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ON DOUGLAS SPACE WITH AN APPROXIMATE INFINITE SERIES (α, β)–METRIC

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Abstract. We deal with a Finsler space F^n with an approximate infinite series (α, β)-metric $L(\alpha, \beta) = \beta \sum_{k=0}^{r} \left(\frac{\alpha}{\beta}\right)^k$, where $\alpha < \beta$. We introduced a Finsler space F^n with an infinite series (α, β)-metric $L(\alpha, \beta) = \dfrac{\beta^2}{\beta - \alpha}$ and investigated various geometrical properties at [6]. The purpose of the present paper is devoted to finding the condition for a Finsler space F^n with an approximate infinite series (α, β)-metric above to be a Douglas space.

1. Introduction

A Finsler metric $L(\alpha, \beta)$ in a differentiable manifold M^n is called an (α, β)-metric, if L is a positively homogeneous function of degree one of a Riemannian metric $\alpha = (a_{ij}(x)y^i y^j)^{1/2}$ and a one-form $\beta = b_i(x)y^i$ on M^n. The interesting and important examples of an (α, β)-metric are Randers metric $\alpha + \beta$, Kropina metric α^2/β and Matsumoto metric $\alpha^2/(\alpha - \beta)$. The notion of an (α, β)-metric was introduced by M. Matsumoto (cf. [10]) and has been studied by many authors.

A Finsler space is called a Berwald space if the Berwald connection is linear. Berwald spaces are specially interesting and important, because the connection is linear, and many examples of Berwald spaces have been known.

The notion of a Douglas space was introduced by S. Bácsó and M. Matsumoto [2] as a generalization of a Berwald space from the viewpoint of

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geodesic equations. It is remarkable that a Finsler space is a Douglas space, if and only if the Douglas tensor vanishes identically.

The present paper is devoted to finding the condition that the Finsler space F^n with the approximate infinite series (α,β)-metric $L=\beta\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k$ be a Douglas space (Theorem 3.1, 3.2).

2. Preliminaries

Let us consider the *r-th series* (α,β)*-metric*

$$L(\alpha,\beta)=\beta\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k, \tag{2.1}$$

where we assume $\alpha<\beta$.

Then the metric above(cf. [12]) is called an *approximate infinite series* (α,β)*-metric* or the *rth approximate infinite series* (α,β)*-metric.*

If $r=1$, then $L=\alpha+\beta$ is a Randers metric. The conditions that the Randers space be a Berwald space and a Douglas space are found in [11], respectively. If $r=2$, then $L=\alpha+\beta+\dfrac{\alpha^2}{\beta}$ is treated in [9] as an (α,β)-metric that a locally Minkowski space is flat-parallel. If $r=\infty$, then this metric (2.1) is expressed as the form

$$L(\alpha,\beta)=\lim_{r\to\infty}\beta\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k=\frac{\beta^2}{\beta-\alpha}. \tag{2.2}$$

Then the metric above ([6]) is called an *infinite series* (α,β)*-metric.* We have not at all investigated the geometrical meaning on the metric above by this time. It is remarkable that this metric (2.2) is the difference between a Randers metric and a Matsumoto metric.

On the other hand, the geodesics of a Finsler space $F^n=(M^n,L)$ are given by the system of differential equations including the function

$$4G^i(x,y)=g^{ij}(y^r\dot{\partial}_j\partial_r L^2-\partial_j L^2).$$

For an (α, β)-metric $L(\alpha, \beta)$ the space $R^n = (M^n, \alpha)$ is called the *associated Riemannian space* with $F^n = (M^n, L(\alpha, \beta))$ ([1], [7]). The covariant differentiation with respect to the Levi-Civita connection $\gamma_j{}^i{}_k(x)$ of R^n is denoted by (;). We put $(a^{ij}) = (a_{ij})^{-1}$, and use the symbols as follows:

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \;\; s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \;\; r^i{}_j = a^{ir} r_{rj}, \;\; s^i{}_j = a^{ir} s_{rj},$$
$$r_j = b_r r^r{}_j, s_j = b_r s^r{}_j, \;\; b^i = a^{ir} b_r, \;\; b^2 = a^{rs} b_r b_s.$$

According to [8], if $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, where $\gamma^2 = b^2\alpha^2 - \beta^2$, then the function $G^i(x, y)$ of F^n with an (α, β)-metric is written in the form

$$2G^i = \gamma_0{}^i{}_0 + 2B^i,$$
$$B^i = \frac{\alpha L_\beta}{L_\alpha} s^i{}_0 + C^* \left\{ \frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right) \right\}, \tag{2.3}$$

where $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$ and $L_{\alpha\alpha} = \partial^2 L/\partial\alpha\partial\alpha$, the subscript 0 means the contraction by y^i and we put

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2s_0\alpha L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}. \tag{2.4}$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity. For example, $\gamma_0{}^i{}_0$ is $hp(2)$.

