JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 4, December 2009

FORMULAS OF GALOIS ACTIONS OF SOME CLASS INVARIANTS OVER QUADRATIC NUMBER FIELDS WITH DISCRIMINANT $D \equiv 1 \pmod{12}$

Daeyeol Jeon*

Abstract. A class invariant is the value of a modular function that generates a ring class field of an imaginary quadratic number field such as the singular moduli of level 1. In this paper, using Shimura Reciprocity Law, we compute the Galois actions of some class invariants from the generalized Weber functions $\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2$ and $\mathfrak{g}_3$ over quadratic number fields with discriminant $D \equiv 1 \pmod{12}$.

1. Introduction

Let K be an imaginary quadratic number field with the discriminant D with ring of integer $\mathcal{O} = \mathbb{Z}[\theta]$ where

$$(1.1)\qquad \theta := \begin{cases} \frac{\sqrt{D}}{2}, & \text{if } D \equiv 0 \pmod{4}; \\ \frac{-1+\sqrt{D}}{2}, & \text{if } D \equiv 1 \pmod{4}. \end{cases}$$

Then the theory of complex multiplication states that the modular invariant $j(\mathcal{O}) = j(\theta)$ generates the *ring class field* $H_{\mathcal{O}}$ over K with degree $[H_{\mathcal{O}} : K] = h(\mathcal{O})$, the class number of $\mathcal{O}$, and the conjugates of $j(\theta)$ under the action of $Gal(H_{\mathcal{O}}/K)$ are *singular moduli* $j(\tau)$, where $\tau := \tau_Q$ is the Heegner point determined by $Q(\tau_Q, 1) = 0$ for a positive definite integral primitive binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $D = b^2 - 4ac$. CM-theory also tells that the minimal polynomial over $\mathbb{Q}$ of $j(\mathcal{O})$ that is called the *Hilbert class polynomial*

Received October 06, 2009; Accepted November 11, 2009.
2000 Mathematics Subject Classification: Primary 11R37; Secondary 11F03.
Key words and phrases: class invariants, Hilbert class polynomials, Galois actions, Weber functions.
This work was supported by the Korea Science and Engineering Foundation(KOSEF) grant funded by the Korea government(MEST)(No. 2009-0060674).

for the order $\mathcal{O}$ has integer coefficients [14]. For example, the class polynomial for $D=-71$ is

$$(1.2)\quad \begin{aligned}\mathcal{H}_{-71}(X) &= X^7+313645809715X^6-3091990138604570X^5\\ &\quad +98394038810047812049302X^4\\ &\quad -823534263439730779968091389X^3\\ &\quad +5138800366453976780323726329446X^2\\ &\quad -425319473946139603274605151187659X\\ &\quad +737707086760731113357714241006081263.\end{aligned}$$

Computing Hilbert class polynomials is very important in number theory and its application to cryptography ([1], [5]), but a serious drawback of the polynomials is huge coefficients as seen in (1.2) even for fairly small discriminant $D=-71$. A remedy for this hinderance is using other modular functions than j-function whose values at CM points also generate the ring class fields $H_{\mathcal{O}}$ but produce much smaller coefficients for Hilbert class polynomials. This method dates back to early 1900. In his Lehrbuch der Algebra [16], H. Weber calls the value of a modular function $f(\theta)$ a *class invariant* if we have

$$K(f(\theta))=K(j(\theta))$$

and gives several examples such as a holomorphic cube root $\gamma_2:\mathbb{H}\to\mathbb{C}$ of j-function and a modular function $\mathfrak{f}_2:\mathbb{H}\to\mathbb{C}$ of level 48. The function values $\zeta_3\gamma_2(\theta)$ and $\zeta_{48}\mathfrak{f}_2(\theta)$ at $\theta=\frac{-1+\sqrt{-71}}{2}$ are both class invariants . While the coefficients of $\mathcal{H}_{-71}(X)$ are enormously large, the minimal polynomials of the class invariants are

$$\begin{aligned}\mathcal{H}_{-71}^{\zeta_3\gamma_2(\theta)}(X) &= X^7+6745X^6-327467x^5+51857115X^4\\ &\quad +2319299751X^3+41264582513X^2\\ &\quad -307873876442X+903568991567\end{aligned}$$

and

$$\mathcal{H}_{-71}^{\zeta_{48}\mathfrak{f}_2(\theta)}(X)=X^7+X^6-X^5-X^4-X^3+X^2+2X-1.$$

