JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 4, December 2009

CASTELNOUVO-MUMFORD REGULARITY OF GRADED MODULES HAVING A LINEAR FREE PRESENTATION

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Abstract. In this paper we investigate the upper bound on the Castelnuovo-Mumford regularity of a graded module with linear free presentation. Let M be a finitely generated graded module over a polynomial ring R with zero dimensional support. We prove that if M is generated by elements of degree $d \geq 0$ with a linear free presentation

$$\bigoplus^{p} R(-d-1) \xrightarrow{\phi} \bigoplus^{q} R(-d) \to M \to 0,$$

then the Castelnuovo-Mumford regularity of M is at most $d+q-1$. As an important application, we can prove *vector bundle technique*, which was used in [11], [13], [17] as a tool for obtaining several remarkable results.

1. Introduction

Let $R = K[x_0, \ldots, x_n]$ be the coordinate ring of $\mathbb{P}^n$ where K is an algebraically closed field of characteristic zero. David Mumford [16] defined the regularity of a coherent sheaf on projective space in order to generalize the idea of Castelnuovo's base point free pencil trick, which is useful in controlling the vanishing of cohomology of a sheaf. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is d-regular if

$$H^i(\mathbb{P}^n, \mathcal{F}(d-i) = 0) \quad \text{for every } i \geq 1.$$

We will write $\operatorname{reg}(\mathcal{F})$ for the least number d such that $\mathcal{F}$ is d-regular.

Received September 25, 2009; Accepted November 24, 2009.
2000 Mathematics Subject Classification: Primary 14N05, 13D02; Secondary 14M17.
Key words and phrases: Castelnuovo-Mumford regularity, linear free presentation, local cohomology, Eagon-Northcott complex, Fitting ideal.

The definition for a finitely generated graded R-module M, which extends that for sheaves, was given by Eisenbud and Goto [6]. Let

$$\mathbb{F}_\bullet : 0 \longrightarrow F_t \xrightarrow{\phi_t} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

be the minimal free resolution of M where $F_i = \oplus_j R(-j)^{\beta_{i,j}}$. The regularity of M is defined by the supremum of the numbers $j - i$ with non-vanishing betti number $\beta_{i,j}$. We will write $\mathrm{reg}(M)$ for this number. The most important case is when M is the defining ideal I_X of a closed subscheme $X \subset \mathbb{P}^n$. In this case, $\mathrm{reg}(I_X)$ is called the regularity of X and we denote it $\mathrm{reg}(X)$.

In [2], Bayer and Mumford have shown that Castelnuovo-Mumford regularity is exactly same as the maximum degree of the reduced Gröbner bases of a homogeneous ideal I in generic coordinates and they considered it as an important measure of how complicated the ideal I is. In this context, there has been several results which establish bounds for the Castelnuovo-Mumford regularity. In the worst case, a doubly exponential bound for the regularity in terms of the maximum degree of generators was given by Giusti [8] and Galligo [9] in characteristic 0 and recently, by Caviglia-Sbarra [3] for arbitrary field. Mayer and Meyer [15] showed this bound is the best possible. However, many results shows that the bound on the regularity of a closed subscheme $X \subset \mathbb{P}^n$ is much smaller, like the n^{th} power of the maximum degree of generators of I_X or better, if X is nice geometric property. (see [1], [2], [10], [11], [12], [14], [18], and [19]).

The goal of this paper is to give a bound of Castelnuovo-Mumford regularity of a finitely generated graded R-module M in terms of the number of generators and their degrees. We are mainly concerned with the case that M is supported in dimension zero and it has a linear free presentation. As an important application, we will prove *vector bundle technique*, which was used in [11], [13], [17] as a tool for obtaining several remarkable results.