From the former of (2.3) the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$ of F^n with an (α, β)-metric is given by

$$G^i{}_j = \dot\partial_j G^i = \gamma_0{}^i{}_j + B^i{}_j,$$
$$G_j{}^i{}_k = \dot\partial_k G^i{}_j = \gamma_j{}^i{}_k + B_j{}^i{}_k,$$

where we put $B^i{}_j = \dot\partial_j B^i$ and $B_j{}^i{}_k = \dot\partial_k B^i{}_j$. $B^i(x, y)$ is called the *difference vector* ([8]). On account of [8], $B_j{}^i{}_k$ is determined by

$$L_\alpha B_j{}^t{}_i y^j y_t + \alpha L_\beta (B_j{}^t{}_i b_t - b_{j;i}) y^j = 0, \tag{2.5}$$

where $y_k = a_{ik} y^i$. Consequently we have

Theorem 2.1. ([5]) *The Finsler space F^n $(n > 2)$ with an approximate infinite series (α, β)-metric (2.1) is a Berwald space if and only if $b_{j:i} = 0$, and then the Berwald connection is essentially Riemannian $(\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0)$.*

A Finsler space F^n with an (α, β)-metric is a Douglas space, if and only if $B^{ij} \equiv B^i y^j - B^j y^i$ is $hp(3)$ ([2]). From the latter of (2.3) B^{ij} is written as follows:

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i{}_0 y^j - s^j{}_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i). \tag{2.6}$$

We shall state the following lemma for later:

Lemma 2.2. ([3]) *If $\alpha^2 \equiv 0 (mod\, \beta)$, that is, $a_{ij}(x) y^i y^j$ contains $b_i(x) y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x) y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

3. Douglas space

In the present section, we find the condition that a Finsler space F^n with the rth approximate infinite series (α, β)-metric (2.1) be a Douglas space. In the n-dimensional Finsler space F^n with the rth approximate infinite series (α, β)-metric (2.1), we have

$$\begin{aligned} L_\alpha &= \sum_{k=0}^{r} k \left(\frac{\alpha}{\beta}\right)^{k-1}, \quad L_\beta = -\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k}, \\ L_{\alpha\alpha} &= \frac{1}{\beta}\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-2}. \end{aligned} \tag{3.1}$$

Substituting (3.1) into (2.6), we have

$$
\begin{aligned}
&2\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1}\left\{\beta^3\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1} + \alpha(b^2\alpha^2-\beta^2)\sum_{k=0}^{r} k(k-1)\right.\\
&\left.\left(\frac{\alpha}{\beta}\right)^{k-2}\right\}B^{ij} + 2\alpha\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k}\left\{\beta^3\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1}\right.\\
&\left. + \alpha(b^2\alpha^2-\beta^2)\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-2}\right\}(s^i{}_0y^j - s^j{}_0y^i)\\
&- \alpha^3\sum_{k=0}^{r} k(k-1)\left(\frac{\alpha}{\beta}\right)^{k-2}\left\{r_{00}\sum_{k=0}^{r} k\left(\frac{\alpha}{\beta}\right)^{k-1}\right.\\
&\left. + 2\alpha s_0\sum_{k=0}^{r}(k-1)\left(\frac{\alpha}{\beta}\right)^{k}\right\}(b^iy^j - b^jy^i) = 0.
\end{aligned}
\tag{3.2}
$$

We shall divide our consideration into two cases of which r is even or odd.

(1) Case of $r = 2h$, where h is a positive integer.