Despite a long history of the problem (see [3, p. 378] for list of a few of references), one began to treat class invariants in a systemic and algorithmic way only after Shimura Reciprocity Law [13] became available. The reciprocity law provides not only a method of systematically determining whether $f(\theta)$ is a class invariant but also a description of the Galois conjugates of $f(\theta)$ under $Gal(H_{\mathcal{O}}/K)$. This tool is well illustrated in several works by Reinier M. Bröker, Alice Gee, and Peter

Stevenhagen in [2, 7, 8, 9, 15]. Bröker's Ph. D thesis [2] discusses p-adic theory of class invariants as well.

Gee determined the class invariants from the generalized Weber functions $\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ by using the Shimura Reciprocity Law as follows:

Theorem 1.1. [8, p. 73, Theorem 1] *Let K be an imaginary quadratic number field of discriminant D with the ring of integer $\mathcal{O} = \mathbb{Z}[\theta]$. Suppose $\theta = \frac{-B+\sqrt{D}}{2}$ as defined in (1.1). Evaluating the function in the following table at θ gives an integral generator for $H_{\mathcal{O}}$ over K.*

	$D \equiv 1\,(9)$	$D \equiv 4\,(9)$	$D \equiv 7\,(9)$	$D \equiv 3\,(9)$	$D \equiv 6\,(9)$
$D \equiv 1\,(4)$	$\zeta_3\mathfrak{g}_0^2$ $\zeta_3^2\mathfrak{g}_1^2$	$\mathfrak{g}_0^2$ $\mathfrak{g}_1^2$	$\zeta_3^2\mathfrak{g}_0^2$ $\zeta_3\mathfrak{g}_1^2$	$\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6$	$\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2$
$D \equiv 0\,(8)$	$\zeta_3^2\zeta_4\mathfrak{g}_1^2$ $\zeta_3\zeta_4\mathfrak{g}_2^2$	$\zeta_3\zeta_4\mathfrak{g}_1^2$ $\zeta_3^2\mathfrak{g}_2^2$	$\zeta_4\mathfrak{g}_1^2$ $\zeta_4\mathfrak{g}_2^2$	$\frac{1}{3\sqrt{3}}\mathfrak{g}_0^6$	$\frac{1}{\sqrt{3}}\mathfrak{g}_0^2$
$D \equiv 4\,(8)$	$\zeta_3\mathfrak{g}_1^4$ $\zeta_3^2\mathfrak{g}_2^4$	$\zeta_3^2\mathfrak{g}_1^4$ $\zeta_3\mathfrak{g}_2^4$	$\mathfrak{g}_1^4$ $\mathfrak{g}_2^4$	$\frac{1}{3^3}\mathfrak{g}_0^{12}$	$\frac{1}{3}\mathfrak{g}_0^4$

However, she did not give the explicit Galois actions of the class invariants induced by the table above. As it is very useful to obtain Galois actions of class invariants in determining Hilbert class polynomials that are important in computational number theory, we compute the actions of the class invariants given in the table in the case when $D \equiv 1 \pmod{12}$.

2. Preliminary

2.1. Shimura reciprocity law

Let F_N be the modular function field of level N defined over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N-th root of unity. That is, if $f \in F_N$, then f is meromorphic on the completed upper half plane $\overline{\mathbb{H}}$, has a Laurent series expansion in the variable $q^{1/N}$ centered at $q = 0$ with coefficients in $\mathbb{Q}(\zeta_N)$ and is invariant with respect to the matrix group

$$\Gamma(N) := \ker[\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}(\mathbb{Z}/N\mathbb{Z})].$$

The second main theorem of complex multiplication tells us that the value of a modular function $f \in F_N$ at θ lies in the ray class field of conductor N for the order $\mathcal{O} = \mathbb{Z}[\theta]$ of an imaginary quadratic number field K and the ray class field is generated by the values $g(\theta)$ for the functions $g \in F_N$ with no poles at θ, where θ is defined in (1.1)

