JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 4, December 2009

SOLUTION AND STABILITY OF MIXED TYPE FUNCTIONAL EQUATIONS

Kil-Woung Jun*, Il-Sook Jung**, and Hark-Mahn Kim***

Abstract. In this paper we establish the general solution of the following functional equation with mixed type of quadratic and additive mappings

$$f(mx+y)+f(mx-y)+2f(x)=f(x+y)+f(x-y)+2f(mx),$$

where $m \geq 2$ is a positive integer, and then investigate the generalized Hyers–Ulam stability of this equation in quasi-Banach spaces.

1. Introduction and preliminaries

In 1940, S. M. Ulam [13] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$?

In 1941, D.H. Hyers [7] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies

Received October 10, 2009; Accepted November 11, 2009.

2000 Mathematics Subject Classification: Primary 39B82; Secondary 47B48.

Key words and phrases: generalized Hyers-Ulam stability, quasi-Banach space, p-Banach space.

Correspondence should be addressed to Hark-Mahn Kim, hmkim@cnu.ac.kr.

This study was financially supported by research fund of Chungnam National University in 2008.

Hyers inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in E$. In this case there exists a unique additive mapping $L : E \to E'$, defined by $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, such that

$$\|f(x) - L(x)\| \le \varepsilon$$

for all $x \in E$. A generalized version of Hyers' theorem for approximate additive mappings was given by T. Aoki [1] and D.G. Bourgin [4]. In 1978, Th.M. Rassias [9] introduced the unbounded Cauchy difference to be controlled by a sum of powers of norms like $\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p), p < 1$ and then provided a generalization of Hyers' theorem by allowing the unique additive mapping to be linear. In 1991, Z. Gajda [6] following the same approach as in Th.M. Rassias [9], gave an affirmative solution to this question for $p > 1$. It was proved by Z. Gajda [6] as well as by Th.M. Rassias and P. Semrl [10] that one cannot prove the stability theorem when $p = 1$.

The following functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a quadratic functional equation, and every solution of the equation (1.1) is said to be a quadratic mapping. A Hyers–Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space [12]. S. Czerwik [5] and C. Borelli and G.L. Forti [3] have established the generalized Hyers–Ulam stability of the quadratic functional equation (1.1).

In this paper, we deal with the next functional equation deriving from quadratic and additive mappings:

$$\begin{aligned} f(mx+y) + f(mx-y) + 2f(x) \\ = f(x+y) + f(x-y) + 2f(mx) \end{aligned} \tag{1.2}$$

where $m \ge 2$ is a positive integer. The general solution and generalized Hyers–Ulam stability for Eq. (1.2) with a special case $m = 2$ has been investigated in the reference [8]. It is easy to see that the mapping $Q(x) = B(x, x)$ for a symmetric bi-additive mapping B and an additive mapping A are solutions of Eq. (1.2). The main purpose of this paper is to establish the general solution of Eq. (1.2) and investigate the generalized Hyers–Ulam stability for Eq. (1.2) in quasi-Banach spaces.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1. (See [2, 11].) Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space.*

By the Aoki-Rolewicz theorem [11], each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

2. Solution of Eq. (1.2)

Throughout this section, X and Y will be real vector spaces. Before taking up the main subject in this section, we shall need the following two lemmas.

Lemma 2.1. *If an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the equation (1.2) for all $x, y \in X$, then f is quadratic.*

Proof. First, we note that the lemma is true for $m = 2$ in view of [8]. Thus we assume by induction that Lemma 2.1 is true for all $2, \cdots, m$. Now, if we replace y by $x + y$ in (1.2), we get by the evenness of f

$$f((m+1)x+y) + f((m-1)x-y) + 2f(x) = f(2x+y) + f(y) + 2f(mx) \tag{2.1}$$

for all $x, y \in X$. Replacing y by $-y$ in (2.1), we get by the evenness of f

$$f((m+1)x-y) + f((m-1)x+y) + 2f(x) = f(2x-y) + f(y) + 2f(mx) \tag{2.2}$$

for all $x, y \in X$. If we add (2.1) to (2.2) and use the inductive argument, we have

$$(2.3) \qquad \begin{aligned} &f((m+1)x+y)+f((m+1)x-y) \\ &\quad = 2f(y)+4f(mx)+2f(2x)-4f(x)-2f((m-1)x) \end{aligned}$$

for all $x, y \in X$. Letting $y = 0$ in (2.3), we get

$$2f((m+1)x) = 4f(mx)+2f(2x)-4f(x)-2f((m-1)x)$$

for all $x \in X$. Thus we see from (2.3) that

$$f(u+y)+f(u-y) = 2f(u)+2f(y), \; u := (m+1)x$$

for all $u, y \in X$. Therefore the mapping $f : X \to Y$ is quadratic. □

COROLLARY 2.2. *If an even mapping $f : X \to Y$ satisfies the equation (1.2) for all $x, y \in X$, then $g : X \to Y$ is quadratic, where $g(x) := f(x) - f(0), x \in X$.*