Then the equation (3.2) is rewritten as follows:

$$
\begin{aligned}
&2\beta\sum_{k=0}^{2h} k\alpha^{k-1}\beta^{2h-k}\left\{\beta^2\sum_{k=0}^{2h} k\alpha^{k-1}\beta^{2h-k}\right.\\
&\left. + \alpha(b^2\alpha^2-\beta^2)\sum_{k=0}^{2h} k(k-1)\alpha^{k-2}\beta^{2h-k}\right\}B^{ij}\\
&+ 2\alpha\sum_{k=0}^{2h}(k-1)\alpha^{k}\beta^{2h-k}\left\{\beta^2\sum_{k=0}^{2h} k\alpha^{k-1}\beta^{2h-k}\right.\\
&\left. + \alpha(b^2\alpha^2-\beta^2)\sum_{k=0}^{2h} k(k-1)\alpha^{k-2}\beta^{2h-k}\right\}(s^i{}_0y^j - s^j{}_0y^i)\\
&- \alpha^3\sum_{k=0}^{2h} k(k-1)\alpha^{k-2}\beta^{2h-k}\left\{\beta r_{00}\sum_{k=0}^{2h} k\alpha^{k-1}\beta^{2h-k}\right.\\
&\left. +2\alpha s_0\sum_{k=0}^{2h}(k-1)\alpha^{k}\beta^{2h-k}\right\}(b^iy^j - b^jy^i) = 0.
\end{aligned}
$$

The above is rewritten as the following form

$$
\begin{aligned}
(3.3)\quad & \Big[2\beta^3\big\{(M^2+\alpha^2K^2)+2\alpha KM\big\}+2\beta(b^2\alpha^2-\beta^2)\big\{\alpha^2(KP+MQ)\\
&+\alpha(MP+\alpha^2KQ)\big\}\Big]B^{ij}+\Big[2\beta^2\big\{\alpha^2(KL+\alpha^2MN)\\
&+\alpha(LM+\alpha^4KN)\big\}+2\alpha^2(b^2\alpha^2-\beta^2)\big\{(LP+\alpha^4NQ)\\
&+\alpha(LQ+\alpha^2NP)\big\}\Big](s^i{}_0y^j-s^j{}_0y^i)-\Big[\alpha^2\beta r_{00}\big\{\alpha^2(MQ+KP)\\
&+\alpha(MP+\alpha^2KQ)\big\}+2\alpha^4s_0\big\{(LP+\alpha^4NQ)\\
&+\alpha(LQ+\alpha^2NP)\big\}\Big](b^iy^j-b^jy^i)=0,
\end{aligned}
$$

where

$$
(3.4)\quad \begin{aligned}
K&=\sum_{k=1}^{h}2k\alpha^{2k-2}\beta^{2h-2k}, & L&=\sum_{k=0}^{h}(2k-1)\alpha^{2k}\beta^{2h-2k},\\
M&=\sum_{k=0}^{h-1}(2k+1)\alpha^{2k}\beta^{2h-2k-1}, & N&=\sum_{k=1}^{h-1}2k\alpha^{2k-2}\beta^{2h-2k-1}
\end{aligned}
$$

and

$$
(3.5)\quad \begin{aligned}
P&=\sum_{k=1}^{h}2k(2k-1)\alpha^{2k-2}\beta^{2h-2k},\\
Q&=\sum_{k=1}^{h-1}(2k+1)2k\alpha^{2k-2}\beta^{2h-2k-1}.
\end{aligned}
$$

Suppose that the Finsler space F^n is a Douglas space, that is, B^{ij} are $hp(3)$. Separating (3.3) in the rational and irrational terms of y^i, we have

$$
\begin{aligned}
(3.6)\quad & 2\beta\left\{\beta^2(M^2+\alpha^2K^2)+\alpha^2(b^2\alpha^2-\beta^2)(KP+MQ)\right\}B^{ij}\\
&+2\alpha^2\big\{\beta^2(KL+\alpha^2MN)+(b^2\alpha^2-\beta^2)(LP\\
&+\alpha^4NQ)\big\}(s^i{}_0y^j-s^j{}_0y^i)-\alpha^4\big\{\beta r_{00}(KP+MQ)\\
&+2s_0(LP+\alpha^4NQ)\big\}(b^iy^j-b^jy^i)+\alpha\Big[2\beta\big\{2\beta^2KM\\
&+(b^2\alpha^2-\beta^2)(MP+\alpha^2KQ)\big\}B^{ij}+2\big\{\beta^2(LM+\alpha^4KN)\\
&+\alpha^2(b^2\alpha^2-\beta^2)(LQ+\alpha^2NP)\big\}(s^i{}_0y^j-s^j{}_0y^i)\\
&-\alpha^2\left\{\beta r_{00}(MP+\alpha^2KQ)+2\alpha^2s_0(LQ+\alpha^2NP)\right\}(b^iy^j-b^jy^i)\Big]\\
&=0,
\end{aligned}
$$

