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MULTIPLIERS OF WEIGHTED BLOCH SPACES AND BESOV SPACES

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Abstract. Let $M(X)$ be the space of all pointwise multipliers of Banach space X. We will show that, for each $\alpha > 1$, $M(\mathfrak{B}_\alpha) = M(\mathfrak{B}_{\alpha,0}) = H^\infty(B)$. We will also show that, for each $0 < \alpha < 1$, $M(\mathfrak{B}_\alpha)$ and $M(\mathfrak{B}_{\alpha,0})$ are Banach algebras. It is established that certain inclusion relationships exist between the weighted Bloch spaces and holomorphic Besov spaces.

1. Introduction

Throughout this paper, C^n will be the Cartesian product of n copies of complex plane C. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in C^n, the inner product is defined by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $\| z \|^2 = \langle z, z\rangle$.

Let B be the open unit ball in the complex space C^n. For $z \in B, \xi \in C^n$, set

$$b_B{}^2(z,\xi) = \frac{n+1}{(1-\| z \|^2)^2}[\,(1-\| z \|^2)\| \xi \|^2 + |\langle z,\xi\rangle|^2\,] .$$

If $\gamma : [0,1] \to B$ is a continuously differentiable curve, the Bergman length of γ is defined by

$$|\gamma|_B = \int_0^1 b_B(\gamma(t), \gamma'(t))dt.$$

For $z, w \in B$, we define

$$\beta(z,w) = \inf\{|\gamma|_B : \gamma(0) = z, \gamma(1) = w\}$$

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where the infimum is taken over all continuously differentiable curves from z to w. Here, β is called the Bergman metric on B.

If $f \in H(B)$, $H(B)$ is the set of holomorphic functions on B, then the quantity Qf is defined by

$$Qf(z) = \sup_{\|\xi\|=1} \frac{|\langle \nabla f(z), \xi \rangle|}{b_B(z,\xi)}, \quad z \in B, \quad \xi \in C^n$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})$ is the holomorphic gradient of f. The quantity Qf is invariant under the group $Aut(B)$ of holomorphic automorphisms of B. Namely, $Q(f \circ \varphi) = (Qf) \circ \varphi$ for all $\varphi \in Aut(B)$. A holomorphic function $f : B \to C$ is called a Bloch function if

$$\sup_{z \in B} Qf(z) < \infty \ .$$

As for as we know, the theory of Bloch functions on bounded homogeneous domains were the first studied in [4]. In [10], Timoney showed that the linear space of all holomorphic functions $f : B \to C$ with

$$\sup_{z \in B} (1 - \| z \|^2) \| \nabla f(z) \| < \infty$$

is equivalent to the space $\mathfrak{B}$ of Bloch functions on B. The little Bloch space $\mathfrak{B}_0$ is the subspace of $\mathfrak{B}$ consisting of those functions $f : B \to C$ with

$$\lim_{\|z\| \to 1} (1 - \| z \|^2) \| \nabla f(z) \| = 0.$$

For each $\alpha > 0$, the weighted Bloch space of B, denoted by $\mathfrak{B}_\alpha$, consists of all holomorphic functions $f : B \to C$ with

$$\sup_{z \in B} (1 - \| z \|^2)^\alpha \| \nabla f(z) \| < \infty \ .$$

The corresponding little Bloch space $\mathfrak{B}_{\alpha,0}$ is defined by the functions f in $\mathfrak{B}_\alpha$ such that

$$\lim_{\|z\| \to 1} (1 - \| z \|^2)^\alpha \| \nabla f(z) \| = 0.$$

Clearly, both $\mathfrak{B}_\alpha$ and $\mathfrak{B}_{\alpha,0}$ are increasing function spaces of $\alpha > 0$. In particular, $\mathfrak{B}_1 = \mathfrak{B}$ and $\mathfrak{B}_{1,0} = \mathfrak{B}_0$.