LEMMA 2.3. *If an odd mapping $f : X \to Y$ satisfies (1.2) for all $x, y \in X$, then f is additive.*

Proof. The proof is very similar to that of Lemma 2.1. □

THEOREM 2.4. *A mapping $f : X \to Y$ satisfies the equation (1.2) for all $x, y \in X$ if and only if there exist a symmetric bi-additive mapping $B : X \times X \to Y$ and an additive mapping $A : X \to Y$ such that $f(x) = B(x,x) + A(x) + f(0)$ for all $x \in X$.*

Proof. If there exist a symmetric bi-additive mapping $B : X \times X \to Y$ and an additive mapping $A : X \to Y$ such that $f(x) = B(x,x) + A(x) + f(0)$ for all $x \in X$, then it is easy to see

$$\begin{aligned} f(mx+y)+f(mx-y) &= 2m^2B(x,x)+2B(y,y)+2mA(x)+2f(0) \\ &= f(x+y)+f(x-y)+2f(mx)-2f(x) \end{aligned}$$

for all $x, y \in X$. Therefore the mapping $f : X \to Y$ satisfies (1.2).

Conversely, let f satisfy the equation (1.2). Then if we decompose f into the even part f_e and the odd part f_o by putting

$$f_e(x) = \frac{f(x)+f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x)-f(-x)}{2}$$

for all $x \in X$, it is easy to see that the mappings f_e and f_o satisfy the equation (1.2). Hence by Corollary 2.2 and Lemma 2.3 we obtain that the mappings $f_e - f(0)$ and f_o are quadratic and additive, respectively. Therefore there exists a symmetric bi-additive mapping $B : X \times X \to Y$

and an additive mapping $A : X \to Y$ such that $f_e(x) = B(x,x) + f(0)$ and $f_o(x) = A(x)$ for all $x \in X$. So we have

$$f(x) = f_e(x) + f_o(x) = B(x,x) + A(x) + f(0)$$

for all $x \in X$. □

3. Generalized Hyers–Ulam stability of Eq. (1.2)

Throughout this section, assume that X is a quasi-normed space with quasi-norm $\|\cdot\|$ and that Y is a p-Banach space with p-norm $\|\cdot\|$. Let $K \geq 1$ be the modulus of concavity of $\|\cdot\|$.

In this section, using an idea of direct method we prove the generalized stability of Eq. (1.2) in the spirit of Hyers, Ulam and Rassias. For convenience, we denote the following difference operator Df of a given mapping $f : X \to Y$ as

$$\begin{aligned} &Df(x,y) \\ &:= f(mx+y) + f(mx-y) + 2f(x) - f(x+y) - f(x-y) - 2f(mx) \end{aligned}$$

for all $x, y \in X$, $m \geq 2$ is a positive integer. The operator Df is called the approximate remainder and acts as a perturbation of the equation (1.2).

First, we are going to prove the generalized Hyers–Ulam stability of the equation (1.2) for an even function with approximate conditions.

Theorem 3.1. *Suppose that there exists a mapping $\varphi : X \times X \to [0,\infty)$ for which an even mapping $f : X \to Y$ satisfies the approximate conditions*

$$\|Df(x,y)\| \leq \varphi(x,y), \tag{3.1}$$

$$\overline{\Phi}_0(x,y) := \sum_{i=0}^{\infty} \frac{1}{4^{ip}} \varphi(2^i x, 2^i y)^p < \infty \tag{3.2}$$

for all $x, y \in X$. Then the limit

$$Q_0(x) := \lim_{n\to\infty} \frac{f(2^n mx) - f(2^n x)}{4^n}$$

exists for all $x \in X$ and $Q_0 : X \to Y$ is a unique quadratic mapping satisfying the equation (1.2) and the approximation

$$\|f(x) - f(mx) - Q_0(x)\| \leq \frac{K}{4}\Big[\overline{\Phi}_0(x,x) + \overline{\Phi}_0(x,mx)\Big]^{\frac{1}{p}} \tag{3.3}$$

for all $x \in X$.