because α is irrational and K, L, M, N, P, Q are rational polynomials of y^i. Hence the equation (3.6) is divided into two equations as follows:

$$\begin{aligned}&2\beta\left\{\beta^2(M^2+\alpha^2K^2)+\alpha^2(b^2\alpha^2-\beta^2)(KP+MQ)\right\}B^{ij}\\&+2\alpha^2\left\{\beta^2(KL+\alpha^2MN)+(b^2\alpha^2-\beta^2)(LP+\alpha^4NQ)\right\}(s^i{}_0y^j\\&-s^j{}_0y^i)-\alpha^4\left\{\beta r_{00}(KP+MQ)+2s_0(LP+\alpha^4NQ)\right\}(b^iy^j-b^jy^i)\\&=0,\end{aligned}\tag{3.7}$$

$$\begin{aligned}&2\beta\left\{2\beta^2KM+(b^2\alpha^2-\beta^2)(MP+\alpha^2KQ)\right\}B^{ij}\\&+2\left\{\beta^2(LM+\alpha^4KN)+\alpha^2(b^2\alpha^2-\beta^2)(LQ+\alpha^2NP)\right\}(s^i{}_0y^j\\&-s^j{}_0y^i)-\alpha^2\left\{\beta r_{00}(MP+\alpha^2KQ)+2\alpha^2s_0(LQ+\alpha^2NP)\right\}(b^iy^j\\&-b^jy^i)=0.\end{aligned}\tag{3.8}$$

Eliminating B^{ij} from (3.7) and (3.8), we obtain

$$R(s^i{}_0y^j-s^j{}_0y^i)+\alpha^2S(b^iy^j-b^jy^i)=0,\tag{3.9}$$

where

$$\begin{aligned}R=\;&2\alpha^2\{\beta^2(KL+\alpha^2MN)+(b^2\alpha^2-\beta^2)(LP+\alpha^4NQ)\}\\&\{2\beta^2KM+(b^2\alpha^2-\beta^2)(MP+\alpha^2KQ)\}\\&-2\{\beta^2(M^2+\alpha^2K^2)+\alpha^2(b^2\alpha^2-\beta^2)(KP+MQ)\}\\&\{\beta^2(LM+\alpha^4KN)+\alpha^2(b^2\alpha^2-\beta^2)(LQ+\alpha^2NP)\},\\S=\;&\{\beta^2(M^2+\alpha^2K^2)+\alpha^2(b^2\alpha^2-\beta^2)(KP+MQ\}\\&\{\beta r_{00}(MP+\alpha^2KQ)+2\alpha^2s_0(LQ+\alpha^2NP)\}\\&-\alpha^2\{\beta r_{00}(MQ+KP)+2s_0(LP+\alpha^4NQ)\}\\&\{2\beta^2KM+(b^2\alpha^2-\beta^2)(MP+\alpha^2KQ)\}.\end{aligned}\tag{3.10}$$

Transvection of (3.9) by b_iy_j leads to

$$Rs_0+S\gamma^2=0.\tag{3.11}$$

The terms of (3.11) which does not contain α^2 are found in $2\beta^{8h}(\beta s_0 - r_{00})$. Hence there exists $hp(8h) : V_{8h}$ such that

$$2\beta^{8h}(\beta s_0 - r_{00}) = \alpha^2 V_{8h}. \tag{3.12}$$

Then it will be better to divide our consideration into two cases as follows:

(A) $V_{8h} = 0$ and (B) $V_{8h} \neq 0$, $\alpha^2 \not\equiv 0$ (mod. β).

First, the case of (A) leads to $r_{00} = \beta s_0$, that is, $2r_{ij} = b_i s_j + b_j s_i$. Therefore, substituting $r_{00} = \beta s_0$ into (3.11), we have

$$s_0(R + \gamma^2 S_1) = 0, \tag{3.13}$$

where

$$\begin{aligned} S_1 = &\{\beta^2(M^2 + \alpha^2 K^2) + \alpha^2\gamma^2(KP + MQ)\}\{\beta^2(MP + \alpha^2 KQ) \\ &+ 2\alpha^2(LQ + \alpha^2 NP)\} - \alpha^2\{\beta^2(KP + MQ) + 2(LP + \alpha^4 NQ)\} \\ &\{2\beta^2 KM + \gamma^2(MP + \alpha^2 KQ)\}. \end{aligned}$$