2. Preliminaries

Many of the preliminaries we will discuss in this section are based on the fundamental work of Eisenbud and Goto which describes Castelnuovo-Mumford regularity using local cohomology. For precise statements, we give notations and recall definitions. Through the paper, we use the following usual notations:

- $R = K[x_0, \cdots, x_n]$: a polynomial ring in $n+1$ variables over K

- $\mathfrak{m} = (x_0, \ldots, x_n)$: the homogeneous maximal ideal of R.
- $\mathcal{F}$: a coherent sheaf on $\mathbb{P}^n$.
- M : a finitely generated graded R-module.

The most important characterization of Castelnuovo-Mumford regularity is cohomological. One way to state it is to use local cohomology. Eisenbud-Goto gave a connection between the regularity for a coherent sheaf and the regularity of graded modules in [6] using local cohomology. For a graded R-module M, the zeroth local cohomology of M is

$$H^0_{\mathfrak{m}}(M) := \{a \in M \mid \mathfrak{m}^d a = 0 \quad \text{for some } d\}.$$

Since $H^0_{\mathfrak{m}}(M)$ is an left exact functor, it is natural to study its derived functors. For every $i \geq 1$, the i-th local cohomology of M is defined by the derived functors of $H^0_{\mathfrak{m}}(M)$, which we call $H^i_{\mathfrak{m}}(M)$.

The local cohomology is related to the sheaf cohomology in a simple way. Using Čech cohomology one can prove the following result.

PROPOSITION 2.1 (Local cohomology and Sheaf Cohomology). *Let M be a graded R-module, and let $\mathcal{F}$ be the corresponding quasi-coherent sheaf on $\mathbb{P}^n$. Then we have*

(a) *There is an exact sequence of graded R-modules*

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \bigoplus_d H^0(\mathcal{F}(d)) \to H^1_{\mathfrak{m}}(M) \to 0.$$

(b) *For every $i \geq 2$,*

$$H^i_{\mathfrak{m}}(M) = \bigoplus_d H^{i-1}(\mathbb{P}^n, \mathcal{F}(d)).$$

Proof. See Proposition A1.11. in [5]. □

Since the local cohomology is actually dual to the homology of the complex $\mathrm{Hom}(\mathbf{F}, R)$, where $\mathbf{F}$ is a free resolution of M, the regularity can be formulated in terms of local cohomology.

THEOREM 2.2 (Characterization of Regularity: Cohomology). *Let M be a finitely generated graded R-module and let $r_i = \max\{e \mid H^i_{\mathfrak{m}}(M)_e \neq 0\}$ for each $i \geq 0$. Then the followings are equivalent.*

(a) $\mathrm{reg}(M) \leq d$.
(b) $r_i + i \leq d$ *for all* $i \geq 0$.
(c) $r_0 \leq d$ *and* $H^i_{\mathfrak{m}}(M)_{d-i+1} = 0$ *for all* $i > 0$.

Proof. See Theorem 4.3 in [5]. □

From the associated long exact sequence in local cohomology and Theorem 2.2 we can easily see the following results (Corollary 4.4 in [5] and Corollary 20.19. in [4]).

Corollary 2.3. *If M is a graded R-module of finite length, then*

$$\operatorname{reg}(M) = \max\{\, d \mid M_d \neq 0 \,\}$$

Corollary 2.4. *Suppose that $0 \to M' \to M \to M'' \to 0$ is an exact sequence of finitely generated graded R-modules. Then we have*

(a) $\operatorname{reg}(M') \leq \max\{\operatorname{reg}(M), \operatorname{reg}(M'') + 1\}$;
(b) $\operatorname{reg}(M) \leq \max\{\operatorname{reg}(M'), \operatorname{reg}(M'')\}$
(c) $\operatorname{reg}(M'') \leq \max\{\operatorname{reg}(M), \operatorname{reg}(M') - 1\}$

Definition 2.5. Let M be a finitely generated S-modlule, with free presentation

$$R^p \xrightarrow{\phi} R^q \to M \to 0.$$

Then the ideal generated by $q \times q$ minors, written $I_q(\phi)$, depends only on the map of free modules ϕ not on the choice of bases. We call it *the zeroth Fitting ideal* of the map ϕ.

There is a close relation between the annihilator and the zeroth Fitting ideal. We will use the following result in this paper.

