JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **22**, No. 4, December 2009

PARTIAL FRACTION DECOMPOSITION FROM A LINEAR-ALGEBRAIC VIEWPOINT

Jeong Keun Lee*, Jun Soo Choa**, Min Shik Cho***, and Dong Hwan Han****

Abstract. We show that to every real polynomial of degree n, there corresponds a certain basis for the space of polynomials of degree less than or equal to $(n-1)$. As an application, we give a new proof for the existence and uniqueness of the partial fraction decomposition of a rational function.

On my website, I happened to see a post asking a clear explanation: Why should we begin with the identity of the form

$$\frac{x-1}{x^2(x+1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1} (\ a, b, c \text{ real})$$

when we integrate $\int \frac{x-1}{x^2(x+1)} dx$? Apparently this question seemed easy to answer but it is not as simple as we expect.

In classroom, we are encountered with the problem of decomposing a rational function into the partial fractions. For example, this occurs when we integrate a rational function or solve by Laplace transform an initial value problem for an ordinary differential equation with constant coefficients. Most college textbooks [3, 6] emphasize on the actual computation of coefficients in the partial fraction decomposition and it seems difficult to find an explanation about the basic structure behind it.

The partial fraction decomposition of a rational function has been dealt with in several papers [1, 2, 5]. The conventional approaches found in references are essentially based on the Euclid division algorithm. Our

---

Received August 08, 2009; Accepted November 06, 2009.
2000 Mathematics Subject Classification: Primary 15A03.
Key words and phrases: partial fraction decomposition, basis, linear combination, rational functions.
Correspondence should be addressed to Jeong Keun Lee, jklee@sunmoon.ac.kr
*The author appreciates SUN MOON UNIVERSITY for the financial support during his sabbatical year 2007 September-2008 August.
***He was partially supported by Korea National University of Education in the program 2009.

approach uses an elementary theory of a finite dimensional vector space to establish the existence and the uniqueness of the partial fraction decomposition for a rational function. The key steps in our result consist in constructing a particular set of polynomials from the denominator of a rational function, which becomes a basis for a certain vector space of polynomials, and finding a linear combination of the numerator in a rational function with respect to this basis. This particular basis is completely determined by the factorization of the denominator of a rational function.

We begin by mentioning the well-known fact that every polynomial with real coefficients can be factorized into the product of linear and quadratic polynomials. It is an immediate consequence of the famous 'Fundamental theorem of algebra' which states that every polynomial of degree greater than or equal to 1 has at least one complex zero.

THEOREM 1 (Construction of a basis). *Let n denote the degree of a polynomial $q(x)$ and $\mathcal{P}_{n-1}$ be the vector space of polynomials of degree less than or equal to $n-1$.*

(a) *If $a_i (1 \le i \le n)$ are distinct real numbers and $q(x) = \prod_{i=1}^{n}(x-a_i)$, then $\{q(x)/(x-a_i) | 1 \le i \le n\}$ is a basis for $\mathcal{P}_{n-1}$.*

(b) *If $q(x) = (x-a)^n$ for some real number a, then $\left\{ \frac{q(x)}{(x-a)^j} | 1 \le j \le n \right\}$ is a basis for $\mathcal{P}_{n-1}$.*

(c) *If $q(x) = \left((x-\xi)^2+\eta^2\right)^m$ for real numbers ξ and $\eta \neq 0$ and $2m = n$, then $\left\{ \frac{q(x)}{((x-\xi)^2+\eta^2)^j} \right\}_{j=1}^{m}$ and $\left\{ \frac{(x-\xi)q(x)}{((x-\xi)^2+\eta^2)^j} \right\}_{j=1}^{m}$ form a basis for $\mathcal{P}_{n-1}$.*