Let us define a norm on $\mathfrak{B}_\alpha$ as follows;

$$\| f \|_\alpha = |f(0)| + \sup\{(1 - \| w \|^2)^\alpha \| \nabla f(w) \| : w \in B\}.$$

It was proved in [8] that the space $\mathfrak{B}_\alpha$ is a Banach space with respect to the above norm for each $\alpha > 0$. It was also proved in [8] that the weighted little Bloch space $\mathfrak{B}_{\alpha,0}$ is the closure of the set of polynomials in the norm topology of $\mathfrak{B}_\alpha$ for each $\alpha \geq 1$.

Let X be a Banach space of all holomorphic functions on B. We say that an holomorphic function f is a pointwise multiplier of X if $fg \in X$ for all $g \in X$. Let $M(X)$ be the space of all pointwise multipliers of Banach space X and $H^\infty(B)$ be the space of all bounded holomorphic functions in B.

In [12], the authors found the very important properties associated with $M(\mathfrak{B})$ and $M(\mathfrak{B}_0)$(See Theorem 2.1). In §2, we will prove that, for each $\alpha > 1$, $M(\mathfrak{B}_\alpha) = M(\mathfrak{B}_{\alpha,0}) = H^\infty(B)$. We will also prove that, for each $0 < \alpha < 1$, $M(\mathfrak{B}_\alpha) = \mathfrak{B}_\alpha$ and $M(\mathfrak{B}_{\alpha,0}) = \mathfrak{B}_{\alpha,0}$.

Let ν be the Lebesgue measure in C^n normalized by $\nu(B) = 1$. Let p and s be given with $0 < p < \infty$ and $p, s \in R$ where R is a set of real numbers. The holomorphic Besov p-spaces $\mathbb{B}_p^s(B)$ with weight s is defined by the space of all holomorphic functions f on the unit ball B such that

$$\| f \|_{p,s} = \left\{ \int_B (Qf)^p(z)(1- \| z \|^2)^s d\lambda(z) \right\}^{\frac{1}{p}} < \infty.$$

Here $d\lambda(z) = (1- \| z \|^2)^{-n-1} d\nu(z)$ is an invariant volume measure with respect to the Bergman metric on B. In particular, if $s = 0$, the spaces $\mathbb{B}_p = \mathbb{B}_p^0$ are invariant under the action of $Aut(B)$. If $p = \infty$, then the corresponding space is the Bloch space of holomorphic functions on the unit ball B(See [6, 7]).

In §3, certain inclusion relationships between the weighted Bloch spaces and holomorphic Besov spaces are established.

2. Multipliers in $\mathfrak{B}_\alpha$ and $\mathfrak{B}_{\alpha,0}$.

THEOREM 2.1. *Suppose $f \in H(B)$, then the following conditions are equivalent.*

(1)$fBMO \subset BMO$.

(2)$f\mathfrak{B} \subset \mathfrak{B}$.

(3)$f\mathfrak{B}_0 \subset \mathfrak{B}_0$.

(4)$f \in H^\infty$ *and* $(1- \| z \|^2) \| \nabla f(z) \| \log \left(\dfrac{1}{1- \| z \|^2} \right)$ *is bounded in* B.

Proof. See Theorem 11 in [12]. □

From the Theorem 2.1, we can see that $M(\mathfrak{B})$ and $M(\mathfrak{B}_0)$ are all equivalent to the following set,

$$\{f \in H^\infty(B) : (1- \| z \|^2) \| \nabla f(z) \| \log \left(\frac{1}{1- \| z \|^2} \right) \text{ is bounded in } B\}.$$