Theorem 2.6. *If M is generated by t elements as a R-module, then*

$$(\operatorname{ann}M)^t \subset I_q(\phi) \subset \operatorname{ann}M.$$

Proof. Refer to Propostion 20.7. in [4]. □

Let $F = R(-1)^p$ and $G = R^q$ where $p \geq q$. Consider the Eagon-Northcott complex of a graded preserving map $F \xrightarrow{\phi} G$, which furnishes an approximation to a resolution of the zeroth Fitting ideal $I_q(\phi)$ (see page 209 in [4]):

$$\begin{aligned}\mathbf{EN}(\phi): \quad & 0 \to (\operatorname{Sym}_{p-q}G)^* \otimes \textstyle\bigwedge^p F \to (\operatorname{Sym}_{p-q-1}G)^* \otimes \textstyle\bigwedge^{p-1} F \\ & \to \cdots \to (\operatorname{Sym}_2 G)^* \otimes \textstyle\bigwedge^{q+2} F \to G^* \otimes \textstyle\bigwedge^{q+1} F \\ & \to \textstyle\bigwedge^q F \xrightarrow{\wedge^q \phi} \textstyle\bigwedge^q G \cong R \to R/I_q(\phi) \to 0.\end{aligned}$$

Note that each free part of Eagon-Northcott complex is given by

$$(\operatorname{Sym}_j G)^* \cong \bigoplus^{\binom{q+j-1}{j}} R \quad \text{and} \quad \bigwedge^i F \cong \bigoplus^{\binom{p}{i}} R(-i),$$

and thus we have $(\mathrm{Sym}_j G)^* \otimes \bigwedge^{q+j-1} F \cong \bigoplus^{\binom{q+j-1}{j}\binom{p}{q+j-1}} R(-j-q+1)$ such that its regularity is at least $j+q-1$.

To bound the regularity of the Fitting ideal of a graded module M with linear free presentation will be the technical heart of the paper. For the proof of the main result we will use the following theorem, which can be found in [5].

THEOREM 2.7 (Theorem A2.59. in [5]). *Let $F = R^p \xrightarrow{\phi} G = R^q$ with $p \geq q$ be a map of free R-modules. Then the zeroth Fitting ideal $I_q(\phi)$ annihilates the homology of $\mathbf{EN}(\phi)$.*

3. Regularity of the annihilator of a graded module

In this section, we will show that the regularity bound for Fitting ideal of a graded module with linear free presentation can be given by applying Lemma 3.1 to the Eagon-Northcott complex. From this, we will give an upper bound for the regularity of the annihilator of a graded module. The following lemma is the pivotal result of this paper.

LEMMA 3.1. *Suppose that*

$$\mathbb{F}_\bullet : 0 \longrightarrow F_t \xrightarrow{\phi_t} \cdots \xrightarrow{\phi_3} F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

is a complex of finitely generated graded S-modules, and there is an integer d such that

- *the homology $H_i(\mathbb{F}_\bullet)$ is supported in dimension zero for all $i \geq 1$; and*
- $\mathrm{reg}(F_i) \leq d + i$ *for all* $i \geq 0$.

Then $\mathrm{reg}(\mathrm{coker}\,\phi_1) \leq d$ *and* $\mathrm{reg}(\mathrm{im}\,\phi_1) \leq d+1$.

Proof. Let us try to prove by induction on t. Suppose that $t = 0$ such that we have a map $0 \xrightarrow{\phi_1} F_0$. Then $F_0 \cong \mathrm{coker}\ \phi_1$ and thus $\mathrm{reg}(\mathrm{coker}\ \phi_1) = \mathrm{reg}(F_0) \leq d$.