Proof. If we put $g(x) := f(x) - f(0)$, then $g(0) = 0$ and the mapping $g : X \to Y$ also satisfies the inequality

$$\|Dg(x, y)\| \leq \varphi(x, y)$$

for all $x, y \in X$. So we may assume $f(0) = 0$ without loss of generality. By replacing y by x in (3.1), we get

$$\|f((m+1)x) + f((m-1)x) + 2f(x) - f(2x) - 2f(mx)\| \leq \varphi(x, x) \tag{3.4}$$

for all $x \in X$. Replacing y by mx in (3.1), we have

$$\|f(2mx) + 2f(x) - f((m+1)x) - f((m-1)x) - 2f(mx)\| \leq \varphi(x, mx) \tag{3.5}$$

for all $x \in X$. It follows from (3.4) and (3.5) that

$$\|f(2mx) + 4f(x) - f(2x) - 4f(mx)\| \leq K[\varphi(x, x) + \varphi(x, mx)]$$

for all $x \in X$. Letting $g(x) := f(x) - f(mx)$ and $\phi(x) := \varphi(x, x) + \varphi(x, mx)$, one has the crucial inequality

$$\|4g(x) - g(2x)\| \leq K\phi(x) \tag{3.6}$$

for all $x \in X$. If we replace x in (3.6) by $2^i x$ and divide both sides of (3.6) by 4^{i+1}, then we have

$$\left\|\frac{g(2^{i+1}x)}{4^{i+1}} - \frac{g(2^i x)}{4^i}\right\| \leq \frac{K}{4^{i+1}}\phi(2^i x)$$

for all $x \in X$ and all nonnegative integers i. Since Y is a p-Banach space,

$$\left\|\frac{g(2^l x)}{4^l} - \frac{g(2^{n+1}x)}{4^{n+1}}\right\|^p \leq \sum_{i=l}^{n}\left\|\frac{g(2^{i+1}x)}{4^{i+1}} - \frac{g(2^i x)}{4^i}\right\|^p \leq \left(\frac{K}{4}\right)^p \sum_{i=l}^{n}\frac{\phi(2^i x)^p}{4^{ip}} \tag{3.7}$$

for all nonnegative integers l and n with $n \geq l$ and $x \in X$. Since $0 < p \leq 1$, it follows from (3.2) that

$$\phi(x)^p \leq \varphi(x, x)^p + \varphi(x, mx)^p, \quad \text{and} \quad \sum_{i=0}^{\infty}\frac{\phi(2^i x)^p}{4^{ip}} < \infty$$

for all $x \in X$. Therefore we conclude from (3.7) that a sequence $\{\frac{1}{4^n}g(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}g(2^n x)\}$ converges for all $x \in X$. So one can define a mapping $Q_0 : X \to Y$ by

$$Q_0(x) := \lim_{n\to\infty} \frac{g(2^n x)}{4^n} = \lim_{n\to\infty} \frac{f(2^n x) - f(2^n mx)}{4^n}$$

for all $x \in X$. Then letting $l = 0$ and passing the limit $n \to \infty$ in (3.7), we get

$$\begin{aligned}\|g(x) - Q_0(x)\|^p &\le \left(\frac{K}{4}\right)^p \sum_{i=0}^{\infty} \frac{1}{4^{ip}} \phi(2^i x)^p \\ &\le \left(\frac{K}{4}\right)^p \sum_{i=0}^{\infty} \frac{1}{4^{ip}} \Big[\varphi(2^i x, 2^i x)^p + \varphi(2^i x, 2^i mx)^p\Big],\end{aligned}$$

which implies the approximation (3.3) for all $x \in X$.

Now, we show that Q_0 is quadratic. It follows from (3.1) and (3.2) that

$$\begin{aligned}&\|DQ_0(x,y)\|^p = \lim_{n\to\infty} \frac{1}{4^{np}} \|Dg(2^n x, 2^n y)\|^p \\ &\le \lim_{n\to\infty} \frac{K^p}{4^{np}} \Big[\|Df(2^n mx, 2^n my)\|^p + \|Df(2^n x, 2^n y)\|^p\Big] \\ &\le \lim_{n\to\infty} \frac{K^p}{4^{np}} \Big[\varphi(2^n mx, 2^n my)^p + \varphi(2^n x, 2^n y)^p\Big] = 0\end{aligned}$$

for all $x, y \in X$. Therefore the mapping $Q_0 : X \to Y$ satisfies (1.2). Since $Q_0(0) = 0$, we see from Lemma 2.1 that the mapping Q_0 is quadratic.