THEOREM 2.2. *If* $f \in \mathfrak{B}_\alpha, \alpha > 0$, *then*

$$|f(z)| \leq |f(0)| + \| f \|_\alpha (1- \| z \|^2)^{-\alpha}.$$

Proof. See Lemma 1 in [8]. □

By M_ϕ, we denote the operator of multiplication by ϕ, that is, $M_\phi f = \phi f (f \in \mathfrak{B}_\alpha)$. The set of all multipliers of $\mathfrak{B}_\alpha$ will be denoted by $M(\mathfrak{B}_\alpha)$. An application of the closed graph theorem shows that M_ϕ is a bounded linear transformation on $\mathfrak{B}_\alpha$. Hence it has a finite norm $\| M_\phi \|$. Since $1 \in \mathfrak{B}_\alpha$, we have $\phi \in \mathfrak{B}_\alpha$ and so ϕ is analytic in B.

The following result shows that multipliers of $\mathfrak{B}_\alpha$ and $\mathfrak{B}_{\alpha,0}$ are bounded functions.

LEMMA 2.3. *We let* $H^\infty(B)$ *denote the space of bounded holomorphic functions in* B. *If* $f \in M(\mathfrak{B}_\alpha)$ *or* $f \in M(\mathfrak{B}_{\alpha,0})$, *then* $f \in H^\infty(B)$.

Proof. Fix a point $z \in B$. By the Theorem 2.2, the function e_z with $e_z(f) = f(z)$ for $f \in \mathfrak{B}_\alpha$ is continuous. By the Riesz representation theorem, there exists $h_z \in \mathfrak{B}_\alpha$ such that $e_z(f) = f(z) = \langle f, h_z \rangle$. This implies that

$$|f(z)g(z)| = |\langle M_f g, h_z \rangle| \leq \| M_f \| \| g \|_{\mathfrak{B}_\alpha} \| h_z \|_{\mathfrak{B}_\alpha}$$

for $g \in \mathfrak{B}_\alpha$. Taking the supremum of the above inequality, we get

$$|f(z)| \sup\{|g(z)| : \| g \|_{\mathfrak{B}_\alpha = 1}\} \leq \| M_f \| \| h_z \|_{\mathfrak{B}_\alpha}.$$

This implies that $|f(z)| \leq \| M_f \|$. □

THEOREM 2.4. *Let* $0 < \alpha \leq 2$. *Let* λ *be any real number satisfying the following properties:*

(1) $0 \leq \lambda \leq \alpha$ *if* $0 < \alpha < 1$;

(2) $0 < \lambda < 1$ *if* $\alpha = 1$;

(3) $\alpha - 1 \leq \lambda \leq 1$ *if* $1 < \alpha \leq 2$.

Then an analytic function f *on* B *is in* $\mathfrak{B}_\alpha$ *if and only if*

$$\sup_{z,w \in B, z \neq w} (1 - |z|^2)^\lambda (1 - |w|^2)^{\alpha - \lambda} \frac{|f(z) - f(w)|}{|z - w|} < \infty.$$

Proof. See Theorem 2 in [11]. □

LEMMA 2.5. *If $f \in L^1_{\mu_\alpha}(B) \cap H(B), \alpha > -1$, then*

$$f(z) = c_\alpha \int_B \frac{(1-\| w \|^2)^\alpha}{(1-\langle z,w\rangle)^{n+\alpha+1}} f(w) d\nu(w).$$

Proof. See Theorem 2 in [8]. □

THEOREM 2.6. *For $z \in B$, c is real, $t > -1$, define*

$$I_{c,t}(z) = \int_B \frac{(1-\| w \|^2)^t}{|1-\langle z,w\rangle|^{n+1+c+t}} d\nu(w), \quad z \in B.$$

Then,

(i) $I_{c,t}(z)$ is bounded in B if $c < 0$;

(ii) $I_{0,t}(z) \sim -\log(1-\| z \|^2)$ as $\| z \| \to 1^-$;

(iii) $I_{c,t}(z) \sim (1-\| z \|^2)^{-c}$ as $\| z \| \to 1^-$ if $c > 0$.