For $t > 0$, note that we have the long exact sequence in local cohomology $H^i_{\mathfrak{m}}(-)$ from

$$0 \to H_1(\mathbb{F}_\bullet) \to \mathrm{coker}\ \phi_2 \to \mathrm{im}\ \phi_1 \to 0,$$

where $\mathfrak{m}$ is the unique homogeneous maximal ideal of R. Since $H_1(\mathbb{F}_\bullet)$ is supported in dimension zero, we have $H^i_{\mathfrak{m}}(H_1(\mathbb{F}_\bullet)(s)) = 0$ for all $i \geq 1$. Hence the map

$$H^i_{\mathfrak{m}}(\mathrm{coker}\ \phi_2(s)) \to H^i_{\mathfrak{m}}(\mathrm{im}\ \phi_1(s))$$

is surjective for every $i \geq 0$ and s. By Theorem 2.2, we have that $\mathrm{reg}(\mathrm{im}\ \phi_1) \leq \mathrm{reg}(\mathrm{coker}\ \phi_2)$ and, by induction argument on t, $\mathrm{reg}(\mathrm{coker}\ \phi_2) \leq d+1$. Hence we see $\mathrm{reg}(\mathrm{im}\ \phi_1) \leq d+1$. Applying Corollary 2.4 to the short exact sequence

$$0 \to \mathrm{im}\,\phi_1 \to F_0 \to \mathrm{coker}\,\phi_1 \to 0, \tag{3.1}$$

we conclude that

$$\mathrm{reg}(\mathrm{coker}\,\phi_1) \leq \max\{\mathrm{reg}(F_0), \mathrm{reg}(\mathrm{im}\ \phi_1) - 1\} \leq d,$$

and we are done. □

From Lemma 3.1, we derive a general bound on the regularity of Fitting ideals.

Proposition 3.2. *Let M be a finitely generated graded S-module having a linear free presentaion*

$$\bigoplus^{p} R(-1) \xrightarrow{\phi} \bigoplus^{q} R \to M \to 0,$$

such that $\dim M = 0$. *Then we have* $\mathrm{reg}(I_q(\phi)) \leq q$.

Proof. Consider a free presentation of M

$$F = \bigoplus^{p} R(-1) \xrightarrow{\phi} G = \bigoplus^{q} R \to M \to 0.$$

Let $I_q(\phi)$ be the zeroth Fitting ideal of the map ϕ. By assumption, we know $\dim(M) = \dim(R/\mathrm{ann}(M)) = 0$. Applying Theorem 2.6 to $\mathrm{ann}(M)$ and $I_t(\phi)$, we see at once that

$$\dim(R/I_q(\phi)) = \dim(R/\mathrm{ann}(M)) = \dim(M) = 0.$$

Suppose that

$$\mathbb{E}_\bullet : 0 \longrightarrow E_t \xrightarrow{\phi_t} \cdots \xrightarrow{\phi_3} E_2 \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} S \to S/I_q(\phi)) \to 0$$

is the Eagon-Northcott complex $\mathbf{EN}(\phi)$, where $E_i = (\mathrm{Sym}_{i-1}G)^* \otimes \bigwedge^{q-1+i} F$. By Theorem 2.7, the ideal $I_q(\phi)$ annihilates the homology of $\mathbf{EN}(\phi)$ and so $I_q(\phi) \subset \mathrm{ann}(H_i(\mathbf{EN}(\phi)))$. This implies that, for all $i \geq 1$,

$$\dim H_i(\mathbf{EN}(\phi)) \leq \dim R/I_q(\phi) = 0.$$

Hence the homology $H_i(\mathbb{E}_\bullet)$ is supported in dimension zero for all $i \geq 1$ and $\mathrm{reg}(E_i) = q + i - 1$, so we may take $d = q - 1$ in Lemma 3.1 and the result follows. □

Corollary 3.3. *With the same assumption as in Proposition 3.2,*

$$\mathrm{reg}(\mathrm{ann}(M)) \leq q.$$

Proof. Let $\mathfrak{m}$ be the unique maximal homogeneous ideal of R. Since M is a module of dimension zero, we see that $\operatorname{ann}(M)$ and $I_q(\phi)$ are homogeneous ideal of R supported in dimension zero. By Corollary 2.3, we have

$$\begin{aligned}\operatorname{reg}(\operatorname{ann}(M)) &= 1 + \max\{e \mid R/\operatorname{ann}(M)_e = 0\} = \min\{e \mid \operatorname{ann}(M)_e = \mathfrak{m}_e\} \\ \operatorname{reg}(I_q(\phi)) &= 1 + \max\{e \mid R/I_q(\phi))_e = 0\} = \min\{e \mid I_q(\phi))_e = \mathfrak{m}_e\}.\end{aligned}$$