To prove the uniqueness of Q_0, let $T : X \to Y$ be another quadratic mapping satisfying (3.3). We observe that $T(2^n x) = 4^n T(x),\ n \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{1}{4^{np}} \sum_{i=0}^{\infty} \frac{1}{4^{ip}} \varphi(2^{n+i} x, 2^{n+i} y)^p = \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{4^{ip}} \varphi(2^i x, 2^i y)^p = 0$$

for all $x, y \in X$ and so

$$\lim_{n\to\infty} \frac{1}{4^{np}} \overline{\Phi}_0(2^n x, 2^n y) = 0$$

for all $x, y \in X$. Thus it follows from (3.3) and the last relation that

$$\begin{aligned}&\|Q_0(x) - T(x)\|^p = \lim_{n\to\infty} \frac{1}{4^{np}} \|g(2^n x) - T(2^n x)\|^p \\ &\le \left(\frac{K}{4}\right)^p \lim_{n\to\infty} \frac{1}{4^{np}} \Big[\overline{\Phi}_0(2^n x, 2^n x) + \overline{\Phi}_0(2^n x, 2^n mx)\Big] = 0\end{aligned}$$

for all $x \in X$. The proof is complete. □

THEOREM 3.2. *Suppose that there exists a mapping $\varphi : X \times X \to [0, \infty)$ for which an even mapping $f : X \to Y$ satisfies the approximate conditions*

$$\|Df(x,y)\| \le \varphi(x,y), \tag{3.8}$$

$$\overline{\Phi}_1(x,y) := \sum_{i=1}^{\infty} 4^{ip} \varphi\Big(\frac{x}{2^i}, \frac{y}{2^i}\Big)^p < \infty \tag{3.9}$$

for all $x, y \in X$. In this case the limit

$$Q_1(x) := \lim_{n\to\infty} 4^n \left[f\left(\frac{x}{2^n}\right) - f\left(\frac{mx}{2^n}\right)\right]$$

exists for all $x \in X$ and $Q_1 : X \to Y$ is a unique quadratic mapping satisfying the equation (1.2) and the estimation

$$\|f(x) - f(mx) - Q_1(x)\| \le \frac{K}{4}\left[\overline{\Phi}_1(x,x) + \overline{\Phi}_1(x,mx)\right]^{\frac{1}{p}} \tag{3.10}$$

for all $x \in X$.

Proof. If we replace x in (3.6) by $\frac{x}{2^{i+1}}$ and multiply both sides of (3.6) by 4^i, then we have

$$\left\|4^{i+1} g\left(\frac{x}{2^{i+1}}\right) - 4^i g\left(\frac{x}{2^i}\right)\right\| \le 4^i K \phi\left(\frac{x}{2^{n+1}}\right)$$

for all $x \in X$ and all nonnegative integers i. Since Y is a p-Banach space,

$$\begin{aligned}\left\|4^{n+1} g\left(\frac{x}{2^{n+1}}\right) - 4^l g\left(\frac{x}{2^l}\right)\right\|^p &\le \sum_{i=l}^{n} \left\|4^i g\left(\frac{x}{2^i}\right) - 4^{i+1} g\left(\frac{x}{2^{i+1}}\right)\right\|^p \\ &\le K^p \sum_{i=l}^{n} 4^{ip} \phi\left(\frac{x}{2^{i+1}}\right)^p\end{aligned}$$

for all nonnegative integers l and n with $n \ge l$ and $x \in X$. Since $\sum_{i=1}^{\infty} 4^{ip} \phi\left(\frac{x}{2^i}\right)^p < \infty$ for all $x \in X$, it follows that a sequence $\{4^n g\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence in Y for all $x \in X$. So one can define a mapping $Q_1 : X \to Y$ by $Q_1(x) := \lim_{n\to\infty} 4^n \left[f\left(\frac{x}{2^n}\right) - f\left(\frac{mx}{2^n}\right)\right]$ for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Now, we are going to prove the generalized Hyers–Ulam stability of the equation (1.2) for an odd function with approximate conditions.

THEOREM 3.3. *Suppose that there exists a mapping $\varphi : X \times X \to [0, \infty)$ for which an odd mapping $f : X \to Y$ satisfies the approximate* *conditions*

$$\|Df(x,y)\| \le \varphi(x,y), \tag{3.11}$$

$$\overline{\Psi}_0(x,y) := \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \varphi(2^i x, 2^i y)^p < \infty \tag{3.12}$$

for all $x, y \in X$. Then the limit

$$A_0(x) := \lim_{n\to\infty} \frac{f(2^n x) - f(2^n mx)}{2^n}$$

exists for all $x \in X$ and $A_0 : X \to Y$ is a unique additive mapping satisfying the equation (1.2) and the approximation

$$\|f(x) - f(mx) - A_0(x)\| \le \frac{K^4}{2} \Big[\overline{\overline{\Psi}}_0(x)\Big]^{\frac{1}{p}} \tag{3.13}$$

for all $x \in X$, where

$$\begin{aligned} \overline{\overline{\Psi}}_0(x) \;\; := \;\; & \overline{\Psi}_0\Big(\frac{x}{2}, \frac{2m+1}{2}x\Big) + \overline{\Psi}_0\Big(\frac{x}{2}, \frac{2m-1}{2}x\Big) \\ & + \overline{\Psi}_0\Big(\frac{x}{2}, \frac{3mx}{2}\Big) + \overline{\Psi}_0\Big(\frac{x}{2}, \frac{mx}{2}\Big) + \overline{\Psi}_0(x,x), \quad x \in X. \end{aligned}$$