Proof. See Proposition 1.4.10 of [9]. □

THEOREM 2.7. *Suppose $1 < \alpha \leq 2$. Then f is in $\mathfrak{B}_\alpha$ if and only if f is holomorphic and $(1-\| z \|^2)^{\alpha-1}|f(z)|$ is bounded on B.*

Proof. By Theorem 2.4,

$$\sup_{z,w\in B, z\neq w} (1-|z|^2)^\lambda (1-|w|^2)^{\alpha-\lambda} \frac{|f(z)-f(w)|}{|z-w|} < M$$

for some constant M. For $\lambda = 1$ and $z = 0$,

$$\sup_{w\in B, w\neq 0} (1-|w|^2)^{\alpha-1} \frac{|f(0)-f(w)|}{|w|} < M.$$

This implies that $\sup_{w\in B_n, w\neq 0}(1-|w|^2)^{\alpha-1}|f(0)-f(w)| < M$.

Conversely, if $(1-\| z \|^2)^{\alpha-1}|f(z)| \leq M$ for some constant $M > 0$, then

$$f(z) = c_{\alpha-1} \int_B \frac{(1-\| w \|^2)^{\alpha-1}}{(1-\langle z,w\rangle)^{n+\alpha}} f(w) d\nu(w)$$

by Lemma 2.6. Differentiating under the integral sign, we obtain

$$\frac{\partial f}{\partial z_j}(z) = c_{\alpha-1}$$

$$\int_B \frac{(n+\alpha)(1-\langle z,w\rangle)^{n+\alpha-1}(-\bar{w}_j)(1-\| w \|^2)^{\alpha-1} f(w)}{(1-\langle z,w\rangle)^{2(n+\alpha)}} d\nu(w)$$

for $j = 1, 2, \cdots, n$. This shows that

$$\| \nabla f(z) \| \leq c_{\alpha-1}(n+\alpha) M \int_B \frac{1}{|1-\langle z,w\rangle|^{n+\alpha+1}} d\nu(w).$$

By Theorem 2.7, there exists a constant $C > 0$ such that

$$\| \nabla f(z) \| \leq \ CM(1 - \| z \|^2)^{-\alpha}$$

for all $z \in B$. This clearly shows that f is in $\mathfrak{B}_\alpha$. □

Recall that $H^\infty(B)$ is the space of bounded holomorphic functions in B.

THEOREM 2.8. *If* $\alpha > 1$, *then* $M(\mathfrak{B}_\alpha) = M(\mathfrak{B}_{\alpha,0}) = H^\infty(B)$.

Proof. Suppose that $f \in M(\mathfrak{B}_\alpha)$ and $g \in \mathfrak{B}_{\alpha,0}$.

$$\lim_{z \to 1^-} (1- \| z \|^2)^{\alpha-1} |f(z)g(z)|$$

$$\leq C \lim_{z \to 1^-} (1- \| z \|^2)^{\alpha-1} |g(z)|$$

for some constant C by the Lemma 2.3. Since g is in $\mathfrak{B}_{\alpha,0}$,

$$\lim_{z \to 1^-} (1- \| z \|^2)^{\alpha-1} |g(z)| = 0.$$

This shows that $f \in M(\mathfrak{B}_{\alpha,0})$.

If f is in $M(\mathfrak{B}_{\alpha,0})$, f is bounded function by the Lemma 2.3. If g is in $\mathfrak{B}_\alpha$, then $(1- \| z \|^2)^{\alpha-1}|g(z)|$ is bounded on B by Theorem 2.7. This shows that

$$\begin{aligned} &\sup\{(1- \| z \|^2)^{\alpha-1} |f(z)g(z)| : z \in B\} \\ &\leq \sup\{(1- \| z \|^2)^{\alpha-1} |g(z)| : z \in B\} \| f \|_\infty \\ &\leq C \end{aligned}$$

