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# THE GENERALIZED FERNIQUE'S THEOREM FOR ANALOGUE OF WIENER MEASURE SPACE

Kun Sik Ryu*

Abstract. In 1970, Fernique proved that there is a positive real number $\alpha$ such that $\int_{\mathbb{B}} \exp\{\alpha\|x\|^2\}dP(x)$ is finite where $(\mathbb{B}, P)$ is an abstract Wiener measure space and $\|\cdot\|$ is a measurable norm on $(\mathbb{B}, P)$ in [2, 3]. In this article, we investigate the existence of the integral $\int_{\mathcal{C}} \exp\{\alpha(\sup_t |x(t)|)^p\}dm_\varphi(x)$ where $(\mathcal{C}, m_\varphi)$ is the analogue of Wiener measure space and $p$ and $\alpha$ are both positive real numbers.

## 1. Preliminaries

In 1970, Skorokhod proved that there is a positive real number $p_1$ such that $\int_{\mathbb{B}} \exp\{p_1 \parallel x \parallel\}dP(x)$ is finite in [6] and at the same time, Fernique showed independently that there is a positive real number $p_2$ such that $\int_{\mathbb{B}} \exp\{p_2 \parallel x \parallel^2\}dP(x)$ is finite in [1] where $\mathbb{B}$ is an abstract Wiener space, $P$ is an abstract Wiener measure in $\mathbb{B}$ and $\parallel \cdot \parallel$ is a measurable norm on $(\mathbb{B}, P)$. These two theorems play a very important role in the theory of abstract Wiener space.

In 2002, the author and Dr. Im presented the definition and the theories of analogue of Wiener measure $m_\varphi$, which is a kind of generalization of concrete Wiener measure in [4].

In this article, we prove the existence of the integral $\int_{\mathcal{C}} \exp\{\alpha(\sup_t |x(t)|)^p\}\, dm_\varphi(x)$ where $(\mathcal{C}, m_\varphi)$ is the analogue of Wiener measure space and $p$ and $\alpha$ are both positive real numbers. Indeed, the supremum norm on $\mathcal{C}$ is a measurable norm by Lemma 2.9 in [5]. The work here is patterned to some extend on earlier work by Skorokhod in 1970, but the present setting requires a number of new concepts and results associated with analogue of Wiener measure.

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Let $\varphi$ be a Borel probability measure on $\mathbb{R}$. Let $\mathcal{C}$ be the space of all real-valued continuous functions on $[0,1]$.

For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le 1$, let $J_{\vec{t}} : \mathcal{C} \to \mathbb{R}^{n+1}$ be a function with

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n)) . \tag{1.1}$$

For Borel subsets $B_0, B_1, B_2, \cdots, B_n$ of $\mathbb{R}$, we let

$$\begin{aligned} &M_\varphi(J_{\vec{t}}^{-1}(\prod_{k=0}^{n} B_k)) \\ &= \int_{B_0} \Big[ \int_{\prod_{k=1}^{n} B_k} \frac{1}{\prod_{k=1}^{n} \sqrt{2\pi(t_k - t_{k-1})}} \exp\{-\frac{1}{2}\sum_{k=1}^{n} \frac{(u_k - u_{k-1})^2}{t_k - t_{k-1}}\} \\ &\quad d(\prod_{k=1}^{n} m_L)(u_1, u_2, \cdots, u_n)\Big] d\varphi(u_0). \end{aligned} \tag{1.2}$$

Then there is a unique probability measure $m_\varphi$ on the $\sigma$-algebra generated by $J_{\vec{t}}^{-1}(\prod_{k=0}^{n} B_k)$ type sets such that

$$M_\varphi(J_{\vec{t}}^{-1}(\prod_{k=0}^{n} B_k)) = m_\varphi(J_{\vec{t}}^{-1}(\prod_{k=0}^{n} B_k)).$$

This measure $m_\varphi$ is called the analogue of Wiener measure on $\mathcal{C}$ associated with $\varphi$.

From the change of variables formula, we have the following theorem.