Proof. By letting $y = 3mx, y = (2m+1)x$ and $y = (2m-1)x$, respectively in (3.11), we get the inequalities

$$\begin{aligned} \|f(4mx) - f(2mx) - f((3m+1)x) + f((3m-1)x) \\ -2f(mx) + 2f(x)\| \le \varphi(x, 3mx), \end{aligned} \tag{3.14}$$

$$\begin{aligned} \|f((3m+1)x) - f((m+1)x) - f(2(m+1)x) + f(2mx) \\ -2f(mx) + 2f(x)\| \le \varphi(x, (2m+1)x), \end{aligned} \tag{3.15}$$

$$\begin{aligned} \|f((3m-1)x) - f((m-1)x) - f(2mx) + f(2(m-1)x) \\ -2f(mx) + 2f(x)\| \le \varphi(x, (2m-1)x) \end{aligned} \tag{3.16}$$

for all $x \in X$. It follows from (3.14), (3.15) and (3.16),

$$\begin{aligned} &\|f(4mx) + f(2mx) + 2f(x) - 2f(mx) - f(2(m+1)x) \\ &\quad -f(2(m-1)x) - f((m+1)x) + f((m-1)x)\| \\ &\le K^2[\varphi(x,(2m+1)x) + \varphi(x,(2m-1)x) + \varphi(x,3mx)] \end{aligned} \tag{3.17}$$

for all $x \in X$. By letting $x := 2x, y := 2x$ in (3.11), we get the inequality

$$\begin{aligned} &\|f(2(m+1)x) + f(2(m-1)x) - 4f(x) - 2f(2mx) + 2f(2x)\| \\ &\le \varphi(2x, 2x), \quad x \in X. \end{aligned} \tag{3.18}$$

By letting $y = mx$ in (3.11), we get the inequality

$$\|f(2mx) - f((m+1)x) + f((m-1)x) - 2f(mx) + 2f(x)\| \leq \varphi(x, mx), \quad x \in X. \tag{3.19}$$

It follows from (3.17), (3.18) and (3.19) that

$$\begin{aligned}&\|f(4mx) - 2f(2mx) - f(4x) + 2f(2x)\| \\ &\leq K^4[\varphi(x,(2m+1)x) + \varphi(x,(2m-1)x) + \varphi(x, 3mx) \\ &\qquad +\varphi(2x, 2x) + \varphi(x, mx)]\end{aligned}$$

for all $x \in X$. Letting $g(x) := f(x) - f(mx)$, we see that the last inequality can be written by

$$\begin{aligned}\|2g(x) - g(2x)\| \leq K^4\Big[&\varphi\left(\frac{x}{2}, \frac{2m+1}{2}x\right) + \varphi\left(\frac{x}{2}, \frac{2m-1}{2}x\right) \\ &+\varphi\left(\frac{x}{2}, \frac{3mx}{2}\right) + \varphi\left(\frac{x}{2}, \frac{mx}{2}\right) + \varphi(x,x)\Big]\end{aligned} \tag{3.20}$$

for all $x \in X$. For notational convenience, let's define a mapping ψ as

$$\begin{aligned}\psi(x) := \varphi\left(\frac{x}{2}, \frac{2m+1}{2}x\right) + \varphi\left(\frac{x}{2}, \frac{2m-1}{2}x\right) + \varphi\left(\frac{x}{2}, \frac{3mx}{2}\right) \\ +\varphi\left(\frac{x}{2}, \frac{mx}{2}\right) + \varphi(x,x)\end{aligned}$$

for all $x \in X$. Since Y is a p-Banach space, we see from (3.20)

$$\begin{aligned}\left\|\frac{g(2^{n+1}x)}{2^{n+1}} - \frac{g(2^l x)}{2^l}\right\|^p &\leq \sum_{i=l}^{n}\left\|\frac{g(2^{i+1}x)}{2^{i+1}} - \frac{g(2^i x)}{2^i}\right\|^p \\ &\leq \frac{K^{4p}}{2^p}\sum_{i=l}^{n}\frac{1}{2^{ip}}\psi(2^i x)^p\end{aligned} \tag{3.21}$$

for all non-negative integers l and n with $n \geq l$ and $x \in X$. Since

$$\begin{aligned}\psi(x)^p \leq\ &\varphi\left(\frac{x}{2}, \frac{2m+1}{2}x\right)^p + \varphi\left(\frac{x}{2}, \frac{2m-1}{2}x\right)^p \\ &+\varphi\left(\frac{x}{2}, \frac{3mx}{2}\right)^p + \varphi\left(\frac{x}{2}, \frac{mx}{2}\right)^p + \varphi(x,x)^p\end{aligned} \tag{3.22}$$