Let K^{ab} denote the maximal abelian extension of K. For $f \in F_N$, if $f(\theta)$ lies in $H_{\mathcal{O}}$, then all automorphisms in $\mathrm{Gal}(K^{ab}/H_{\mathcal{O}})$ act trivially on $f(\theta)$. The Shimura Reciprocity Law states that the image of $f(\theta)$ under the inverse image of the Artin map of $\mathrm{Gal}(K^{ab}/H_{\mathcal{O}})$ can be obtained as the value at θ of a modular function that is conjugate to f over $\mathbb{Q}(j)$.

We follow the exposition in [4] that we can easily employ to prove of our results. Let $\mathcal{Q}_D^0$ be the set of primitive quadratic forms and $C(D) = \mathcal{Q}_D^0/\Gamma(1)$ denote the form class group of discriminant D. One obtains a complete set of representatives in $C(D)$ by choosing the reduced forms $[a, b, c]$ such that

$$|b| \le a \le c \quad \text{and} \quad b \ge 0 \quad \text{if either} \quad |b| = a \quad \text{or} \quad a = c.$$

The class of $[a, b^{-1}, c]$ is the inverse of $[a, b, c]$ in $C(D)$.

Given $f \in F_N$, if $f(\theta) \in H_{\mathcal{O}}$, there is an explicit formula for computing the action of $C(D)$ on $f(\theta)$ which is a consequence of the Shimura Reciprocity Law. This is given as follows:

Theorem 2.1. [4, 7] *Let $\mathbb{Z}[\theta]$ be the ring of integers of an imaginary quadratic number field K of discriminant D and let $Q = [a, b, c]$ be a primitive quadratic form of discriminant D. Let $\theta = \frac{-B+\sqrt{D}}{2}$ as defined in (1.1) and $\tau_Q = \frac{-b+\sqrt{D}}{2a}$. Let $M = M_{[a,b,c]} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ be given as follows: For $D \equiv 0 \pmod 4$,*

$$M \equiv \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \pmod{p^{r_p}} \ \text{ if } p \nmid a; \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \pmod{p^{r_p}} \ \text{ if } p \mid a \text{ and } p \nmid c; \\ \begin{pmatrix} -\frac{b}{2} - a & -\frac{b}{2} - c \\ 1 & -1 \end{pmatrix} & \pmod{p^{r_p}} \ \text{ if } p \mid a \text{ and } p \mid c, \end{cases} \tag{2.1}$$

and for $D \equiv 1 \pmod 4$,

$$M \equiv \begin{cases} \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} & \pmod{p^{r_p}} \ \text{ if } p \nmid a; \\ \begin{pmatrix} \frac{-b-1}{2} & -c \\ 1 & 0 \end{pmatrix} & \pmod{p^{r_p}} \ \text{ if } p \mid a \text{ and } p \nmid c; \\ \begin{pmatrix} \frac{-b-1}{2} - a & -\frac{1-b}{2} - c \\ 1 & -1 \end{pmatrix} & \pmod{p^{r_p}} \ \text{ if } p \mid a \text{ and } p \mid c. \end{cases} \tag{2.2}$$

where p runs over all prime factors of N and $p^{r_p} || N$. Then the Galois action of the class of $[a, -b, c]$ in $C(D)$ with respect to the Artin map is given by

$$f(\theta)^{[a,-b,c]} = f^M(\tau_Q)$$

for any $f \in F_N$ such that $f(\theta) \in H_{\mathcal{O}}$. Here f^M denotes the image of f under the action of M.