for some constant C. This shows that $fg \in \mathfrak{B}_\alpha$ and $f \in M(\mathfrak{B}_\alpha)$. It is clear that $H^\infty(B) \subset M(\mathfrak{B}_{\alpha,0})$. This shows that if $\alpha > 1$, then

$$M(\mathfrak{B}_\alpha) = M(\mathfrak{B}_{\alpha,0}) = H^\infty(B).$$

□

THEOREM 2.9. *Let* $0 < \alpha < 1$. *Let* f *be analytic on* B. *Then* f *is in* $\mathfrak{B}_\alpha$ *if and only if*

$$\sup_{z,w \in B, z \neq w} (1 - |z|^2)^\alpha \frac{|f(z) - f(w)|}{|z - w|} < \infty.$$

Proof. See Corollary 3 in [11]. □

THEOREM 2.10. *For* $\alpha \in (0, 1)$ *and* $f \in \mathfrak{B}_\alpha$, f *is bounded* .

Proof. By Theorem 2.9, there is a constant M such that $\frac{|f(w)-f(0)|}{|w|} < M$ for every $w \neq 0$ in B, i.e., $|f(w)| \leq M + |f(0)|$ for all $w \in B$. This shows that f is bounded on B. □

THEOREM 2.11. *If* $0 < \alpha < 1$, *then*

$$M(\mathfrak{B}_\alpha) = \mathfrak{B}_\alpha, \quad M(\mathfrak{B}_{\alpha,0}) = \mathfrak{B}_{\alpha,0}.$$

Proof. Since $f \in \mathfrak{B}_\alpha (0 < \alpha < 1)$, then f is bounded in B by Theorem 2.10. For g in $\mathfrak{B}_\alpha$,

$$\begin{aligned}
&(1- \parallel z \parallel^2)^\alpha \parallel \nabla (fg)(z) \parallel \\
&\leq (1- \parallel z \parallel^2)^\alpha \parallel \nabla f(z) \parallel |g(z)| + (1- \parallel z \parallel^2)^\alpha \parallel \nabla g(z) \parallel |f(z)| \\
&\leq M((1- \parallel z \parallel^2)^\alpha \parallel \nabla f(z) \parallel + (1- \parallel z \parallel^2)^\alpha \parallel \nabla g(z) \parallel)
\end{aligned}$$

for some constant M. Since f and g are in $\mathfrak{B}_\alpha$,

$$\sup\{(1- \parallel z \parallel^2)^\alpha \parallel \nabla (fg)(z) \parallel z \in B\} < +\infty.$$

This shows that if $f \in \mathfrak{B}_\alpha$, then $fg \in \mathfrak{B}_\alpha$ for all $g \in \mathfrak{B}_\alpha$. This shows that $M(\mathfrak{B}_\alpha) = \mathfrak{B}_\alpha$.

Suppose that $f \in \mathfrak{B}_{\alpha,0}$, If $g \in \mathfrak{B}_{\alpha,0}$, then

$$\begin{aligned}
&\lim_{z \to 1^-} (1- \parallel z \parallel^2)^\alpha \parallel \nabla f(z) \parallel = 0, \\
&\lim_{z \to 1^-} (1- \parallel z \parallel^2)^\alpha \parallel \nabla g(z) \parallel = 0.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\lim_{z \to 1^-} (1- \parallel z \parallel^2)^\alpha \parallel \nabla (fg)(z) \parallel \\
&\leq M(\lim_{z \to 1^-} (1- \parallel z \parallel^2)^\alpha \parallel \nabla f(z) \parallel + \lim_{z \to 1^-} (1- \parallel z \parallel^2)^\alpha \parallel \nabla g(z) \parallel) \\
&= 0
\end{aligned}$$

for some constant M. This shows that if $f \in \mathfrak{B}_{\alpha,0}$, then $fg \in \mathfrak{B}_{\alpha,0}$ for all $g \in \mathfrak{B}_{\alpha,0}$. This shows that $M(\mathfrak{B}_{\alpha,0}) = \mathfrak{B}_{\alpha,0}$ if $0 < \alpha < 1$.