for all $x \in X$, it follows from (3.12) and (3.22) that the following series $\sum_{i=0}^{\infty} \frac{1}{2^{ip}}\psi(2^i x)^p < \infty$ for all $x \in X$. Therefore we conclude from (3.21) that a sequence $\{\frac{1}{2^n}g(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}g(2^n x)\}$ converges for all $x \in X$. So one can define a mapping $A_0 : X \to Y$ by $A_0(x) := \lim_{n\to\infty} \frac{f(2^n x)-f(2^n mx)}{2^n}$ for all $x \in X$. Then it follows from (3.12) that

$$\begin{aligned} \|DA_0(x,y)\|^p &= \lim_{n\to\infty} \frac{1}{2^{np}} \|Dg(2^n x, 2^n y)\|^p \\ &\le \lim_{n\to\infty} \frac{K^p}{2^{np}} \Big[\|Df(2^n mx, 2^n my)\|^p + \|Df(2^n x, 2^n y)\|^p \Big] \\ &\le \lim_{n\to\infty} \frac{K^p}{2^{np}} \Big[\varphi(2^n mx, 2^n my)^p + \varphi(2^n x, 2^n y)^p \Big] = 0 \end{aligned}$$

for all $x, y \in X$. Therefore the mapping $A_0 : X \to Y$ satisfies the equation (1.2). Since f is an odd function, the mapping $A_0 : X \to Y$ is odd. Therefore we get from Lemma 2.3 that the mapping $A_0 : X \to Y$ is additive. Further, letting $l = 0$ and passing the limit $n \to \infty$ in (3.21), we get

$$\|g(x) - A_0(x)\|^p = \frac{K^{4p}}{2^p} \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \psi(2^i x)^p,$$

which yields the approximation (3.13).

To prove the uniqueness of A_0, let $T : X \to Y$ be another additive mapping satisfying (3.13). Since

$$\lim_{n\to\infty} \frac{1}{2^{np}} \sum_{i=0}^{\infty} \frac{1}{2^{ip}} \varphi(2^{n+i} x, 2^{n+i} y)^p = \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{1}{2^{ip}} \varphi(2^i x, 2^i y)^p = 0$$

for all $x \in X$, one has $\lim_{n\to\infty} \frac{1}{2^{np}} \overline{\overline{\overline{\Psi}}}_o(2^n x) = 0$ for all $x \in X$. Therefore, it follows from (3.13) that

$$\begin{aligned} \|A_0(x) - T(x)\|^p &= \lim_{n\to\infty} \frac{1}{2^{np}} \|g(2^n x) - T(2^n x)\|^p \\ &\le \frac{K^{4p}}{2^p} \lim_{n\to\infty} \frac{1}{2^{np}} \overline{\overline{\overline{\Psi}}}_o(2^n x) = 0, \quad x \in X, \end{aligned}$$

which proves the uniqueness of A_0. This completes the proof. □

THEOREM 3.4. *Suppose that there exists a mapping $\varphi : X \times X \to [0, \infty)$ for which an odd mapping $f : X \to Y$ satisfies the approximate conditions*

$$\|Df(x,y)\| \le \varphi(x,y), \tag{3.23}$$

$$\overline{\Psi}_1(x,y) := \sum_{i=1}^{\infty} 2^{ip} \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right)^p < \infty \tag{3.24}$$

for all $x, y \in X$. Then the limit

$$A_1(x) := \lim_{n\to\infty} 2^n \left[f(\frac{x}{2^n}) - f(\frac{mx}{2^n}) \right]$$

exists for all $x \in X$ and $A_1 : X \to Y$ is a unique additive mapping satisfying the equation (1.2) and the approximation

$$\|f(x) - f(mx) - A_1(x)\| \le \frac{K^4}{2} \left[\overline{\overline{\Psi}}_1(x) \right]^{\frac{1}{p}} \tag{3.25}$$

for all $x \in X$, where

$$\begin{aligned} \overline{\overline{\Psi}}_1(x) \quad := \quad & \overline{\Psi}_1\Big(\frac{x}{2}, \frac{2m+1}{2}x\Big) + \overline{\Psi}_1\Big(\frac{x}{2}, \frac{2m-1}{2}x\Big) \\ & + \overline{\Psi}_0\Big(\frac{x}{2}, \frac{3mx}{2}\Big) + \overline{\Psi}_1\Big(\frac{x}{2}, \frac{mx}{2}\Big) + \overline{\Psi}_1(x, x), \quad x \in X. \end{aligned}$$