If $R + \gamma^2 S_1 = 0$ in (3.13), then we obtain

$$\begin{aligned} R + \gamma^2 S_1 &= b^2\alpha^2 S_1 + R - \beta^2 S_1 \\ &= b^2\alpha^2 S_1 + (\alpha^2 R_1 + 2\beta^{8h+1}) - (\alpha^2\beta^2 S_2 + 2\beta^{8h+1}) \\ &= \alpha^2(b^2 S_1 + R_1 - \beta^2 S_2) = 0, \end{aligned}$$

where

$$\begin{aligned} R_1 = &\,2\{\beta^2(KL + \alpha^2 MN) + \gamma^2(LP + \alpha^4 NQ)\}\{2\beta^2 KM \\ &+ \gamma^2(MP + \alpha^2 KQ)\} - 2\{\beta^2 K^2 + \gamma^2(KP + MQ)\} \\ &\{\beta^2(LM + \alpha^4 KN) + \alpha^2\gamma^2(LQ + \alpha^2 NP)\} \\ &- 2\beta^2 M^2\{\alpha^2\beta^2 KN + \gamma^2(LQ + \alpha^2 NP)\} \\ &- 2\beta^4 M^3 L_1 + 2\beta^{2h+4}\{3\beta^{4h-2} M_1 + 3\alpha^2\beta^{2h-1} M_1^2 + \alpha^4 M_1^3\}, \end{aligned}$$

$$
\begin{aligned}
S_2 = {} & \{\beta^2K^2+\gamma^2(KP+MQ)\}\{\beta^2(MP+\alpha^2KQ)\\
& +2\alpha^2(LQ+\alpha^2NP)\}-\{\beta^2(KP+MQ)+2(LP+\alpha^4NQ)\}\\
& \{2\beta^2KM+\gamma^2(MP+\alpha^2KQ)\}+\beta^2M^2\{\beta^2KQ\\
& +2(LQ+\alpha^2NP)\}+\beta^4M^3P_1+2\beta^{2h+2}(\alpha^4M_1^3\\
& +3\alpha^2\beta^{2h-1}M_1^2+3\beta^{4h-2}M_1),\\
L_1 = {} & \sum_{k=1}^{h}(2k-1)\alpha^{2k-2}\beta^{2h-2k},\\
M_1 = {} & \sum_{k=1}^{h-1}(2k+1)\alpha^{2k-2}\beta^{2h-2k-1},\\
P_1 = {} & \sum_{k=2}^{h}2k(2k-1)\alpha^{2k-4}\beta^{2h-2k}.
\end{aligned}
$$

Thus the term of $b^2S_1+R_1-\beta^2S_2=0$ which does not contain α^2 is $2(b^2-7)\beta^{8h-1}$. Thus there exists $hp(8h-3): V_{8h-3}$ such that

$$2(b^2-7)\beta^{8h-1}=\alpha^2V_{8h-3},$$

where we assume $b^2\neq 7$. Hence we have $V_{8h-3}=0$, which leads to a contradiction, that is, $R+\gamma^2S_1\neq 0$. Therefore, we have $s_0=0$ from (3.13) and we obtain $r_{00}=0$ easily. Substituting $s_0=0$ and $r_{00}=0$ into (3.10), we have

$$R(s^i{}_0y^j-s^j{}_0y^i)=0. \tag{3.14}$$

If $R=0$, then from the former of (3.10), we have

$$
\begin{aligned}
& 2\alpha^2\{\beta^2(KL+\alpha^2MN)+\gamma^2(LP+\alpha^4NQ)\}\{2\beta^2KM\\
& +\gamma^2(MP+\alpha^2KQ)\}-2\{\beta^2(M^2+\alpha^2K^2)+\alpha^2\gamma^2(KP+MQ)\}\\
& \{\beta^2(LM+\alpha^4KN)+\alpha^2\gamma^2(LQ+\alpha^2NP)\}=0.
\end{aligned} \tag{3.15}
$$

The term of (3.15) which does not contain α^2 is $2\beta^{8h+1}$. Thus there exists $hp(8h-1): V_{8h-1}$ such that

$$2\beta^{8h+1}=\alpha^2V_{8h-1},$$

from which we have $V_{8h-1} = 0$. It is a contradiction, that is, $R \neq 0$. Therefore we obtain $s^i{}_0 y^j - s^j{}_0 y^i = 0$ in (3.14). Transvection of this equation by y_j gives $s^i{}_0 = 0$, which implies $s_{ij} = 0$. Consequently, we have $r_{ij} = s_{ij} = 0$, that is, $b_{i:j} = 0$ is obtained.