□

3. Relationships between the spaces $\mathfrak{B}_\alpha$ and $\mathbb{B}_p^s$.

Let a be in B and P_a be the orthogonal projection of C^n onto the subspace generated by a, which is given by $P_0 = 0$, and

$$P_a z = \frac{\langle z, a \rangle}{\langle a, a \rangle} a, \quad if \quad a \neq 0.$$

Let $Q_a = I - P_a$. Define φ_a on B by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1-|a|^2} Q_a z}{1 - \langle z, a\rangle}.$$

THEOREM 3.1. *φ_a belongs to $Aut(B)$ and satisfies*
(1) $\varphi_a(0) = a, \varphi_a(a) = 0$ *and* $\varphi_a(\varphi_a(z)) = z$,
(2) $1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1-\|a\|^2)(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)(1-\langle a,w\rangle)} \quad (z, w \in \overline{B})$,
(3) $1 - \| \varphi_a(z) \|^2 = \frac{(1-\|a\|^2)(1-\|z\|^2)}{|1-\langle z,a\rangle|^2} \quad (a \in B, z \in \overline{B})$.

Proof. See Theorem 2.2.2 of [9]. □

THEOREM 3.2. *Let $1 < p < \infty$, and $f \in H(B)$. If s is a real number such that $-np < s < n$, then there is some constant $M > 0$*

$$Qf(z) \le C(1- \| z \|^2)^{\frac{-s}{p}} \| f \|_{p,s}, \quad z \in B.$$

Proof. Let $f \in H(B), \zeta \in B$, and $t \in [0,1]$. By the mean value theorem,

$$f(t\zeta) = \int_B f \circ \varphi_{t\zeta}(w) d\nu(w),$$

and

$$\langle \nabla f(0), \zeta\rangle = \left[\frac{d}{dt} f(t\zeta)\right]_{t=0} = \int_B \nabla f(-w) \left[\frac{d}{dt}\varphi_{t\zeta}(w)\right]_{t=0} d\nu(w).$$

A formal computation shows that

$$a_\zeta(w) = \left[\frac{d}{dt}\varphi_{t\zeta}(w)\right]_{t=0} = \zeta - \langle w, \zeta\rangle w$$

and a_ζ is a bounded holomorphic mapping from B into C^n that satisfies $a_\zeta(0) = \zeta$. Therefore,

$$\langle \nabla f(0), \zeta\rangle = \int_B \langle \nabla f(-w), a_\zeta(w)\rangle \frac{b_B(-w, a_\zeta(w))}{b_B(-w, a_\zeta(w))} d\nu(w),$$

which implies

$$|\langle \nabla f(0), \zeta\rangle| \le M \int_B \frac{Qf(w)}{1 - \| w \|^2} d\nu(w)$$

for some constant M. The latter inequality follows from the following inequalities for the Bergman metric:

$$\frac{\| \xi \|}{\sqrt{1- \| w \|^2}} \le b_B(w, \xi) \le \frac{\| \xi \|}{1- \| w \|^2}, \ w \in B, \ \xi \in C^n.$$

Thus,

$$Qf(0) \leq C \int_B Qf(w)(1 - \parallel w \parallel^2)^n d\lambda(w).$$

Note that Qf is invariant, i.e., $Q(f \circ \varphi) = (Qf) \circ \varphi$ for all $\varphi \in Aut(B)$. Replacing f by $f \circ \varphi_z$ in the above equation, we obtain

$$Qf(z) = Q(f \circ \varphi_z)(0) \leq C \int_B Q(f \circ \varphi_z)(w)(1 - \parallel w \parallel^2)^n d\lambda(w).$$