The action of M depends only on $M_{p^{r_p}}$ for all primes $p|N$ where $M_{p^{r_p}} \in \mathrm{GL}_2(\mathbb{Z}/p^{r_p}\mathbb{Z})$ is the reduction modulo p^{r_p} of M. Every $M_{p^{r_p}}$ with determinant x decomposes as $M_{p^{r_p}} = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/p^{r_p}\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z}/p^{r_p}\mathbb{Z})$ is generated by $S_{p^{r_p}}$ and $T_{p^{r_p}}$, it suffices to find the action of $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^{r_p}}$, $S_{p^{r_p}}$ and $T_{p^{r_p}}$ on f for all $p|N$. For $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^{r_p}}$, the action on f is given by lifting the automorphism of $\mathbb{Q}(\zeta_N)$ determined by

$$\zeta_{p^{r_p}} \mapsto \zeta_{p^{r_p}}^x \quad \text{and} \quad \zeta_{q^{r_q}} \mapsto \zeta_{q^{r_q}}$$

for all prime factors $q|N$ with $q \neq p$. In order that the actions of the matrices at different primes commute with each other, we lift $S_{p^{r_p}}$ and $T_{p^{r_p}}$ to matrices in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ such that they reduce to the identity matrix in $\mathrm{SL}_2(\mathbb{Z}/q^{r_q}\mathbb{Z})$ for all $q \neq p$.

2.2. Weber functions

Recall that the normalized Eisenstein series

$$g_2(z) = 60 \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+nz)^4}$$

$$g_3(z) = 140 \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+nz)^6}$$

are modular forms of weights 4 and 6, respectively. The Dedekind-eta function

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \text{with} \quad q = e^{2\pi i z}$$

is holomorphic and non-zero for z in the complex upper half plane $\mathbb{H}$ and $\Delta(z) = \eta^{24}(z)$ is modular form of weight 12 with no poles or zeros on $\mathbb{H}$. The classical j-invariant is defined for $z \in \mathbb{H}$ by

$$j(z) = 12^3\frac{g_2^3(z)}{(2\pi)^{12}\Delta(z)} = 12^3 + 6^6\frac{g_3^2(z)}{(2\pi)^{12}\Delta(z)}$$

is invariant under the group $\Gamma(1)$, and Weber functions are given by

$$\gamma_2(z) = \frac{12g_2(z)}{(2\pi)^4\eta^8(z)},\ \gamma_3(z) = \frac{216g_3(z)}{(2\pi)^6\eta^{12}(z)},$$

$$\mathfrak{f}(z) = \zeta_{48}^{-1}\frac{\eta(\frac{z+1}{2})}{\eta(z)},\ \mathfrak{f}_1(z) = \frac{\eta(\frac{z}{2})}{\eta(z)},\ \mathfrak{f}_2(z) = \sqrt{2}\frac{\eta(2z)}{\eta(z)}.$$

One can generalize Weber functions by taking the holomorphic 24-th root of the Siegel function

$$\phi = n^{12}\frac{\Delta(A\tau)}{\Delta(\tau)},$$

where $A = \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix}$ for $k \in \mathbb{Z}$ and a positive integer n. If $n = 2$, we have Weber functions and if $n = 3$, we consider

(2.3)
$$\mathfrak{g}_0(z) = \frac{\eta(\frac{z}{3})}{\eta(z)},\ \mathfrak{g}_1(z) = \zeta_{24}^{-1}\frac{\eta(\frac{z+1}{3})}{\eta(z)},\ \mathfrak{g}_2(z) = \frac{\eta(\frac{z+2}{3})}{\eta(z)},\ \mathfrak{g}_3(z) = \sqrt{3}\frac{\eta(3z)}{\eta(z)}.$$

The Siegel function has a long story [6, 12] and the study on its 24-th root of unity for arbitrary n has been made pretty recently [8, 10, 11].

Note that the functions in (2.3) are modular of level 72. For the generating matrices S, $T \in \Gamma(1)$ given by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the transformation rules $\eta \circ S(z) = \sqrt{-iz}\eta(z)$ and $\eta \circ T(z) = \zeta_{24}\eta(z)$ hold. Hence

(2.4)
$$\begin{aligned}(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ S &= (\mathfrak{g}_3, \zeta_{24}^{-2}\mathfrak{g}_2, \zeta_{24}^{2}\mathfrak{g}_1, \mathfrak{g}_0),\\ (\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ T &= (\mathfrak{g}_1, \zeta_{24}^{-2}\mathfrak{g}_2, \mathfrak{g}_0, \zeta_{24}^{2}\mathfrak{g}_3).\end{aligned}$$