(d) *If $q(x) = \prod_{i=1}^{k}(x-a_i)^{n_i} \prod_{i=1}^{\ell}\left((x-\xi_i)^2+\eta_i^2\right)^{m_i}$ for distinct real numbers a_i's and distinct complex numbers $\xi_i + \eta_i\sqrt{-1} (\eta_i \neq 0)$ and $\sum_{i=1}^{k} n_i + 2\sum_{i=1}^{\ell} m_i = n$, then $\left\{ \frac{q(x)}{(x-a_i)^j} \right\}_{i=1,j=1}^{k,\quad n_i}$, $\left\{ \frac{q(x)}{\left((x-\xi_i)^2+\eta_i^2\right)^j} \right\}_{i=1,j=1}^{\ell,\quad m_i}$ and $\left\{ \frac{(x-\xi_i)q(x)}{\left((x-\xi_i)^2+\eta_i^2\right)^j} \right\}_{i=1,j=1}^{\ell,\quad m_i}$ form a basis for $\mathcal{P}_{n-1}$.*

Proof. (a) It suffices to show that

$$\sum_{i=1}^{n} c_i \frac{q(x)}{x-a_i} = 0 \tag{1}$$

implies $c_i = 0$ for each integer $1 \le i \le n$. Putting $x = a_j$ in (1), we can easily see that $c_j = 0$ for each integer j since a_j's are distinct and each polynomial $q(x)/(x - a_i)$ vanishes at $x = a_j$ for all $i \ne j$. Thus (a) is proved.

(b) We prove that

$$\sum_{j=1}^{n} c_j \frac{q(x)}{(x-a)^j} = 0 \tag{2}$$

implies $c_j = 0$ for each integer $1 \le j \le n$. But this is easily seen by noting that (2) is rewritten as

$$\sum_{j=1}^{n} c_j (x-a)^{n-j} = 0.$$

(c) We need to show that

$$\sum_{j=1}^{m} [c_j + (x-\xi)d_j] \left((x-\xi)^2 + \eta^2\right)^{m-j} = 0 \tag{3}$$

implies $c_j = d_j = 0$ for each integer $1 \le j \le m$. Substituting $x = \xi + \eta\sqrt{-1}$ in (3), we know that

$$c_m + \sqrt{-1}\eta d_m = 0,$$

from which it follows that $c_m = d_m = 0$. Dividing the both side of (3) with $c_m = d_m = 0$ by $(x-\xi)^2 + \eta^2$ and substituting $x = \xi + \eta\sqrt{-1}$, we again see that $c_{m-1} = d_{m-1} = 0$. Repeating this process, we see that $c_j = d_j = 0$ for each integer $1 \le j \le m$ and (c) is proved.

(d) Suppose that

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{i,j} \frac{q(x)}{(x-a_i)^j} + \sum_{i=1}^{\ell}\sum_{j=1}^{m_i} \frac{[c_{i,j} + (x-\xi_i)d_{i,j}]q(x)}{\left((x-\xi_i)^2 + \eta_i^2\right)^j} = 0. \tag{4}$$

Applying the argument used in proving (b), we can see that $a_{i,j} = 0$ for each integer $1 \le j \le n_i, 1 \le i \le k$. If we apply the argument used in proving (c) to (4), together with $a_{i,j} = 0$, we see that $c_{i,j} = d_{i,j} = 0$ for each integer $1 \le j \le m_i, 1 \le i \le \ell$. Thus (d) is proved. □

We focus on the problem of expressing a rational function as the partial fraction decomposition.

Let $\{v_i(x)\}_{i=1}^{n}$ be a basis for $\mathcal{P}_{n-1}$ constructed in Theorem 1. The elementary theory of a finite dimensional vector space tells us that for any $p(x) \in \mathcal{P}_{n-1}$, there exists a unique set $\{c_i\}_{i=1}^{n}$of real numbers such that

$$p(x) = c_1 v_1(x) + c_2 v_2(x) + \cdots + c_n v_n(x). \tag{5}$$

Now we obtain the partial fraction decomposition of a rational function $p(x)/q(x)$ if we divide the both sides of (5) by $q(x)$.

More precisely, we have the following result.

