2}} \times 1) = G'(p'_{n+1} \times 1) : A_{n+1} \times S^m \to X_n$ has an lifting $G_{n+1}(1 \times i''_{n+1}) : A_{n+1} \times S^m \to X_{n+1}$ and $k_X^{n+2} G'(p_{k_A^{n+2}} \times 1) \sim *$. Thus $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_{\#}$ is G^{f_n}-primitive. (2) It follows from Theorem 3.4. □

Taking $f = 1_X$, $f' = 1_{X'}$, $l = k$, we can obtain the following corollary given by Haslam [5].

Corollary 3.6. [5] *Let X be space having the homotopy type of* 1-*connected countable CW-complexes and $\{X_n, i_n, p_n\}$ Postnikov systems for X.*

(1) *If X is a G-space, then each X_n is G-space and all the k invariants k_X^{n+2} are G-primitive.*

(2) *If X_{n-1} is a G-space and the k-invariants k_X^{n+1} is G-primitive, then X_n is a G-space, where f_n is an induced map from f.*

References

[1] J. Aguade, *Decomposable free loop spaces*, Canad. J. Math. **39** (1987), 938-955.

[2] B. Eckmann and P. Hilton, *Decomposition homologique d'un polyedre simplement connexe*, ibid, **248** (1959), 2054-2558.

[3] D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840-856.

[4] D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729-756.

[5] H. B. Haslam, *G-spaces and H-spaces*, Ph. D. Thesis, Univ. of California, Irvine, 1969.

[6] P. Hilton, *Homotopy Theory and Duality*, Gordon and Breach Science Pub., 1965.

[7] D. W. Kahn, *Induced maps for Postnikov systems*, Trans. Amer. Math. Soc. **107** (1963), 432-450.

[8] D. W. Kahn, *A note on H-spaces and Postnikov systems of spheres*, Proc. Amer. Math. Soc. **15** (1964), 300-307.

[9] K. L. Lim, *On cyclic maps*, J. Austral. Math. Soc.,(Series A) **32** (1982), 349-357.

[10] K. L. Lim, *Cocyclic maps and coevaluation subgroups*, Canad. Math. Bull. **30** (1987), 63-71.

[11] R. E. Mosher and M. C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row, New York, 1968.

[12] N. Oda, *The homotopy of the axes of pairings* , Canad. J. Math. **17** (1990), 856-868.
[13] M. Postnikov, *On the homotopy type of polyhedra*, Dokl. Akad. Nauk. SSSR **76** (1951), no 6, 789-791.
[14] J. Siegel, *G-spaces, H-spaces and W-spaces*, Pacific J. Math. **31** (1969), 209-214.
[15] M. H. Woo and J. R. Kim, *Certain subgroups of homotopy groups*, J. Korean Math. Soc. **21** (1984), no. 2, 109-120.
[16] M. H. Woo and Y. S. Yoon, *T-spaces by the Gottlieb groups and duality*, J. Austral. Math. Soc., (Series A) **59** (1995), 193-203.
[17] Y. S. Yoon, *Lifting Gottlieb sets and duality*, Proc. Amer. Math. Soc. **119** (1993), no. 4, 1315-1321.
[18] Y. S. Yoon, *The generalized dual Gottlieb sets*, Topology Appl. **109** (2001),173-181.
[19] Y. S. Yoon, *Generalized Gottlieb groups and generalized Wang homomorphisms*, Sci.Math. Japon. **55** (2002), no. 1, 139-148.
[20] Y. S. Yoon, *H^f-spaces for maps and their duals*, J. Korea Soc. Math. Edu. Series B **14** (2007), no. 4, 289-306.
[21] Y. S. Yoon, *Lifting T-structures and their duals*, J. Chungcheong Math. Soc. **20** (2007), no. 3, 245-259.

*
Department of Mathematics Education
Hannam University
Daejeon 306-791, Republic of Korea
E-mail: yoon@hannam.ac.kr