3. Results

In this section, we compute the action of a primitive quadratic form $Q = [a, b, c]$ on the class invariants when $D \equiv 1 \pmod{12}$ in the table in Theorem 1.1. For that we need to find the action of $M_m \in GL_2(\mathbb{Z}/m\mathbb{Z})$ with $m = 8, 9$. Combining Lemma 6 of [7] and the transformation rule (2.4), we obtain the following:

LEMMA 3.1. *The actions of* $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_m$, S_m *and* T_m $(m = 8, 9)$ *on* $\mathfrak{g}_i^2$ $(i = 0, 1, 2, 3)$ *are given by*

	$\mathfrak{g}_0^2$	$\mathfrak{g}_1^2$	$\mathfrak{g}_2^2$	$\mathfrak{g}_3^2$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_8$	$\mathfrak{g}_0^2$	$\mathfrak{g}_1^2$	$\mathfrak{g}_2^2$	$\mathfrak{g}_3^2$
S_8	$-\mathfrak{g}_0^2$	$-\mathfrak{g}_1^2$	$-\mathfrak{g}_2^2$	$-\mathfrak{g}_3^2$
T_8	$-\mathfrak{g}_0^2$	$-\mathfrak{g}_1^2$	$-\mathfrak{g}_2^2$	$-\mathfrak{g}_3^2$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9$, $x = -3k + 1$	$\mathfrak{g}_0^2$	$\zeta_3^{2k}\mathfrak{g}_1^2$	$\zeta_3^{k}\mathfrak{g}_2^2$	$\mathfrak{g}_3^2$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9$, $x = -3k - 1$	$\mathfrak{g}_0^2$	$\zeta_3^{2k}\mathfrak{g}_2^2$	$\zeta_3^{k}\mathfrak{g}_1^2$	$\mathfrak{g}_3^2$
S_9	$-\mathfrak{g}_3^2$	$\zeta_3\mathfrak{g}_2^2$	$\zeta_3^2\mathfrak{g}_1^2$	$-\mathfrak{g}_0^2$
T_9	$-\mathfrak{g}_1^2$	$\zeta_3\mathfrak{g}_2^2$	$-\mathfrak{g}_0^2$	$\zeta_3^2\mathfrak{g}_3^2$

Using this, together with Theorem 2.1, we have the following theorems.

THEOREM 3.2. *Let* $\mathbb{Z}[\theta]$ *be the ring of integers of an imaginary quadratic number field* K *of discriminant* $D \equiv 1 \pmod{36}$ *and let* $Q = [a, b, c]$ *be a reduced primitive quadratic form of discriminant* D. *Let* $\theta = \frac{-1+\sqrt{D}}{2}$, $\tau_Q = \frac{-b+\sqrt{D}}{2a}$ *and* $u = (-1)^{\frac{b+1}{2}+ac+a+c}$. *Then the actions of* $[a, -b, c]$ *on* $\zeta_3\mathfrak{g}_0^2(\theta)$ *are as follows:*

(1) *The case* $3 \nmid a$.

a) *If* $b \equiv 1 \pmod 3$, *then* $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$a \equiv 1 \pmod 9$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 2 \pmod 9$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$

b) *If* $a(b-1) \equiv 1 \pmod 3$, *then* $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$

$a \equiv 1 \pmod 9$	$-u\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$-u\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$-u\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$a \equiv 2 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$-u\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$-u\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$-u\mathfrak{g}_2^2(\tau_Q)$

c) *If* $a(b-1) \equiv -1 \pmod 3$, *then* $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$a \equiv 1 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$-u\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$
$a \equiv 2 \pmod 9$	$-u\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$-u\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$-u\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$

(2) *The cases* $3|a$ *and* $3 \nmid c$.

a) *If* $b \equiv -1 \pmod 3$, *then* $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 2 \pmod 9$	$b \equiv 5 \pmod 9$	$b \equiv 8 \pmod 9$
$c \equiv 1 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 2 \pmod 9$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 4 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 5 \pmod 9$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 7 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$c \equiv 8 \pmod 9$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$

b) *If* $(b+1)c \equiv 1 \pmod 3$, *then* $\zeta_3 \mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$c \equiv 1 \pmod 9$	$-u\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$
$c \equiv 4 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$-u\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$
$c \equiv 7 \pmod 9$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$-u\mathfrak{g}_2^2(\tau_Q)$