Proof. Since Y is a p-Banach space, it is verified by (3.20) that

$$\begin{aligned} \left\| 2^{n+1} g\left(\frac{x}{2^{n+1}}\right) - 2^l g\left(\frac{x}{2^l}\right) \right\|^p &\le \sum_{i=l}^{n} \left\| 2^i g\left(\frac{x}{2^i}\right) - 2^{i+1} g\left(\frac{x}{2^{i+1}}\right) \right\|^p \\ &\le \frac{K^{4p}}{2^p} \sum_{i=l}^{n} 2^{(i+1)p} \psi\left(\frac{x}{2^{i+1}}\right)^p \end{aligned} \tag{3.26}$$

for all nonnegative integers l and n with $n \ge l$ and $x \in X$. Since $\sum_{i=1}^{\infty} 2^{ip} \psi\left(\frac{x}{2^i}\right)^p < \infty$ for all $x \in X$, the condition (3.26) implies that a sequence $\{2^n g\left(\frac{x}{2^n}\right)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n g\left(\frac{x}{2^n}\right)\}$ converges for all $x \in X$. So one can define a mapping $A_1 : X \to Y$ by $A_1(x) := \lim_{n\to\infty} 2^n \left[f(\frac{x}{2^n}) - f(\frac{mx}{2^n})\right]$ for all $x \in X$.

The remaining proof goes through by the similar argument to Theorem 3.3. □

Now, we are ready to prove the generalized Hyers–Ulam stability of the equation (1.2) for a general function with approximate conditions.

THEOREM 3.5. *Suppose that there exists a mapping $\varphi : X \times X \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the approximate condition (3.11) on the difference Df and (3.12) for all $x, y \in X$. Then there exist a quadratic mapping $Q : X \to Y$ and an additive mapping*

$A : X \to Y$ satisfying (1.2) and

$$(3.27)\quad \|f(x) - f(mx) - Q(x) - A(x)\| \\ \le \frac{K^3}{8}\Big[\overline{\Phi}_0(x,x) + \overline{\Phi}_0(x,mx) + \overline{\Phi}_0(-x,-x) + \overline{\Phi}_0(-x,-mx)\Big]^{\frac{1}{p}} \\ + \frac{K^6}{4}\Big[\overline{\overline{\Psi}}_0(x) + \overline{\overline{\Psi}}_0(-x)\Big]^{\frac{1}{p}}$$

for all $x \in X$, where $\overline{\Phi}_0$ and $\overline{\overline{\Psi}}_0$ are defined as in Theorem 3.1 and Theorem 3.3, respectively, for all $x \in X$.

Proof. Let $f_e(x) = \frac{f(x)+f(-x)}{2}$ be the even part of f and $f_o(x) = \frac{f(x)-f(-x)}{2}$ the odd part of f. Then it follows from (3.11) that

$$(3.28)\qquad \|Df_e(x,y)\| \le \frac{K}{2}[\varphi(x,y) + \varphi(-x,-y)],$$

$$(3.29)\qquad \|Df_o(x,y)\| \le \frac{K}{2}[\varphi(x,y) + \varphi(-x,-y)]$$

for all $x, y \in X$. Hence, in view of (3.28) and Theorem 3.1, we see that there exists a unique quadratic mapping $Q : X \to Y$ satisfying

$$(3.30)\quad \|f_e(x) - f_e(mx) - Q(x)\| \\ \le \frac{K^2}{8}\Big[\overline{\Phi}_0(x,x) + \overline{\Phi}_0(x,mx) + \overline{\Phi}_0(-x,-x) + \overline{\Phi}_0(-x,-mx)\Big]^{\frac{1}{p}}$$

for all $x \in X$.

From (3.29) and Theorem 3.3, it follows that there exists a unique additive mapping $A : X \to Y$ satisfying

$$(3.31)\quad \|f_o(x) - f_o(mx) - A(x)\| \le \frac{K^5}{4}\Big[\overline{\overline{\Psi}}_0(x) + \overline{\overline{\Psi}}_0(-x)\Big]^{\frac{1}{p}}$$

for all $x \in X$. Thus it follows from (3.30) and (3.31) that

$$\|f(x) - f(mx) - Q(x) - A(x)\| \\ \le K\|f_e(x) - f_e(mx) - Q(x)\| + K\|f_o(x) - f_o(mx) - A(x)\| \\ \le \frac{K^3}{8}\Big[\overline{\Phi}_0(x,x) + \overline{\Phi}_0(x,mx) + \overline{\Phi}_0(-x,-x) + \overline{\Phi}_0(-x,-mx)\Big]^{\frac{1}{p}} \\ + \frac{K^6}{4}\Big[\overline{\overline{\Psi}}_0(x) + \overline{\overline{\Psi}}_0(-x)\Big]^{\frac{1}{p}}$$