By the change of variables : $\zeta = \varphi_z(w)$ or $w = \varphi_z(\zeta)$, and by Theorem 3.1,

$$Qf(z) \leq C(1 - \parallel z \parallel^2)^n \int_B Qf(\zeta) \frac{(1 - \parallel \zeta \parallel^2)^{n-s}}{|1 - < z, \zeta > |^{2n}} (1 - \parallel \zeta \parallel^2)^s d\lambda(\zeta).$$

By the Hölder inequality,

$$Qf(z) \leq C(1 - \parallel z \parallel^2)^n \parallel f \parallel_{p,s} \left[\int_B \frac{(1 - \parallel \zeta \parallel^2)^{q(n-s)}}{|1 - < z, \zeta > |^{2nq}} (1 - \parallel \zeta \parallel^2)^s d\lambda(\zeta) \right]^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By Theorem 2.6, if $-np < s < n$, then

$$Qf(z) \leq C(1 - \parallel z \parallel^2)^{\frac{-s}{p}} \parallel f \parallel_{p,s} .$$

□

THEOREM 3.3. *Let $1 < p < \infty$ and $-np < s < n$. For $\alpha \geq 1 + \frac{s}{p}$, we have*

$$\mathbb{B}_p^s \subseteq \mathfrak{B}_\alpha.$$

Proof. Since $Qf(z)$ and $(1 - \parallel z \parallel^2) \parallel \nabla f(z) \parallel$ behave equivalently as $\parallel z \parallel \to 1$ on B(See [10]), we may replace $Qf(z)$ in Theorem 3.2 by $(1 - \parallel z \parallel^2) \parallel \nabla f(z) \parallel$ with a different constant C and we have

$$(1 - \parallel z \parallel^2)^\alpha \parallel \nabla f(z) \parallel \leq C(1 - \parallel z \parallel^2)^{\alpha - 1 - \frac{s}{p}} \parallel f \parallel_{p,s} .$$

By choosing $\alpha = 1 + \frac{s}{p}$, we get

$$\parallel f \parallel_\alpha - |f(0)| \leq C \parallel f \parallel_{p,s} .$$

But, since the norm $\parallel \ \parallel_\alpha$ is non-increasing with $\alpha > 0$, we obtain the desired result. □

THEOREM 3.4. *Let $0 < p < \infty$ and $s \in R$. For $\alpha < 1 + \frac{s-n}{p}$,*

$$\mathfrak{B}_\alpha \subseteq \mathbb{B}_p^s.$$

Proof. From the fact that $Qf(z)$ and $(1-\parallel z\parallel^2)\parallel\nabla f(z)\parallel$ behave the same within constants as $\parallel z\parallel\to 1$, we may replace $Qf(z)$ by $(1-\parallel z\parallel^2)\parallel\nabla f(z)\parallel$ in the definition of $\parallel f\parallel_{p,s}$. Namely,

$$\begin{aligned}\parallel f\parallel_{p,s}^p &= \int_B (Qf)^p(z)(1-\parallel z\parallel^2)^s d\lambda(z)\\ &\le C\int_B \left[(1-\parallel z\parallel^2)\parallel\nabla f(z)\parallel\right]^p(1-\parallel z\parallel^2)^s d\lambda(z)\\ &\le C\int_B \left[\frac{(1-\parallel z\parallel^2)^\alpha\parallel\nabla f(z)\parallel}{(1-\parallel z\parallel^2)^{\alpha-1}}\right]^p(1-\parallel z\parallel^2)^s d\lambda(z)\\ &\le C\parallel f\parallel_\alpha^p\int_B(1-\parallel z\parallel^2)^{-p\alpha+p+s-n-1}d\nu(z).\end{aligned}$$

By Theorem 2.7, if $\alpha<1+\frac{s-n}{p}$, then

$$\parallel f\parallel_{p,s}\le C\parallel f\parallel_\alpha$$

which yields the desired result. □

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