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$c \equiv 2 \pmod 9$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$-u\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$
$c \equiv 5 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$-u\mathfrak{g}_2^2(\tau_Q)$
$c \equiv 8 \pmod 9$	$-u\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_2^2(\tau_Q)$

c) *If* $(b+1)c \equiv -1 \pmod 3$, *then* $\zeta_3 \mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$c \equiv 1 \pmod 9$	$-u\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$
$c \equiv 4 \pmod 9$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$-u\mathfrak{g}_1^2(\tau_Q)$
$c \equiv 7 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\mathfrak{g}_1^2(\tau_Q)$

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$c \equiv 2 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\mathfrak{g}_1^2(\tau_Q)$
$c \equiv 5 \pmod 9$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$-u\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$
$c \equiv 8 \pmod 9$	$-u\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_1^2(\tau_Q)$

(3) *The cases* $3|a$ *and* $3|c$.

a) *If* $b \equiv 1 \pmod 9$, *then* $\zeta_3 \mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$

b) *If* $b \equiv 2 \pmod 9$*, then* $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$

c) *If* $b \equiv 4 \pmod 9$*, then* $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$

d) *If* $b \equiv 5 \pmod 9$*, then* $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$

e) *If* $b \equiv 7 \pmod 9$*, then* $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$u\mathfrak{g}_0^2(\tau_Q)$

f) *If* $b \equiv 8 \pmod 9$*, then* $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$	$-u\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$

Proof. As explained before, we compute the action $\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]}$ by using the following equality:

$$\zeta_3\mathfrak{g}_0^2(\theta)^{[a,-b,c]} = (\zeta_3\mathfrak{g}_0^2)^M(\tau_Q),$$

where M is defined in (2.2). Since $(\zeta_3\mathfrak{g}_0^2)^M = \big((\zeta_3\mathfrak{g}_0^2)^{M_8}\big)^{M_9}$, first consider the action $(\mathfrak{g}_0^2)^{M_8}$. In either case of $m = 8$ or 9, the matrix has the following decomposition due to [7, Lemma 6]:

$$(3.1)\quad M_m = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_m S_m T_m^{-a^{-1}} S_m T_m^{-a^{-1}} S_m T_m^{\frac{b-3}{2a}}, & p \nmid a; \\ \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}_m T_m^{\frac{1-b}{2}c} S_m T_m^{c^{-1}} S_m T_m^{c}, & p|a \wedge p \nmid c; \\ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}_m T_m^{(\frac{1-b-2a}{2})d} S_m T_m^{d^{-1}} S_m T_m^{(d-1)}, & p|a \wedge p|c, \end{cases}$$

where $d = a + b + c$ and $p = 2, 3$. By using the decomposition (3.1) and Lemma 3.1, we have the following action:

$$(\mathfrak{g}_0^2)^{M_8} = (-1)^{\frac{b+1}{2}+ac+a+c}\mathfrak{g}_0^2.$$

Thus

$$(\zeta_3\mathfrak{g}_0^2)^{M_8} = u\zeta_3\mathfrak{g}_0^2.$$

The action $(\mathfrak{g}_0^2)^{M_9}$ can be obtained by the same method and a case by case computation, and then the result follows. ☐

By the same method used in the proof of Theorem 3.2, we have the following result:

THEOREM 3.3. *Let $\mathbb{Z}[\theta]$ be the ring of integers of an imaginary quadratic number field K of discriminant $D \equiv 1 (\mathrm{mod}\, 36)$ and let $Q = [a, b, c]$ be a reduced primitive quadratic form of discriminant D. Let $\theta = \frac{-1+\sqrt{D}}{2}$, $\tau_Q = \frac{-b+\sqrt{D}}{2a}$ and $u = (-1)^{\frac{b+1}{2}+ac+a+c}$. Then the actions of $[a, -b, c]$ on $\zeta_3^2\mathfrak{g}_1^2(\theta)$ are as follows:*

(1) *The case $3 \nmid a$.*

a) *If $b \equiv -1 \,(\mathrm{mod}\, 3)$, then $\zeta_3^2\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ is given by the following table:*