THEOREM 2 (Existence and uniqueness of partial fraction decomposition). *Consider a rational function of the form*

$$r(x) := \frac{p(x)}{q(x)}, \quad \deg q(x) = n, \; p(x) \in \mathcal{P}_{n-1}.$$

If $q(x)$ *is factorized into the following form*

$$q(x) = \prod_{i=1}^{k} (x - a_i)^{n_i} \prod_{i=1}^{\ell} \left((x - \xi_i)^2 + \eta_i^2\right)^{m_i}$$

for distinct real numbers a_i'*s and distinct complex numbers* $\xi_i + \eta_i\sqrt{-1}$ $(\eta_i \neq 0)$*, then there exist unique constants* $a_{i,j}$ $(1 \leq j \leq n_i, 1 \leq i \leq k)$, $c_{i,j}$ *and* $d_{i,j}$ $(1 \leq j \leq m_i, 1 \leq i \leq \ell)$ *such that*

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{a_{i,j}}{(x - a_i)^j} + \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{c_{i,j} + (x - \xi_i) d_{i,j}}{\left((x - \xi_i)^2 + \eta_i^2\right)^j}. \tag{6}$$

Proof. By Theorem 1 (d), there exist unique real constants $a_{i,j}$ $(1 \leq j \leq n_i, 1 \leq i \leq k)$, $c_{i,j}$ and $d_{i,j}$ $(1 \leq j \leq m_i, 1 \leq i \leq \ell)$ such that

$$p(x) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{i,j} \frac{q(x)}{(x - a_i)^j} + \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{[c_{i,j} + (x - \xi_i) d_{i,j}] q(x)}{\left((x - \xi_i)^2 + \eta_i^2\right)^j} \tag{7}$$

since $p(x) \in \mathcal{P}_{n-1}$. Dividing both sides of (7) by $q(x)$, we obtain (6) and the proof is completed. □

If $(x - a)^m | q(x)$ but $(x - a)^{m+1} \nmid q(x)$, then to a factor $(x - a)^m$ of $q(x)$ there corresponds a fractional function of the form

$$\sum_{j=1}^{m} \frac{c_j}{(x - a)^j}. \tag{8}$$

Similarly, if $\left((x - \xi)^2 + \eta^2\right)^m | q(x)$ but $((x-\xi)^2+\eta^2)^{m+1} \nmid q(x)$ $(\eta \neq 0)$, then to the factor $\left((x - \xi)^2 + \eta^2\right)^m$ there corresponds the partial fraction of the form

$$\sum_{j=1}^{m} \frac{c_j + (x-\xi)d_j}{((x-\xi)^2+\eta^2)^j}. \tag{9}$$

(8) (respectively, (9)) is sometimes called a partial fraction corresponding to a factor $(x-a)^m$ (respectively, $\left((x-\xi)^2+\eta^2\right)^m$). Thus we have decomposed a rational function into the sum of partial fractions corresponding to each factor of the denominator of a rational function.

Finally we proceed to determine the coefficients in the partial fraction decomposition (6). Although it is not the main topic of this article, we believe it is worthwhile to include it for the self-containedness of this discussion.

Theorem 3 (Coefficients in partial fraction decomposition). *Let a polynomial $q(x)$ be the denominator of a rational function $r(x)$.*

(a) *If $(x-a)^m | q(x)$ but $(x-a)^{m+1} \nmid q(x)$, then the coefficients c_j's in the partial fraction (8) are given by*

$$c_m = r(x)(x-a)^m\big|_{x=a},$$
$$c_j = \left(r(x) - \sum_{i=j+1}^{m} \frac{c_i}{(x-a)^i} \right)(x-a)^j \Bigg|_{x=a} \qquad (j = m-1, m-2, \cdots, 1).$$

(b) *If $\left((x-\xi)^2+\eta^2\right)^m | q(x)$ but $((x-\xi)^2+\eta^2)^{m+1} \nmid q(x)$ $(\eta \neq 0)$, then the coefficients c_j's and d_j's in the partial fraction (9) are given by*

$$c_m + d_m\eta\sqrt{-1} = r(x)[(x-\xi)^2+\eta^2]^m\big|_{x=\xi+\eta\sqrt{-1}},$$
$$c_j + d_j\eta\sqrt{-1} = \left(r(x) - \sum_{i=j+1}^{m} \frac{c_i+(x-\xi)d_i}{[(x-\xi)^2+\eta^2]^i} \right)[(x-\xi)^2+\eta^2]^j \Bigg|_{x=\xi+\eta\sqrt{-1}}$$
$$(j = m-1, m-2, \cdots, 1).$$

