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HARMONIC TRANSFORMATIONS OF THE HYPERBOLIC PLANE

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Abstract. Let (H, g) denote the upper half plane in R^2 with the Riemannian metric $g := ((dx)^2 + (dy)^2)/y^2$. First of all we get a necessary and sufficient condition for a diffeomorphism ϕ of (H, g) to be a harmonic map. And, we obtain the fact that if a diffeomorphism ϕ of (H, g) is a harmonic function, then the following facts are equivalent:
(1) ϕ is a harmonic map;
(2) ϕ is an affine transformation;
(3) ϕ is an isometry (motion).

1. Introduction

Let H be the hyperbolic plane. Considering the points of H as all complex numbers with positive imaginary parts, we can view a diffeomorphism ϕ of H to be a complex-valued function defined on the upper half-plane H, and set $z' = \phi(z)$. If $\phi(z)$ is an analytic function of z, then ϕ is said to be a *holomorphic transformation* of H. Let $Hol(H)$ (resp. $Iso(H)$) be the group of all holomorphic transformations (resp. all motions (isometries)) of H. As is known in the theory of complex variables, the group $Hol(H)$ coincides with the group of all linear fractional transformations ([1, p. 182])

$$z(\in H) \longmapsto (az+b)/(cz+d) \in H, \text{where} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R).$$

On the other hand, $\theta : z \longmapsto -\bar{z}$ is a diffeomorphism of H such that $\theta \in Iso(H)$ and $\theta^2 = I$. Then, the following relation between $Iso(H)$ and $Hol(H)$ is well known.

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Theorem 1.1. *The group $Hol(H)$ is a normal subgroup of $Iso(H)$ of index 2, and $Iso(H) = Hol(H) + \theta Hol(H)$ holds.*

Let (M, g) be a Riemannian manifold, $Iso(M)$ (resp. $Aff(M)$) the group of all isometries (resp. affine transformations) of (M, g). Then, $Iso(M) \subset Aff(M)$. Moreover, if $\phi : (M, g) \longrightarrow (M, g)$ is an affine map, then ϕ is a harmonic map, but the converse dose not hold, in general.

In this paper, we obtain a necessary and sufficient condition for a diffeomorphism ϕ of the Poincaré upper plane (hyperbolic plane) H to be a harmonic map (cf. Proposition 2.2). And then, we get the following main result (cf. Theorem 2.3).

Theorem 1.2. *Let ϕ be a diffeomorphism of H. Assume ϕ is a harmonic function. Then, the following statements are equivalent*

(i) ϕ *is a harmonic map.*
(ii) ϕ *is an affine transformation.*
(iii) ϕ *is an isometry.*

Here, we have the following question (cf. Remark 2.4):
Does there exist a harmonic transformation of (H, g) which is not isometric?

2. Harmonic transformations of the hyperbolic plane

Let (M, g), (N, h) be two Riemannian manifolds. Let $\phi : M \longrightarrow N$ be a smooth map. Let $E := \phi^{-1}TN$ be the induced bundle by ϕ over M of the tangent bundle TN of N. We denote by $\Gamma(E)$, the space of all sections V of E. We denote by ∇, ${}^N\nabla$ the Levi-Civita connections of (M, g), (N, h), respectively. Then we give the induced connection $\tilde{\nabla}$ on E by

$$(2.1)\quad (\tilde{\nabla}_X V)_x := \tfrac{d}{dt}\, {}^N P_{\phi(\gamma(t))}{}^{-1} V_{\gamma(t)}|_{t=0}, \quad X \in \Gamma(TM),\ V \in \Gamma(E),$$

where $x \in M$, $\gamma(t)$ is a curve through x at $t = 0$ whose tangent vector at x is X_x, and ${}^N P_{\phi(\gamma(t))} : T_{\phi(x)}N \longrightarrow T_{\phi(\gamma(t))}N$ is the parallel displacement along a curve $\phi(\gamma(s))$, $0 \le s \le t$, given by the Levi-Civita connection ${}^N\nabla$ of (N, h).

If the tension field $\tau(\phi)$ of a smooth map ϕ of an m-dimensional Riemannian manifold (M, g) into another Riemannian manifold (N, h) vanishes on M, then ϕ is said to be a harmonic map (cf. [2, 9]) of (M, g) into (N, h), where

$$(2.2)\qquad \tau(\phi) := \textstyle\sum_{i=1}^m (\tilde{\nabla}_{e_i}\phi_* e_i - \phi_* \nabla_{e_i} e_i),$$

for $\{e_i\}_{i=1}^m$ an (locally defined) orthonormal frame on (M, g).