Secondly, we treat the case (B). The equation (3.12) shows that there exists a function $k = k(x)$ satisfying

$$\beta s_0 - r_{00} = k(x)\alpha^2. \tag{3.16}$$

Substituting (3.16) into (3.11) and using (3.10), we have

$$b^2 S + s_0 R_1 - \beta^2 S_3 = 0, \tag{3.17}$$

where

$$\begin{aligned} S_3 = {} & \beta^2 M^2 \{\beta^2 s_0 KQ - k\beta(MP + \alpha^2 KQ) + 2s_0(LQ + \alpha^2 NP)\} \\ & + \{\beta^2 K^2 + \gamma^2(KP + MQ)\}\{\beta(\beta s_0 - k\alpha^2)(MP + \alpha^2 KQ) \\ & + 2\alpha^2 s_0(LQ + \alpha^2 NP)\} - \{\beta(\beta s_0 - k\alpha^2)(MQ + KP) \\ & + 2s_0(LP + \alpha^4 NQ)\}\{2\beta^2 KM + \gamma^2(MP + \alpha^2 KQ)\} \\ & + P_1(\alpha^6\beta^4 s_0 M_1^3 + 3\alpha^4\beta^{2h+3} s_0 M_1^2 + 3\alpha^2\beta^{4h+2} s_0 M_1 \\ & + \beta^{6h+1} s_0) + 2\beta^{2h+2} s_0(\alpha^4 M_1^3 + 3\alpha^2\beta^{2h-1} M_1^2 + 3\beta^{4h-2} M_1). \end{aligned}$$

The term of (3.17) which seemingly does not contain α^2 is included in the term as follows : $2\{(b^2 - 7)s_0 - k\beta\}\beta^{8h-1}$, where we assume $b^2 \neq 7$. Thus there exists $hp(8h-2) : V_{8h-2}$ such that

$$2\{(b^2 - 7)s_0 - k\beta\}\beta^{8h-1} = \alpha^2 V_{8h-2}.$$

From $\alpha^2 \not\equiv$ (mod β), it follows that V_{8h-2} must vanish and hence we have

$$s_0 = \frac{k(x)}{b^2 - 7}\beta. \tag{3.18}$$

From (3.18), we have $s_i = k(x)b_i/(b^2 - 7)$. Transvection of the above by b^i leads to $k(x)b^2 = 0$. Hence we get $k(x) = 0$. Substituting $k(x) = 0$

into (3.16) and (3.18), we obtain $s_0 = 0$ and $r_{00} = 0$. From (3.9), we have $R(s^i{}_0 y^j - s^j{}_0 y^i) = 0$. If $R = 0$, then it is a contradiction. Hence $R \neq 0$. Therefore, we obtain $s^i{}_0 y^j - s^j{}_0 y^i = 0$. Transcevtion of this equation by y_j gives $s^i{}_0 = 0$, that is, $s_{ij} = 0$. Hence both the case (A) and (B) lead to $r_{ij} = 0$ and $s_{ij} = 0$, that is, $b_{i:j} = 0$.

Conversely if $b_{i:j} = 0$, then we have $B^{ij} = 0$ from (2.6), so F^n is a Douglas space.

(2) Case of $r = 2h + 1$, where h is a positive integer.

We find the condition that the odd approximate infinite series (α, β)-metric (2.1) be a Douglas space in the same way as the case of $r = 2h$.