THEOREM 1.1. (The Wiener integration formula for analogue of Wiener measure) *Let $0 = t_0 < t_1 < \cdots < t_n \le 1$. If $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a Borel measurable function, then the following equality holds;*

$$\begin{aligned} &\int_{\mathcal{C}} f(x(t_0), x(t_1), \cdots, x(t_n))\, dm_\varphi(x) \\ &= \frac{1}{\prod_{k=1}^{n} \sqrt{2\pi(t_k - t_{k-1})}} \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \cdots, u_n) \\ &\quad \exp\{-\frac{1}{2}\sum_{k=1}^{n} \frac{(u_k - u_{k-1})^2}{t_k - t_{k-1}}\} d(\prod_{j=1}^{n} m_L \times \varphi)((u_1, u_2, \cdots, u_n), u_0) \end{aligned} \tag{1.3}$$

*where existence of one side implies that of the other and their equality.*

From Corollary 2.9 in [5], we have the following lemma.

LEMMA 1.2.

$$m_\varphi(\{x \text{ in } \mathcal{C}| \sup_{0\le s\le 1} |x(s)-x(0)| \ge K\}) \le \frac{1}{K}\sqrt{\frac{2}{\pi}}\exp\{-\frac{K^2}{2}\} \tag{1.4}$$

*for a positive real number $K$.*

## 2. Main Theorem

In this section, we investigate the existence of the integral $\int_{\mathcal{C}} \exp\{\alpha(\sup_{0\le s\le 1} |x(s)|)^p\} dm_\varphi(x)$ for two positive real numbers $\alpha$, $p$.

THEOREM 2.1. *For $0 < p < 2$, $\int_{\mathcal{C}} \exp\{\alpha(\sup_{0\le s\le 1} |x(s)-x(0)|)^p\} dm_\varphi(x)$ is finite for all positive real number $\alpha$. If $p = 2$ then $\int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s) - x(0)|^p\} dm_\varphi(x)$ is finite for $0 < \alpha < \frac{1}{2}$.*

*Proof.* For any non-negative integer $n$, let

$$\begin{aligned} &A_n \\ &= \{x \text{ in } \mathcal{C}| n^p \le \sup_{0\le s\le 1} |x(s)-x(0)|^p < (n+1)^p\} \\ &= \{x \text{ in } \mathcal{C}| n \le \sup_{0\le s\le 1} |x(s)-x(0)| < n+1\}. \end{aligned}$$

From Lemma 1.2, we have

$$\begin{aligned} &\int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)-x(0)|^p\} dm_\varphi(x) \\ &= \sum_{n=0}^{\infty} \int_{A_n} \exp\{\alpha \sup_{0\le s\le 1} |x(s)-x(0)|^p\} dm_\varphi(x) \\ &\le \sum_{n=0}^{\infty} \exp\{\alpha(n+1)^p\} m_\varphi(\{x \text{ in } \mathcal{C}| \sup_{0\le s\le 1} |x(s)-x(0)|^p \ge n^p\}) \\ &\le \sum_{n=0}^{\infty} \frac{1}{n}\sqrt{\frac{2}{\pi}}\exp\{\alpha(n+1)^p - \frac{n^2}{2}\}. \end{aligned} \tag{2.1}$$

If $0 < p < 2$ then by the root test for series, $\sum_{n=0}^{\infty} \frac{1}{n}\sqrt{\frac{2}{\pi}}\exp\{\alpha(n+1)^p - \frac{n^2}{2}\}$ converges for all real number $\alpha$. If $p = 2$ and $0 < \alpha < \frac{1}{2}$, then by the root test for series, $\sum_{n=0}^{\infty} \frac{1}{n}\sqrt{\frac{2}{\pi}}\exp\{\alpha(n+1)^p - \frac{n^2}{2}\}$ converges. □

THEOREM 2.2. *If $0 < p < 1$ and $\int_{\mathbb{R}} \exp\{2\alpha|u|^p\} d\varphi(u)$ is finite for some positive real number $\alpha$, then $\int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\} dm_\varphi(x)$ is finite.*

*Proof.* Let $p$ be a real number with $0 < p < 1$. By Höder's inequality and the inequality $(|a| + |b|)^p \le |a|^p + |b|^p$,

$$\begin{aligned}
(2.2) \qquad & \int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\} dm_\varphi(x) \\
&\le \int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} (|x(s) - x(0)| + |x(0)|)^p\} dm_\varphi(x) \\
&\le \int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s) - x(0)|^p\} \exp\{\alpha|x(0)|^p\} dm_\varphi(x) \\
&\le \Big[\int_{\mathcal{C}} \exp\{2\alpha \sup_{0\le s\le 1} |x(s) - x(0)|^p\} dm_\varphi(x)\Big]^{\frac{1}{2}} \\
&\quad \Big[\int_{\mathbb{R}} \exp\{2\alpha|u|^p\} d\varphi(u)\Big]^{\frac{1}{2}}.
\end{aligned}$$