	$b \equiv 2 \,(\mathrm{mod}\, 9)$	$b \equiv 5 \,(\mathrm{mod}\, 9)$	$b \equiv 8 \,(\mathrm{mod}\, 9)$
$a \equiv 1 \,(\mathrm{mod}\, 9)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 2 \,(\mathrm{mod}\, 9)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$

$a \equiv 4 \pmod 9$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$

b) *If* $a + b \equiv -1 \pmod 3$, *then* $\zeta_3^2\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$a \equiv 1 \pmod 9$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$u\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$u\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$u\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\zeta_3\mathfrak{g}_1^2(\tau_Q)$

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$a \equiv 2 \pmod 9$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$u\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$u\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$u\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$

c) *If* $a - b \equiv 1 \pmod 3$, *then* $\zeta_3^2\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b \equiv 0 \pmod 9$	$b \equiv 3 \pmod 9$	$b \equiv 6 \pmod 9$
$a \equiv 1 \pmod 9$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$u\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 4 \pmod 9$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$u\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 7 \pmod 9$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$u\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$

	$b \equiv 1 \pmod 9$	$b \equiv 4 \pmod 9$	$b \equiv 7 \pmod 9$
$a \equiv 2 \pmod 9$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$u\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 5 \pmod 9$	$u\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$
$a \equiv 8 \pmod 9$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$u\mathfrak{g}_2^2(\tau_Q)$

(2) *The cases* $3|a$ *and* $3 \nmid c$.

a) *If* $b \equiv 1 \pmod 3$, *then* $\zeta_3^2\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b\equiv 1 \pmod 9$	$b\equiv 4 \pmod 9$	$b\equiv 7 \pmod 9$
$c\equiv 1 \pmod 9$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$c\equiv 2 \pmod 9$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$
$c\equiv 4 \pmod 9$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$c\equiv 5 \pmod 9$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$
$c\equiv 7 \pmod 9$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$c\equiv 8 \pmod 9$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$

b) *If* $b+c\equiv 1 \pmod 3$, *then* $\zeta_3^2\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b\equiv 0 \pmod 9$	$b\equiv 3 \pmod 9$	$b\equiv 6 \pmod 9$
$c\equiv 1 \pmod 9$	$u\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$
$c\equiv 4 \pmod 9$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$u\mathfrak{g}_1^2(\tau_Q)$
$c\equiv 7 \pmod 9$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$u\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$

	$b\equiv 2 \pmod 9$	$b\equiv 5 \pmod 9$	$b\equiv 8 \pmod 9$
$c\equiv 2 \pmod 9$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$u\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$
$c\equiv 5 \pmod 9$	$u\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$
$c\equiv 8 \pmod 9$	$u\zeta_3^2\mathfrak{g}_1^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_1^2(\tau_Q)$	$u\mathfrak{g}_1^2(\tau_Q)$

c) *If* $b-c\equiv 1 \pmod 3$, *then* $\zeta_3^2\mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$b\equiv 2 \pmod 9$	$b\equiv 5 \pmod 9$	$b\equiv 8 \pmod 9$
$c\equiv 1 \pmod 9$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$u\mathfrak{g}_2^2(\tau_Q)$
$c\equiv 4 \pmod 9$	$u\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$
$c\equiv 7 \pmod 9$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$u\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$

	$b\equiv 0 \pmod 9$	$b\equiv 3 \pmod 9$	$b\equiv 6 \pmod 9$
$c\equiv 2 \pmod 9$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$u\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$
$c\equiv 5 \pmod 9$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$	$u\mathfrak{g}_2^2(\tau_Q)$
$c\equiv 8 \pmod 9$	$u\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_2^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_2^2(\tau_Q)$

(3) *The cases* $3|a$ *and* $3|c$.