The equation (3.2) is written as the following form

$$
\begin{aligned}
&2\beta \sum_{k=0}^{2h+1} k\alpha^{k-1}\beta^{2h-k+1}\Bigg\{\beta^2 \sum_{k=0}^{2h+1} k\alpha^{k-1}\beta^{2h-k+1} \\
&+ \alpha\gamma^2 \sum_{k=0}^{2h+1} k(k-1)\alpha^{k-2}\beta^{2h-k+1}\Bigg\}B^{ij} \\
&+ 2\alpha \sum_{k=0}^{2h+1} (k-1)\alpha^k\beta^{2h-k+1}\Bigg\{\beta^2 \sum_{k=0}^{2h+1} k\alpha^{k-1}\beta^{2h-k+1} \\
&+ \alpha\gamma^2 \sum_{k=0}^{2h+1} k(k-1)\alpha^{k-2}\beta^{2h-k+1}\Bigg\}(s^i{}_0 y^j - s^j{}_0 y^i) \\
&- \alpha^3 \sum_{k=0}^{2h+1} k(k-1)\alpha^{k-2}\beta^{2h-k+1}\Bigg\{\beta r_{00} \sum_{k=0}^{2h+1} k\alpha^{k-1}\beta^{2h-k+1} \\
&+ 2\alpha s_0 \sum_{k=0}^{2h+1} (k-1)\alpha^k\beta^{2h-k+1}\Bigg\}(b^i y^j - b^j y^i) = 0.
\end{aligned}
$$

The above is rewritten as follows:

$$
\begin{aligned}
&2\beta(O + \alpha\beta K)\{\beta^2(O + \alpha\beta K) + \alpha\gamma^2(T + \alpha U)\}B^{ij} \\
&+ 2\alpha(\beta L + \alpha^3 K)\{\beta^2(O + \alpha\beta K) + \alpha\gamma^2(T + \alpha U)\}(s^i{}_0 y^j - s^j{}_0 y^i) \\
&- \alpha^3(T + \alpha U)\{\beta r_{00}(O + \alpha\beta K) + 2\alpha s_0(\beta L + \alpha^3 K)\}(b^i y^j - b^j y^i) \\
&= 0,
\end{aligned}
\tag{3.19}
$$

where K and L are defined by (3.5) and we put

$$O=\sum_{k=0}^{h}(2k+1)\alpha^{2k}\beta^{2h-2k} \tag{3.20}$$

and

$$\begin{aligned} T&=\sum_{k=1}^{h}2k(2k-1)\alpha^{2k-2}\beta^{2h-2k+1},\\ U&=\sum_{k=1}^{h}(2k+1)2k\alpha^{2k-2}\beta^{2h-2k}. \end{aligned} \tag{3.21}$$

Suppose that the Finsler space F^n is a Douglas space, that is, B^{ij} are $hp(3)$. Separating (3.19) in the rational and irrational terms of y^i, we obtain

$$\begin{aligned} &2\beta\{O(\beta^2O+\alpha^2\gamma^2U)+\alpha^2\beta K(\beta^3K+\gamma^2T)\}B^{ij}\\ &+2\alpha^2\{\beta L(\beta^3K+\gamma^2T)+\alpha^2K(\beta^2O+\alpha^2\gamma^2U)\}(s^i{}_0y^j-s^j{}_0y^i)\\ &-\alpha^4\{\beta T(\beta r_{00}K+2s_0L)+U(\beta r_{00}O+2\alpha^4s_0K)\}(b^iy^j-b^jy^i)\\ &+\alpha\Big[2\beta\{O(\beta^3K+\gamma^2T)+\beta K(\beta^2O+\alpha^2\gamma^2U)\}B^{ij}\\ &+2\{\beta L(\beta^2O+\alpha^2\gamma^2U)+\alpha^4K(\beta^3K+\gamma^2T)\}(s^i{}_0y^j-s^j{}_0y^i)\\ &-\{\alpha^2T(\beta r_{00}O+2\alpha^4s_0K)+\alpha^4\beta U(\beta r_{00}K+2s_0L)\}(b^iy^j-b^jy^i)\Big]\\ &=0. \end{aligned} \tag{3.22}$$

Since α is irrational and K, L, O, T and U are rational polynomials of y^i, the equation (3.22) is divided into two equations in the following forms

$$\begin{aligned} &2\beta\{O(\beta^2O+\alpha^2\gamma^2U)+\alpha^2\beta K(\beta^3K+\gamma^2T)\}B^{ij}\\ &+2\alpha^2\{\beta L(\beta^3K+\gamma^2T)+\alpha^2K(\beta^2O+\alpha^2\gamma^2U)\}(s^i{}_0y^j-s^j{}_0y^i)\\ &-\alpha^4\{\beta T(\beta r_{00}K+2s_0L)+U(\beta r_{00}O+2\alpha^4s_0K)\}(b^iy^j-b^jy^i)\\ &=0, \end{aligned} \tag{3.23}$$

$$\begin{aligned} &2\beta\{O(\beta^3K+\gamma^2T)+\beta K(\beta^2O+\alpha^2\gamma^2U)\}B^{ij}+2\{\beta L(\beta^2O\\ &+\alpha^2\gamma^2U)+\alpha^4K(\beta^3K+\gamma^2T)\}(s^i{}_0y^j-s^j{}_0y^i)-\{\alpha^2T(\beta r_{00}O\\ &+2\alpha^4s_0K)+\alpha^4\beta U(\beta r_{00}K+2s_0L)\}(b^iy^j-b^jy^i)=0. \end{aligned} \tag{3.24}$$