Proof. (a) We write $r(x)$ as the form

$$r(x) = \sum_{j=1}^{m} \frac{c_j}{(x-a)^j} + R(x), \tag{10}$$

where $R(x)$ is the remainder term not containing $x-a$ in the partial fraction decomposition (6) of $r(x)$. Multiplying the both sides of (10) by $(x-a)^m$ and putting $x=a$, we see that

$$c_m = r(x)(x-a)^m\big|_{x=a}.$$

Now we rewrite (10) as

$$r(x) - \frac{c_m}{(x-a)^m} = \sum_{j=1}^{m-1} \frac{c_j}{(x-a)^j} + R(x). \tag{11}$$

If we put $x = a$ after multiplying the both sides of the expression (11) by $(x-a)^{m-1}$, then we see that

$$c_{m-1} = \left(r(x) - \frac{c_m}{(x-a)^m} \right) (x-a)^{m-1} \Big|_{x=a}.$$

Repeating this process completes the proof of (a).

(b) As in (a), we write $r(x)$ as the form

$$r(x) = \sum_{j=1}^{m} \frac{c_j + (x-\xi)d_j}{[(x-\xi)^2+\eta^2]^j} + \tilde{R}(x), \tag{12}$$

where $\tilde{R}(x)$ is the remainder term not containing $(x-\xi)^2+\eta^2$ in the partial fraction decomposition (6) of $r(x)$. Multiplying the both sides of (12) by $[(x-\xi)^2+\eta^2]^m$ and putting $x = \xi + \eta\sqrt{-1}$, we obtain

$$c_m + d_m \eta \sqrt{-1} = r(x)[(x-\xi)^2+\eta^2]^m \big|_{x=\xi+\eta\sqrt{-1}},$$

from which c_m and d_m follow. Now we rewrite (12) as

$$r(x) - \frac{c_m + (x-\xi)d_m}{[(x-\xi)^2+\eta^2]^m} = \sum_{j=1}^{m-1} \frac{c_j + (x-\xi)d_j}{[(x-\xi)^2+\eta^2]^j} + \tilde{R}(x). \tag{13}$$

If we put $x = \xi + \eta\sqrt{-1}$ after multiplying the both sides of (13) by $[(x-\xi)^2+\eta^2]^{m-1}$, then we obtain

$$\begin{aligned} c_{m-1} &+ d_{m-1}\eta\sqrt{-1} \\ &= \left(r(x) - \frac{c_m + (x-\xi)d_m}{[(x-\xi)^2+\eta^2]^m} \right) [(x-\xi)^2+\eta^2]^{m-1} \Big|_{x=\xi+\eta\sqrt{-1}}, \end{aligned}$$

from which we can obtain c_{m-1} and d_{m-1}. Repeating this process, we can show that for $j = m-1, m-2, \cdots, 1$

$$\begin{aligned} c_j &+ d_j\eta\sqrt{-1} \\ &= \left(r(x) - \sum_{i=j+1}^{m} \frac{c_i + (x-\xi)d_i}{[(x-\xi)^2+\eta^2]^i} \right) [(x-\xi)^2+\eta^2]^j \Bigg|_{x=\xi+\eta\sqrt{-1}}. \end{aligned}$$

Thus (b) is proved. □

Determining the coefficients in the partial fraction decomposition may require a lot of complicated computations and time but is one of the most important parts in applications such as integrating a rational function. See [4, 7] for another algorithms or computational skills.

In the below, we illustrate our results through two explicit examples.