In general, the Riemannian connection ∇ for the Riemannian metric g on a Riemannian manifold (M, g) is given by

$$2\, g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \tag{2.3}$$
$$-g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])$$

for $X, Y, Z \in \mathfrak{X}(M)$ (cf. [3]).

Let H denote the set of all points in the (x,y)-plane with positive y coordinates, and call it the upper half plane. Let $p = (x_o, y_o) \in H$, and let the components of u, $u'(\in T_pH)$ be (ξ, η), (ξ', η'), respectively. We define a Riemannian metric g on H by letting

$$g_p(u, u') = (\xi\xi' + \eta\eta')/{y_o}^2. \tag{2.4}$$

If we consider H to be the set of all complex numbers z with positive imaginary part $I(z)$, then we can write the Riemannian metric g on H as $dz\ d\bar{z}\ /I(Z)^2$. The Riemannian manifold $(H$,$g)$ is called the hyperbolic plane (Poincaré upper plane). Here

$$\{y\ \tfrac{\partial}{\partial x} =: e_1,\ y\ \tfrac{\partial}{\partial y} =: e_2\} \tag{2.5}$$

is an (globally defined) orthonormal frame on $(H$,$g)$. From (2,3) and (2,5), we get

$$\nabla_{e_1} e_1 = e_2, \quad \nabla_{e_1} e_2 = -e_1, \quad \nabla_{e_2} e_1 = \nabla_{e_2} e_2 = 0, \tag{2.6}$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = -\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = (1\,/y)\frac{\partial}{\partial y}, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = (-1\,/y)\frac{\partial}{\partial x},$$

where ∇ is the Levi-Civita connection for the metric g on $(H$,$g)$. Let ϕ be a diffeomorphism of $(H$,$g)$ which is denoted by

$$\phi(x,y) = (u(x,y), v(x,y)) \in H, \quad ((x,y) \in H). \tag{2.7}$$

From (2.5) and (2.7), we obtain

$$\begin{cases} \phi_* e_1 = y\ (u_x \big(\frac{\partial}{\partial x}\big)_{(u,v)} + v_x \Big(\frac{\partial}{\partial y}\Big)_{(u,v)}), \\ \phi_* e_2 = y\ (u_y \big(\frac{\partial}{\partial x}\big)_{(u,v)} + v_y \Big(\frac{\partial}{\partial y}\Big)_{(u,v)}), \end{cases} \tag{2.8}$$

where ϕ_* is the differential map of ϕ, and $(u,v) := (u(x,y), v(x,y)) \in H$ $((x,y) \in H)$. And, from (2.6) and (2.8), we have

$$
(2.9) \qquad \begin{cases} \phi_* \left(\nabla_{e_2} e_1\right) = \phi_* \left(\nabla_{e_2} e_2\right) = 0, \\ \phi_* \left(\nabla_{e_1} e_1\right) = y \ (u_y \left(\frac{\partial}{\partial x}\right)_{(u,v)} + v_y \left(\frac{\partial}{\partial y}\right)_{(u,v)}), \\ \phi_* \left(\nabla_{e_1} e_2\right) = -y \ (u_x \left(\frac{\partial}{\partial x}\right)_{(u,v)} + v_x \left(\frac{\partial}{\partial y}\right)_{(u,v)}). \end{cases}
$$

By virtue of (2.1), (2.5), (2.6) and (2.8), we get

$$
(2.10) \qquad \begin{aligned} \tilde{\nabla}_{e_1} \phi_* e_1 &= y^2 (u_{xx} - \frac{2u_x v_x}{v}) (\partial/\partial x)_{(u,v)} \\ &\qquad + y^2 \{v_{xx} + \frac{(u_x{}^2 - v_x{}^2)}{v}\} (\partial/\partial y)_{(u,v)}, \\ \tilde{\nabla}_{e_1} \phi_* e_2 &= y^2 \{u_{xy} - \frac{(u_x v_y + u_y v_x)}{v}\} (\partial/\partial x)_{(u,v)} \\ &\qquad + y^2 \{v_{xy} + \frac{(u_x u_y - v_x v_y)}{v}\} (\partial/\partial y)_{(u,v)}, \\ \tilde{\nabla}_{e_2} \phi_* e_1 &= y \ \{u_x + y u_{xy} - \frac{y \left(u_x v_y + u_y v_x\right)}{v}\} (\partial/\partial x)_{(u,v)} \\ &\qquad + y \ \{v_x + y \ v_{xy} + \frac{y \left(u_x u_y - v_x v_y\right)}{v}\} (\partial/\partial y)_{(u,v)}, \\ \tilde{\nabla}_{e_2} \phi_* e_2 &= y \ \{u_y + y \ u_{yy} - \frac{2y u_y v_y}{v}\} (\partial/\partial x)_{(u,v)} \\ &\qquad + y \ \{v_y + y \ v_{yy} + \frac{y \left(u_y{}^2 - v_y{}^2\right)}{v}\} (\partial/\partial y)_{(u,v)}. \end{aligned}
$$