for all $x \in X$. □

THEOREM 3.6. *Suppose that there exists a mapping $\varphi : X \times X \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the approximate condition (3.8) on the difference Df and (3.9) for all $x, y \in X$. Then* *there exist a quadratic mapping $Q : X \to Y$ and an additive mapping $A : X \to Y$ satisfying (1.2) and*

$$\begin{aligned} (3.32)\quad & \|f(x) - f(mx) - Q(x) - A(x)\| \\ & \le \frac{K^3}{8}\Big[\overline{\Phi}_1(x,x) + \overline{\Phi}_1(x,mx) + \overline{\Phi}_1(-x,-x) + \overline{\Phi}_1(-x,-mx)\Big]^{\frac{1}{p}} \\ & + \frac{K^6}{4}\Big[\overline{\overline{\Psi}}_1(x) + \overline{\overline{\Psi}}_1(-x)\Big]^{\frac{1}{p}} \end{aligned}$$

for all $x \in X$, where $\overline{\Phi}_1$ and $\overline{\overline{\Psi}}_1$ are defined as in Theorem 3.2 and Theorem 3.4, respectively, for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.5. □

THEOREM 3.7. *Suppose that there exists a mapping $\varphi : X \times X \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies the approximate condition (3.1) on the difference Df and φ satisfies both condition (3.2) and condition (3.24) for all $x, y \in X$. Then there exist a quadratic mapping $Q : X \to Y$ and an additive mapping $A : X \to Y$ satisfying (1.2) and*

$$\begin{aligned} (3.33)\quad & \|f(x) - f(mx) - Q(x) - A(x)\| \\ & \le \frac{K^3}{8}\Big[\overline{\Phi}_0(x,x) + \overline{\Phi}_0(x,mx) + \overline{\Phi}_0(-x,-x) + \overline{\Phi}_0(-x,-mx)\Big]^{\frac{1}{p}} \\ & + \frac{K^6}{4}\Big[\overline{\overline{\Psi}}_1(x) + \overline{\overline{\Psi}}_1(-x)\Big]^{\frac{1}{p}} \end{aligned}$$

for all $x \in X$, where $\overline{\Phi}_0$ and $\overline{\overline{\Psi}}_1$ are defined as in Theorem 3.1 and Theorem 3.4, respectively, for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 3.5. □

The following corollary is an immediate result from Theorem 3.1.

COROLLARY 3.8. *Suppose that there exists a constant $\varepsilon \ge 0$ for which a mapping $f : X \to Y$ satisfies the approximate condition*

$$\|Df(x,y)\| \le \varepsilon$$

for all $x, y \in X$. Then there exist a quadratic mapping $Q : X \to Y$ and an additive mapping $A : X \to Y$ satisfying (1.2) and

$$\|f(x) - f(mx) - Q(x) - A(x)\| \le \frac{\sqrt[p]{4}K^3\varepsilon}{2\sqrt[p]{4^p - 1}} + \frac{\sqrt[p]{10}K^6\varepsilon}{2\sqrt[p]{2^p - 1}}$$

for all $x \in X$.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Japan **2** (1950), 64-66.

[2] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis, Vol. 1*, Colloq. Publ. **48**, Amer. Math. Soc. Providence, 2000.

[3] C. Borelli and G.L. Forti, *On a general Hyers-Ulam stability result*, Internat. J. Math. Math. Sci. **18** (1995), 229-236.

[4] D.G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223-237.

[5] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[6] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14** (1991), 431–434.

[7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. **27** (1941), 222-224.

[8] A. Najati and M. Moghimi, *Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces*, J. Math. Anal. Appl. **337** (2008), no. 1, 399–415.

[9] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.

[10] Th. M. Rassias and P. Šemrl, *On the behaviour of mappings which do not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.

[11] S. Rolewicz, *Metric Linear Spaces*, Reidel and Dordrecht, and PWN-Polish Sci. Publ. 1984.

[12] F. Skof, *Sull' approssimazione delle applicazioni localmente δ-additive*, Atti Accad. Sci. Torino Cl Sci. Fis. Mat. Natur. **117** (1983), 377-389.

[13] S. M. Ulam, *A collection of the mathematical problems*, Interscience Publ. New York, 1960.

*
Department of Mathematics
Chungnam National University
Daejeon 305–764, Republic of Korea
E-mail: kwjun@math.cnu.ac.kr

**
Department of Mathematics
Chungnam National University
Daejeon 305–764, Republic of Korea
E-mail: jilsook@hanmail.net

***
Department of Mathematics
Chungnam National University
Daejeon 305–764, Republic of Korea
E-mail: hmkim@cnu.ac.kr