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STUDY ON UNIVALENT HARMONIC MAPPINGS

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Abstract. In this paper, we obtain some coefficient bounds of harmonic univalent mappings on $\Delta = \{z : |z| > 1\}$ which are starlike, convex, or convex in one direction.

1. Introduction

A harmonic univalent orientation-preserving mapping f from $\Delta = \{z : |z| > 1\}$ onto a simply connected domain in the extended complex plane $\mathbb{C}^*$ containing the point at infinity, which keep infinity fixed, is of the form

$$f(z) = h(z) + \overline{g(z)} + Alog|z| \tag{1.1}$$

where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $A \in \mathbb{C}$. The orientation-preserving and univalent properties imply that the Jacobian $|f_z|^2 - |f_{\bar{z}}|^2$ is positive and so

$$|f_{\bar{z}}(z)| = \left| g'(z) + \frac{\overline{A}}{2z} \right| < |f_z(z)| = \left| h'(z) + \frac{A}{2z} \right|.$$

The set of all harmonic, orientation-preserving, univalent mappings with the form (1.1) is denoted here by Σ. The study of harmonic univalent orientation-preserving mappings defined on $\Delta = \{z : |z| > 1\}$ with $f(\infty) = \infty$ was started by Hengartner and Schober[2]. Especially they studied the subclass Σ_R of all functions in Σ that map $\Delta = \{z : |z| > 1\}$ onto the

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complement of a real line segment. They showed that functions in Σ_R can be represented, but in a nonlinear fashion, in terms of probability measures on the unit circle, and they also obtained some sharp coefficient estimates as applications.

In this paper, we consider harmonic orientation-preserving univalent mappings f in Σ which are starlike, convex, or convex in one direction. In next section, we obtain inequality

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) \leq 1 + 2Re\{b_1\}$$

when $f \in \Sigma$ is starlike, and show that this estimate is sharp for $f(z) = z - \frac{1}{\bar{z}} + 2log|z|$, which maps Δ onto $\mathbb{C}\backslash\{0\}$. In Theorem 2.5 and Theorem 2.6, we also obtain some coefficient estimates for $f \in \Sigma$ which is convex or convex in one direction by using the fact that for an analytic function $\varphi(\zeta)$ with $Re\{\varphi(\zeta)\} > 0$ and $\varphi(0) = 1$ in $|\zeta| < 1$, there exists a bounded regular function $\omega(\zeta)$, with $\omega(0) = 0$ and $|\omega(\zeta)| < 1$ in $|\zeta| < 1$, such that

$$\varphi(\zeta) = \frac{1 + \omega(\zeta)}{1 - \omega(\zeta)}.$$

2. Mappings to special domains

Definition 2.1. If for each r ($r > 1$) the image curve $f(|z| = r)$ has the property that each straight line parallel to some fixed direction cuts $f(|z| = r)$ in at most two points, we say that $f(z)$ is convex in that direction.

Definition 2.2. Let f be a mapping in Σ.

(1) f is starlike if each radial line from the origin hits the boundary $\partial f(\Delta)$ in exactly one point[3].

(2) f is convex if f maps Δ onto a domain whose complement is convex.

Theorem 2.3. *If $f \in \Sigma$ is starlike, then*

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) \leq 1 + 2Re\{b_1\}.$$

This estimate is sharp for $f(z) = z - \frac{1}{\bar{z}} + 2log|z|$, *which maps* Δ *onto* $\mathbb{C}\backslash\{0\}$.