EXAMPLE 1. *Find the partial fraction decomposition of the following rational function*

$$r_1(x) = \frac{2x+1}{x^2(x+1)}.$$

<u>Solution</u> By Theorem 1, we can see that $q(x)/x$, $q(x)/x^2$ and $q(x)/(x+1)$ is a basis for $\mathcal{P}_2$, where $q(x) = x^2(x+1)$. Thus there are unique constants c_1, c_2 and d_1 such that

$$2x + 1 = c_1 \frac{q(x)}{x} + c_2 \frac{q(x)}{x^2} + d_1 \frac{q(x)}{x+1} \tag{14}$$

By dividing the both sides of (14) by $q(x)$, we have the following partial fraction decomposition for $r_1(x)$

$$\frac{2x+1}{x^2(x+1)} = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{d_1}{x+1},$$

which is the sum of partial fractions corresponding to each factor of the denominator $q(x)$. The coefficients c_1, c_2 and d_1 are given by the followings:

$$\begin{aligned} c_2 &= r_1(x)x^2\big|_{x=0} = \left.\frac{2x+1}{x+1}\right|_{x=0} = 1, \\ c_1 &= \left(r_1(x) - \frac{c_2}{x^2}\right)\Big|_{x=0} = \left.\frac{1}{x+1}\right|_{x=0} = 1, \\ d_1 &= r_1(x)(x+1)\big|_{x=-1} = \left.\frac{2x+1}{x^2}\right|_{x=-1} = -1. \end{aligned}$$

□

EXAMPLE 2. *Find the partial fraction decomposition of the following rational function*

$$r_2(x) = \frac{2x-3}{x(x^2+1)}.$$

<u>Solution</u> Let $q(x) = x(x^2+1)$. Since $q(x)/x$, $q(x)/(x^2+1)$ and $xq(x)/(x^2+1)$ is a basis for $\mathcal{P}_2$ by Theorem 1, there are unique constants a_1, c_1 and d_1 such that

$$2x + 1 = a_1 \frac{q(x)}{x} + c_1 \frac{q(x)}{x^2+1} + d_1 \frac{xq(x)}{x^2+1} \tag{15}$$

By dividing the both sides of (15) by $q(x)$, we have the following partial fraction decomposition for $r_2(x)$

$$\frac{2x-3}{x(x^2+1)} = \frac{a_1}{x} + \frac{c_1}{x^2+1} + \frac{d_1 x}{x^2+1}.$$

Applying Theorem 3, we have

$$a_1 = r_2(x)x|_{x=0} = \left.\frac{2x-3}{x^2+1}\right|_{x=0} = -3,$$

$$c_1 + d_1\sqrt{-1} = r_2(x)(x^2+1)\big|_{x=\sqrt{-1}} = \left.\frac{2x-3}{x}\right|_{x=\sqrt{-1}} = 2 + 3\sqrt{-1}.$$

Thus $a_1 = -3$, $c_1 = 2$ and $d_1 = 3$. □

References

[1] A. W. Boldyreff, *Decomposition of rational fractions into partial fractions*, Mathematics Magazine **24** (1951), no. 3, 139-146.

[2] H. J. Hamilton, *The partial fraction decomposition of a rational function*, Mathematics Magazine **45** (1972), no. 3, 117-119.

[3] A. Jeffrey, Advanced engineering mathematics, Harcourt/Academic Press, 2002.

[4] P. T. Joshi, *Efficient techniques for partial fractions*, The Two-year college Mathematics Journal **14** (1983), no. 2, 110-118.

[5] P. Schultz, *An algebraic approach to partial fractions*, The Two-Year College Mathematics Journal **14** (1983), no. 4, 346-348.

[6] G. B. Thomas and R. L. Finney, Calculus and Analytic Geometry, 9th ed. , Addison-Wesley, 1996.

[7] Joseph Wiener, *An algebraic approach to partial fractions*, The College Mathematics Journal **17** (1986), no. 1, 71-72.

*
Department of mathematics
Sun Moon University
Asan 336-708, Republic of Korea
E-mail: jklee@sunmoon.ac.kr

**
Department of mathematics education
Sungkyunkwan University,
Seoul 110-745, Republic of Korea
E-mail: jschoa@skku.ac.kr

***
Department of mathematics education
Korean National University of Education
Cheongwon 363-791, Republic of Korea,
E-mail: mscho@knue.ac.kr

****
Department of mathematics
Sun Moon University
Asan 336-708, Republic of Korea
E-mail: dhhan@sunmoon.ac.kr