Then it follows directly from Theorem 2.6 and Proposition 3.2 that

$$\operatorname{reg}(\operatorname{ann}(M)) \leq \operatorname{reg}(I_q(\phi)) \leq q,$$

as we wished. □

4. Regularity of a graded module with linear free presentation

Now we are ready to prove the main result in this paper.

THEOREM 4.1. *If M is a finitely generated graded R-module having a linear free presentaion*

$$\bigoplus^{p} R(-d-1) \xrightarrow{\phi} \bigoplus^{q} R(-d) \to M \to 0,$$

such that $\dim M = 0$ *then we have* $\operatorname{reg}(M) \leq d + q - 1$.

Proof. Note that we may assume that $d = 0$ and it suffices to show that the regularity is at least $q-1$ since the module $M(d)$ is generated by elements of degree 0 and $\operatorname{reg}(M(d)) = \operatorname{reg}(M) - d$. For the proof, we do induction on $q \geq 1$. If $q = 1$ then there is a homogeneous ideal $J \subset R$ generated by elements of degree 1 such that $M \cong R/J$ and we have

$$\operatorname{reg}(M) = \operatorname{reg}(R/J) = \operatorname{reg}(J) - 1.$$

Note that $\operatorname{reg}(J) = 0$ is complete intersection ideal generated by linear forms whose the minimal free resolution is given by Koszul complex. Hence we have $\operatorname{reg}(M) = 0 = q - 1$.

Suppose $q > 1$ and let $M/\mathfrak{m}M = \langle \bar{m}_1, \ldots, \bar{m}_t \rangle$, where $t \leq q$. By Nakayama Lemma, we know that $M = Rm_1 + \cdots + Rm_t$. Consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{ann}(m_1) & \to & R & \to & Rm_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & R(-1)^p & \xrightarrow{\phi} & R^q & \to & M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
 & & R(-1)^p & \to & R^{q-1} & \to & M/Rm_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{4.1}
$$

Since $\mathrm{Supp}(M) = \mathrm{Supp}(Rm_1) \cup \mathrm{Supp}(M/Rm_1)$ and M is supported in dimension zero, we see that $\dim(M/Rm_1) = 0$ and $\dim(Rm_1) = 0$. By induction hypothesis, we know that

$$\mathrm{reg}(M/Rm_1) \le q - 2. \tag{4.2}$$

On the other hand, note that $\mathrm{ann}(M) \subset \mathrm{ann}(m_1)$ and they are homogeneous ideal of R supported in dimension zero. By Corollary 2.3 and 3.3, we see that

$$\mathrm{reg}(\mathrm{ann}(m_1)) \le \mathrm{reg}(\mathrm{ann}(M)) \le q. \tag{4.3}$$

Hence $\mathrm{reg}(Rm_1) = \mathrm{reg}(\mathrm{ann}(m_1)) - 1 \le q - 1$, and so we conclude that

$$\mathrm{reg}(M) \le \max\{\mathrm{reg}(Rm_1), \mathrm{reg}(M/Rm_1)\} \le q - 1,$$

as we wished. □

As an important application of Theorem 4.1 we can prove *vector bundle technique*, which is used in [11], [13], [17] as a tool for obtaining several remarkable results.

THEOREM 4.2. *If a coherent sheaf $\mathcal{M}$ on $\mathbb{P}^n$ fits into the following exact sequence*

$$0 \to \mathcal{M} \to \bigoplus^{p} \mathcal{O}_{\mathbb{P}^n}(-d-1) \to \bigoplus^{q} \mathcal{O}_{\mathbb{P}^n}(-d) \to 0. \tag{4.4}$$

Then we have $\mathrm{reg}(\mathcal{M}) \le d + q + 1$.