Thus, from (2.9) and (2.10), we get

Proposition 2.1. *Let ϕ be a diffeomorphism of (H, g). Then, ϕ is an affine transformation if and only if*

$$
(2.11) \qquad \begin{aligned} u_{xx} - \frac{2u_x v_x}{v} &= -\left(u_{yy} - \frac{2u_y v_y}{v}\right) = \frac{u_y}{y}, \\ v_{xx} + \frac{(u_x{}^2 - v_x{}^2)}{v} &= -\left(v_{yy} + \frac{u_y{}^2 - v_y{}^2}{v}\right) = \frac{v_y}{y}, \\ u_{xy} - \frac{(u_x v_y + u_y v_x)}{v} &= \frac{-u_x}{y}, \\ v_{xy} + \frac{(u_x u_y - v_x v_y)}{v} &= \frac{-v_x}{y}. \end{aligned}
$$

By the help of (2.2), (2.9) and (2.10), we get

PROPOSITION 2.2. *Let ϕ be a diffeomorphism of (H, g). Then, a necessary and sufficient condition for the map ϕ to be a harmonic map is*

$$\begin{aligned} u_{xx} + u_{yy} - \frac{2(u_x v_x + u_y v_y)}{v} &= 0 \quad \textit{and} \\ v_{xx} + v_{yy} + \frac{({u_x}^2 + {u_y}^2 - {v_x}^2 - {v_y}^2)}{v} &= 0. \end{aligned} \tag{2.12}$$

Now, assume real functions u and v in a diffeomorphism ϕ of (H, g) such that $\phi(x,\ y) = (u(x,\ y),\ v(x,\ y)) \in H$ are harmonic functions, that is,

$$\triangle_g u = \triangle_g v = 0, \tag{2.13}$$

where $\triangle_g := \sum_{i=1}^{2}({e_i}^2 - \nabla_{e_i} e_i)$ is the Laplacian operator on (H, g). A map ϕ satisfying (2.13) is said to be a *harmonic function*. Then, from (2.5), (2.6) and (2.13), we have

$$\triangle_g u = y^2(u_{xx} + u_{yy}) = 0, \quad \triangle_g v = y^2(v_{xx} + v_{yy}) = 0. \tag{2.14}$$

If $\phi : (H, g) \longrightarrow (H, g)$ is a diffeomorphism, then we get at each point $p \in H$

$$u_x v_y - u_y v_x \neq 0. \tag{2.15}$$

From (2.15) and Proposition 2.2, we get the fact that a necessary and sufficient condition for a harmonic function ϕ, which is a diffeomorphism of (H, g), to be a harmonic map is

$$u_x v_x + u_y v_y = 0 \quad \text{and} \quad {u_x}^2 + {u_y}^2 = {v_x}^2 + {v_y}^2. \tag{2.16}$$

For a diffeomorphism ϕ of (H, g), (2.16) holds if and only if

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1} = ({u_x}^2 + {u_y}^2)^{-1} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \tag{2.17}$$

holds. By the help of (2.16) and (2.17), we obtain the fact that a necessary and sufficient condition for a harmonic function ϕ, being a diffeomorphism of (H, g), to be a harmonic map is

$$u_x = \pm v_y \quad \text{and} \quad u_y = \mp v_x \quad \text{(double signs in the same order)}. \tag{2.18}$$

By virtue of Theorem 1.1 and (2.18), we obtain the following

THEOREM 2.3. *Let ϕ be a diffeomorphism of (H, g). Assume the map ϕ is a harmonic function. Then, the following statements are equivalent*:
(1) *the map ϕ is a harmonic map.*
(2) *ϕ is an affine transformation of (H, g).*
(3) *ϕ is an isometry (motion) of (H, g).*

REMARK 2.4. We would like to know whether or not there exists a diffeomorphism ϕ of (H, G) which is not isometric, but satisfies the condition (2.12) appeared in the Proposition 2.2. In other words, the question is as follows:

Does there exist a harmonic transformation of (H, g) *which is not isometric*?

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