Proof. A starlike function is characterized by the condition

$$\frac{\partial}{\partial\theta}\{argf(re^{i\theta})\} > 0$$

for $r > 1$. But $argf(re^{i\theta}) = Im\{logf(re^{i\theta})\}$, so that

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(Im\{logf(re^{i\theta})\}\right) &= Im\left\{\frac{\partial}{\partial\theta}logf(re^{i\theta})\right\}\\ &= Re\left\{\frac{zh' - \overline{zg'}}{f}\right\} > 0.\end{aligned}$$

From this, we have that

$$\left|\frac{1 - \frac{zh'-\overline{zg'}}{f}}{1 + \frac{zh'-\overline{zg'}}{f}}\right| < 1.$$

Thus

$$|f - (zh' - \overline{zg'})|^2 < |f + (zh' - \overline{zg'})|^2. \tag{2.1}$$

An integration of the left side of (2.1) gives

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}) - re^{i\theta}h'(re^{i\theta}) + \overline{re^{i\theta}g'(re^{i\theta})}|^2 \ d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} (f - re^{i\theta}h' + \overline{re^{i\theta}g'})(\bar{f} - \overline{re^{i\theta}h'} + re^{i\theta}g') \ d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} P(\theta)\overline{P(\theta)} \ d\theta\\ &= \sum_{k=1}^{\infty}(1+k)^2|a_k|^2r^{-2k} + \sum_{k=1}^{\infty}(1-k)^2|b_k|^2r^{-2k} + |A|^2(log\ r)^2,\end{aligned}$$

where

$$P(\theta) = \sum_{k=1}^{\infty}(1+k)a_k(re^{i\theta})^{-k} + \sum_{k=1}^{\infty}(1-k)\overline{b_k}(re^{-i\theta})^{-k} + Alog\ r.$$

An integration of the right side of (2.1) gives

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}) + re^{i\theta}h'(re^{i\theta}) - \overline{re^{i\theta}g'(re^{i\theta})}|^2 \ d\theta \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} (f + re^{i\theta}h' - \overline{re^{i\theta}g'})(\bar{f} + \overline{re^{i\theta}h'} - re^{i\theta}g') \ d\theta \\
&\quad = 4r^2 + 4\overline{b_1} + 4b_1 + \sum_{k=1}^{\infty}(1-k)^2|a_k|^2 r^{-2k} \\
&\qquad + \sum_{k=1}^{\infty}(1+k)^2|b_k|^2 r^{-2k} + |A|^2(log\ r)^2.
\end{aligned}$$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} |f - re^{i\theta}h' + \overline{re^{i\theta}g'}|^2 \ d\theta < \frac{1}{2\pi}\int_0^{2\pi} |f + re^{i\theta}h' - \overline{re^{i\theta}g'}|^2 \ d\theta$$

implies that

$$\begin{aligned}
&\sum_{k=1}^{\infty}(1+k)^2|a_k|^2 r^{-2k} + \sum_{k=1}^{\infty}(1-k)^2|b_k|^2 r^{-2k} + |A|^2(log\ r)^2 \\
&\qquad < 4(r^2 + \overline{b_1} + b_1) + \sum_{k=1}^{\infty}(1-k)^2|a_k|^2 r^{-2k} \\
&\qquad\qquad + \sum_{k=1}^{\infty}(1+k)^2|b_k|^2 r^{-2k} + |A|^2(log\ r)^2.
\end{aligned}$$

Simplify this, then we obtain

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) r^{-2k} < r^2 + \overline{b_1} + b_1$$

for $r > 1$. Letting $r \to 1$, we have that

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) \leq 1 + 2Re\{b_1\}.$$

□

THEOREM 2.4. ([4, Theorem 2.5.]) *If $f \in \Sigma$ with real A is convex in the direction of the real axis, then $h - g$ is conformal and convex in the direction of the real axis in Δ.*

THEOREM 2.5. *If $f \in \Sigma$ with $A = 0$ is convex and if the coefficients $a_n,\ b_n$ are all real, then*

$$-1 \le a_1 + b_1 \le 3,$$
$$|a_n + b_n| \le \frac{2\sqrt{1 + a_1 + b_1}}{n} \le \frac{4}{n} \quad for \quad n > 1.$$

Proof. Step 1; Since f is convex, $e^{-i\phi/2} f = e^{-i\phi/2} h + e^{-i\phi/2}\bar{g}$ is also convex. Thus $e^{-i\phi/2} f$ is convex in the direction of the real axis. This implies that $e^{-i\phi/2} h - e^{i\phi/2} g$ is conformal and convex in the direction of the real axis in Δ by Theorem 2.4. Hence, $h - e^{i\phi} g$ is convex in the direction $\phi/2$ and analytic function $h + g = z + \sum_{k=1}^{\infty} (a_k + b_k) z^{-k}$ is convex in the direction $\pi/2$.