The last term is finite by Theorem 2.1 and assumption. □

THEOREM 2.3. *If $1 \le p < 2$ and $\int_{\mathbb{R}} \exp\{2^p\alpha|u|^p\} d\varphi(u)$ is finite, then*

$$\int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\} dm_\varphi(x)$$

*is finite.*

*Proof.* By Höder's inequality and the inequality $(|a|+|b|)^p \le 2^{p-1}(|a|^p + |b|^p)$,

$$\begin{aligned}
(2.3) \qquad & \int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\} dm_\varphi(x) \\
&\le \int_{\mathcal{C}} \exp\{\alpha( \sup_{0\le s\le 1} |x(s) - x(0)| + |x(0)|)^p\} dm_\varphi(x) \\
&\le \int_{\mathcal{C}} \exp\{\alpha 2^{p-1} \sup_{0\le s\le 1} (|x(s) - x(0)|)^p + \alpha 2^{p-1}|x(0)|^p\} dm_\varphi(x) \\
&\le \Big[\int_{\mathcal{C}} \exp\{2^p\alpha \sup_{0\le s\le 1} (|x(s) - x(0)|)^p\} dm_\varphi(x)\Big]^{\frac{1}{2}} \\
&\quad \Big[\int_{\mathbb{R}} \exp\{2^p\alpha|u|^p\} d\varphi(u)\Big]^{\frac{1}{2}}.
\end{aligned}$$

The finiteness of the last term in above come from Theorem 2.1 and assumption, as desired. □

THEOREM 2.4. *If* $\alpha < \frac{1}{2}$ *and* $\int_{\mathbb{R}} \exp\{4\alpha|u|^2\}d\varphi(u)$ *is finite then*

$$\int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^2\} dm_\varphi(x)$$

*is finite.*

*Proof.* By Hölder's inequality and the inequality $(|a|+|b|)^2 \le 2(|a|^2+|b|^2)$,

$$\begin{aligned}
(2.4)\qquad & \int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^2\} dm_\varphi(x) \\
&\le \int_{\mathcal{C}} \exp\{2\alpha \sup_{0\le s\le 1} |x(s)-x(0)|^2\} \exp\{2\alpha|x(0)|^2\} dm_\varphi(x) \\
&\le \Big[ \int_{\mathcal{C}} \exp\{4\alpha \sup_{0\le s\le 1} |x(s)-x(0)|^2\} dm_\varphi(x)\Big]^{\frac{1}{2}} \\
&\quad \Big[ \int_{\mathcal{C}} \exp\{4\alpha|u|^2\} d\varphi(u)\Big]^{\frac{1}{2}}.
\end{aligned}$$

The finiteness of the last term is justified by Theorem 2.1. □

REMARK 2.5. *If* $p > 2$ *and* $\alpha > 0$ *then by Theorem 1.1,*

$$\begin{aligned}
(2.5)\quad & \int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\} dm_\varphi(x) \\
&\ge \int_{\mathcal{C}} \exp\{\alpha|x(1)|^p\} dm_\varphi(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\{\alpha|u_1|^p - \frac{1}{2}(u_1-u_0)^2\} dm_L(u_1) d\varphi(u_0) \\
&\ge \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{|u_1|\le 1} \exp\{\alpha|u_1|^p - \frac{1}{2}(u_1-u_0)^2\} dm_L(u_1) d\varphi(u_0) \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{|u_1|\ge 1} \exp\{\alpha|u_1|^2 - \frac{1}{2}(u_1-u_0)^2\} dm_L(u_1) d\varphi(u_0) \\
&= \ +\infty.
\end{aligned}$$

REMARK 2.6. *Suppose* $\varphi = \delta_0$, *that is,* $(\mathcal{C}, m_\varphi)$ *is the concrete Wiener measure space. Then by the above theorems,* $\int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\} dm_\varphi(x)$ *is finite for* $0 < p < 2$ *and all real number* $\alpha$ *and* $\int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^2\} dm_\varphi(x)$ *is finite for* $\alpha < \frac{1}{2}$. *Moreover,* $\int_{\mathcal{C}} \exp\{\alpha \sup_{0\le s\le 1} |x(s)|^p\} dm_\varphi(x) = +\infty$ *for* $p > 2$ *and* $\alpha > 0$.

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Department of Mathematics Education
Hannam University
Daejeon 306-791, Republic of Korea
*E-mail*: ksr@hannam.ac.kr