a) *If* $b \equiv 1 \pmod 9$, *then* $\zeta_3^2 \mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$

b) *If* $b \equiv 2 \pmod 9$, *then* $\zeta_3^2 \mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$

c) *If* $b \equiv 4 \pmod 9$, *then* $\zeta_3^2 \mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$

d) *If* $b \equiv 5 \pmod 9$, *then* $\zeta_3^2 \mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$	$-u\mathfrak{g}_0^2(\tau_Q)$

e) *If* $b \equiv 7 \pmod 9$, *then* $\zeta_3^2 \mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3^2\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$	$u\zeta_3\mathfrak{g}_3^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$	$u\mathfrak{g}_3^2(\tau_Q)$

f) *If* $b \equiv 8 \pmod 9$, *then* $\zeta_3^2 \mathfrak{g}_1^2(\theta)^{[a,-b,c]}$ *is given by the following table:*

	$c \equiv 0 \pmod 9$	$c \equiv 3 \pmod 9$	$c \equiv 6 \pmod 9$
$a \equiv 0 \pmod 9$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 3 \pmod 9$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$
$a \equiv 6 \pmod 9$	$-u\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3^2\mathfrak{g}_0^2(\tau_Q)$	$-u\zeta_3\mathfrak{g}_0^2(\tau_Q)$

Remark 3.4. The Galois actions of $\mathfrak{g}_0^2(\theta)$ and $\mathfrak{g}_1^2(\theta)$ for the discriminant $D \equiv 13 \pmod{36}$ can be obtained by multiplying proper powers of ζ_3 to the Galois actions in Theorem 3.2 and 3.3, and so are the Galois actions of $\zeta_3^2\mathfrak{g}_0^2(\theta)$ and $\zeta_3\mathfrak{g}_1^2(\theta)$ for the discriminant $D \equiv 25 \pmod{36}$.

References

[1] O. Atkin and F. Morain, *Elliptic curves and primality proving*, Math. Comp. **61** (1993), 29-68.

[2] R. M. Bröker, *Constructing elliptic curves of prescribed order*, Ph.D. Thesis, Universiteit Leiden, 2006.

[3] J. H. Bruinier, P. Jenkins, and K. Ono, *Hilbert class polynomials and traces of singular moduli*, Math. Ann. **334** (2006), 373–393.

[4] H. H. Chan, A. Gee, and V. Tan, *Cubic singular moduli, Ramanujan's class invariants λ_n and the explicit Shimura reciprocity law*, Pacific J. Math. **208** (2003), 23–37.

[5] D. Cox, *Primes of the form $x^2 + ny^2$*, John Wiley & Sons, 1989.

[6] M. Deuring, *Die Klassenkörper der komplexen Multiplikation*, in 'Enz. Math. Wiss. Band I_2' Heft 10, Teil II, Stuttgart 1958.

[7] A. Gee, *Class invariants by Shimura's reciprocity law*, J. Théor. Nombre Bordeaux **11** (1999), 45–72.

[8] A. Gee, *Class fields by Shimura reciprocity*, Ph. D. Thesis, Universiteit van Amsterdam, 2001.

[9] A. Gee and P. Stevenhagen, *Generating class fields using Shimura reciprocity*, Proceedings of the Third International Symposium on Algorithmic Number Theory (Lecture Notes in Computer Sciences 1423 pp. 441-453, Springer-Verlag, 1998).

[10] W. Hart, *Schläfli modular equations for generalized Weber functions*, Ramanujan J. **15** (2008), 435–468.

[11] F. Hajir and F. R. Villegas , *Explicit elliptic units I*, Duke Math. J. **90** (1997), no. 3, 495–521.

[12] S. Lang, *Elliptic Functions*, Springer Graduate Texts in Mathematics, vol. 112, 1987.

[13] G. Shimura, *Introduction to the arithmetic theory of automorphic forms*, Princeton University Press, 1971.

[14] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Springer Graduate Texts in Mathematics, vol. 151, 1994.
[15] P. Stevenhagen, *Hilbert's 12th problem, complex multiplication and Shimura reciprocity*, Class field theory-its centenary and prospect, ed. K. Miyake, *Adv. Studies in pure math.* **30** (2001), 161-176.
[16] H. Weber, *Lehrbuch der Algebra*, dritter Band, Friedrich Vieweg und Sohn, 1908.

*
Department of Mathematics Education
Kongju National University
Kongju 314-701, Republic of Korea
E-mail: dyjeon@kongju.ac.kr