Step 2; Let $G(\zeta) = h(1/\zeta) + g(1/\zeta)$ on $0 < |\zeta| < 1$. Then the function $G(\zeta) = \frac{1}{\zeta} + \sum_{k=1}^{\infty} (a_k + b_k)\zeta^k$ is regular and convex in the direction of the imaginary axis and real on the real axis. Thus, on $|\zeta| = r$ $(0 < r < 1)$,

$$Im\{\zeta G'(\zeta)\} = -\frac{\partial}{\partial\theta} Re\{G(re^{i\theta})\} \begin{cases} > 0 & \text{for } 0 < \theta < \pi \\ < 0 & \text{for } \pi < \theta < 2\pi. \end{cases}$$

Let $K(\zeta) = \zeta G'(\zeta)$ and, define

$$\varphi_\rho(\zeta) = \frac{K(\rho\zeta)}{F(\zeta)} \quad \text{for } 0 < \rho < 1,$$

where $F(\zeta) = -\frac{1}{\zeta} + \zeta = -\frac{1}{\zeta} + \sum_{k=0}^{\infty} \alpha_k \zeta^k$. Then φ_ρ is analytic in $\overline{\mathbb{D}} = \{\zeta : |\zeta| \le 1\}$ except for simple poles at $\zeta = \pm 1$ and

$$Re\{\varphi_\rho(e^{i\theta})\} = \frac{1}{2sin\theta} Im\{K(\rho e^{i\theta})\} > 0.$$

It follows from the maximum principle for harmonic functions that $Re\{\varphi_\rho(\zeta)\} > 0$ for $|\zeta| < 1$. Letting ρ tend to 1, we have that

$$Re\left\{\frac{K(\zeta)}{F(\zeta)}\right\} > 0 \quad \text{for } |\zeta| < 1.$$

From $Re\left\{\frac{K(\zeta)}{F(\zeta)}\right\} = Re\left\{\frac{\zeta G'(\zeta)}{F(\zeta)}\right\} > 0$ and $\frac{\zeta G'(\zeta)}{F(\zeta)} = 1$ at $\zeta = 0$, there exists a bounded regular function $\omega(\zeta)$, with $\omega(0) = 0$ and $|\omega(\zeta)| < 1$ in $|\zeta| < 1$, such that

$$\frac{\zeta G'(\zeta)}{F(\zeta)} = \frac{1 + \omega(\zeta)}{1 - \omega(\zeta)}.$$

This implies that

$$[\zeta F(\zeta)+\zeta^2 G'(\zeta)]\omega(\zeta)=\zeta^2 G'(\zeta)-\zeta F(\zeta).$$

Let $G(\zeta)=\frac{1}{\zeta}+\sum_{k=1}^{\infty}(a_k+b_k)\zeta^k=\frac{1}{\zeta}+\sum_{k=0}^{\infty}c_k\zeta^k$, then we have

$$[-2+\sum_{k=0}^{\infty}(kc_k+\alpha_k)\zeta^{k+1}]\omega(\zeta)=\sum_{k=0}^{\infty}(kc_k-\alpha_k)\zeta^{k+1},$$

$$\begin{aligned}[-2+\sum_{k=0}^{n-1}(kc_k+\alpha_k)\zeta^{k+1}]\omega(\zeta)&=\sum_{k=0}^{n}(kc_k-\alpha_k)\zeta^{k+1}\\&\quad-\sum_{k=n}^{\infty}(kc_k+\alpha_k)\omega(\zeta)\zeta^{k+1}+\sum_{k=n+1}^{\infty}(kc_k-\alpha_k)\zeta^{k+1}\\&=\sum_{k=0}^{n}(kc_k-\alpha_k)\zeta^{k+1}+\sum_{k=n+2}^{\infty}\beta_k\zeta^k,\end{aligned}$$