Proof. Note that it suffices to prove the case of $d = 0$. Consider the following long exact sequence in sheaf cohomology from (4.4):

$$\cdots \to H^i(\mathbb{P}^n, \mathcal{M}(\ell)) \to H^i(\mathbb{P}^n, \mathcal{O}^p_{\mathbb{P}}(\ell)) \to H^i(\mathbb{P}^n, \mathcal{O}^q_{\mathbb{P}}(\ell)) \to \cdots \tag{4.5}$$

Then we have $H^{i+1}(\mathbb{P}^n, \mathcal{M}(t)) = 0$ for all $i \geq 1$ and $t \geq 0$ and the exact sequence

$$0 \to F_0 \to R(-1)^p \to R^q \to F_1 \to 0,$$

where $F_i = \bigoplus_{\ell\in\mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{M}(\ell))$ for each $i \geq 0$. Note that the graded R-module $F_1 = \bigoplus_{\ell\in\mathbb{Z}} H^1(X, \mathcal{M}(\ell))$ has a linear free presentation and it has zero dimensional support by Serre's Vanishing theorem. Hence it follows from Theorem 4.1 that

$$H^1(\mathbb{P}^n, \mathcal{M}(\ell)) = 0 \quad \text{for all } \ell \geq q-1.$$

Hence we conclude that

$$\mathrm{reg}(\mathcal{M}) = \min\{\, \ell + i \mid H^i(\mathbb{P}^n, \mathcal{M}(\ell)) = 0, i \geq 1\} \leq q,$$

which implies $\mathrm{reg}(\mathcal{M}) \leq q$ and the result follows. □

Example 4.3 (Theorem 1.1 in [13]). Let $X \subset \mathbb{P}^n$ be a smooth projective variety embedded by a very ample line bundle $\mathcal{L}$. Suppose that X is projectively normal and the defining ideal I_X has the following minimal free presentation:

$$R(-3)^{\alpha_2} \to R(-2)^{\alpha_1} \to I_X \to 0,$$

which means I_X is generated by quadric polynomials and it has only linear syzygies. Consider an embedding $Y \subset \mathbb{P}(V)$ of X given by a subsystem $V \subset H^0(X, \mathcal{L})$ of codimension t. Then $E = \oplus_{\ell\in\mathbb{Z}} H^0(X, \mathcal{O}_X(\ell))$ can be regarded as a finitely generated graded $S = \mathrm{Sym}(V)$-module and its minimal free presentation is of the form:

$$S(-2)^{\alpha} \to S \oplus S(-1)^t \to E \to 0 \tag{4.6}$$

for some $\alpha > 0$. Hence we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & S & \to & S/I_Y & \to & 0 \\
 & & \downarrow & & \downarrow\varphi & & \\
S(-2)^{\alpha} & \to & S \oplus S(-1)^t & \to & E & \to & 0 \\
\| & & \downarrow & & \downarrow & & \\
S(-2)^{\alpha} & \to & S(-1)^t & \to & \mathrm{coker}\,\varphi & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array} \tag{4.7}$$

Since the map φ is induced by an isomorphic projection $X \to Y \subset \mathbb{P}^{n-1}$ a graded S-module $\operatorname{coker}\varphi$ is supported in dimension zero. Hence we have the regularity of $\operatorname{coker}\varphi$ is at most t by Theorem 4.1 and thus

$$\operatorname{reg}(Y) = \operatorname{reg}(S/I_Y) + 1 \leq \max\{\operatorname{reg}(E) + 1, \operatorname{reg}(\operatorname{coker}\varphi) + 2\}.$$

Since X is projectively normal, we know that $\operatorname{reg}(X) = \operatorname{reg}(E) + 1$ and we conclude that

$$\operatorname{reg}(Y) \leq \max\{\operatorname{reg}(X), t + 2\}.$$

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