where $\sum_{k=n+2}^{\infty}\beta_k\zeta^k$ converges in $|\zeta|<1$. Let $\zeta=re^{i\theta}$ $(r<1)$. Then integrations give

$$\begin{aligned}4+\sum_{k=0}^{n-1}|kc_k+\alpha_k|^2&\geq 4+\sum_{k=0}^{n-1}|kc_k+\alpha_k|^2r^{2k+2}\\&=\frac{1}{2\pi}\int_0^{2\pi}|-2+\sum_{k=0}^{n-1}(kc_k+\alpha_k)\zeta^{k+1}|^2d\theta\\&\geq\frac{1}{2\pi}\int_0^{2\pi}|-2+\sum_{k=0}^{n-1}(kc_k+\alpha_k)\zeta^{k+1}|^2|\omega(\zeta)|^2d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}|\sum_{k=0}^{n}(kc_k-\alpha_k)\zeta^{k+1}+\sum_{k=n+2}^{\infty}\beta_k\zeta^k|^2d\theta\\&\geq\sum_{k=0}^{n}|kc_k-\alpha_k|^2r^{2k+2}.\end{aligned}$$

Letting $r\to 1$, we have that

$$4+\sum_{k=0}^{n-1}|kc_k+\alpha_k|^2\geq\sum_{k=0}^{n}|kc_k-\alpha_k|^2,$$

$$|nc_n - \alpha_n|^2 \le 4 + \sum_{k=0}^{n-1}(|kc_k + \alpha_k|^2 - |kc_k - \alpha_k|^2) \tag{2.2}$$
$$= 4 + 4\sum_{k=0}^{n-1} k \; Re\{c_k \bar{\alpha}_k\}.$$

Since $\alpha_1 = 1$ and $\alpha_n = 0$ for $n > 1$, we obtain from (2.2) with $n = 1$ that

$$|c_1 - 1|^2 \le 4, \quad -1 \le c_1 \le 3$$

and, for $n > 1$,

$$n^2|c_n|^2 \le 4 + 4c_1 \le 16, \quad |c_n| \le \frac{4}{n}.$$

From these, we finally get

$$-1 \le a_1 + b_1 \le 3,$$
$$|a_n + b_n| \le \frac{2\sqrt{1 + a_1 + b_1}}{n} \le \frac{4}{n} \quad \text{for} \quad n > 1.$$

□

THEOREM 2.6. *If $f \in \Sigma$ with real A is an odd function convex in the direction of the real axis and if the coefficients a_n, b_n are all real, then*

$$-1 \le b_1 - a_1 \le 3,$$
$$|a_n - b_n| \le \frac{2\sqrt{1 + b_1 - a_1}}{n} \le \frac{4}{n} \quad \textit{for} \quad \textit{odd } n > 1.$$

Proof. By Theorem 2.4, $h - g$ is conformal and convex in the direction of the real axis in Δ. Let $G(\zeta) = i\{h(\frac{1}{i\zeta}) - g(\frac{1}{i\zeta})\} = \frac{1}{\zeta} - (a_1 - b_1)\zeta + (a_3 - b_3)\zeta^3 - \cdots = \frac{1}{\zeta} + \sum_{k=0}^{\infty} c_k\zeta^k$ on $0 < |\zeta| < 1$. Then $G(\zeta)$ is regular and convex in the direction of the imaginary axis, in $0 < |\zeta| < 1$, and real on the real axis. By following the same process as the step 2 in the proof of Theorem 2.5, we obtain

$$-1 \le b_1 - a_1 \le 3,$$
$$|a_n - b_n| \le \frac{2\sqrt{1 + b_1 - a_1}}{n} \le \frac{4}{n} \quad \text{for} \quad \text{odd } n > 1.$$

□

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