Eliminating B^{ij} from (3.23) and (3.24), we have

$$V(s^i{}_0 y^j - s^j{}_0 y^i) + \alpha^2 W(b^i y^j - b^j y^i) = 0, \tag{3.25}$$

where

$$\begin{aligned} V &= 2\alpha^2\{\beta L(\beta^3 K + \gamma^2 T) + \alpha^2 K(\beta^2 O + \alpha^2\gamma^2 U)\} \\ &\quad \{O(\beta^3 K + \gamma^2 T) + \beta K(\beta^2 O + \alpha^2\gamma^2 U)\} \\ &\quad - 2\{O(\beta^2 O + \alpha^2\gamma^2 U) + \alpha^2\beta K(\beta^3 K + \gamma^2 T)\} \\ &\quad \{\beta L(\beta^2 O + \alpha^2\gamma^2 U) + \alpha^4 K(\beta^3 K + \gamma^2 T)\}, \\ W &= \{O(\beta^2 O + \alpha^2\gamma^2 U) + \alpha^2\beta K(\beta^3 K + \gamma^2 T)\} \\ &\quad \{T(\beta r_{00} O + 2\alpha^4 s_0 K) + \alpha^2\beta U(\beta r_{00} K + 2 s_0 L)\} \\ &\quad - \alpha^2\{\beta T(\beta r_{00} K + 2 s_0 L) + U(\beta r_{00} O + 2\alpha^4 s_0 K)\} \\ &\quad \{O(\beta^3 K + \gamma^2 T) + \beta K(\beta^2 O + \alpha^2\gamma^2 U)\}. \end{aligned} \tag{3.26}$$

Transvection of (3.25) by $b_i y_j$ leads to

$$V s_0 + W\gamma^2 = 0. \tag{3.27}$$

The terms of (3.27) which does not contain α^2 are found in $2\beta^{8h+4}(\beta s_0 - r_{00})$. Therefore there exists $h_p(8h+4) : V_{8h+4}$ such that

$$2\beta^{8h+4}(\beta s_0 - r_{00}) = \alpha^2 V_{8h+4}. \tag{3.28}$$

We assumed that $\alpha^2 \not\equiv 0$ (mod. β). Hence it will be better to divide our consideration into two cases as follows:

$$(\mathrm{A}^0)\ \ V_{8h+4} = 0 \quad \text{and} \quad (\mathrm{B}^0)\ \ V_{8h+4} \neq 0.$$

First, the case of (A^0) leads to $r_{00} = \beta s_0$, that is, $2r_{ij} = b_i s_j + b_j s_i$. Thus, substituting $r_{00} = \beta s_0$ into (3.27) and using the second equation of (3.26), we obtain

$$s_0(V + W_1\gamma^2) = 0, \tag{3.29}$$

where

$$W_1 = \{O(\beta^2 O + \alpha^2\gamma^2 U) + \alpha^2\beta K(\beta^3 K + \gamma^2 T)\}$$
$$\{T(\beta^2 O + 2\alpha^4 K) + \alpha^2\beta U(\beta^2 K + 2L)\}$$
$$- \alpha^2\{\beta T(\beta^2 T + 2L) + U(\beta^2 O + 2\alpha^4 K)\}$$
$$\{O(\beta^3 K + \gamma^2 T) + \beta K(\beta^2 O + \alpha^2\gamma^2 U)\}.$$

If there is $V + W_1\gamma^2 = 0$ in (3.29), then we have

$$\begin{aligned} & V + W_1\gamma^2 \\ &= b^2\alpha^2 W_1 + V - \beta^2 W_1 \\ &= b^2\alpha^2 W_1 + (\alpha^2 V_1 + 2\beta^{8h+5}) - (\alpha^2\beta^2 W_2 + 2\beta^{8h+5}) \\ &= \alpha^2(b^2 W_1 + V_1 - \beta^2 W_2) \\ &= 0, \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} V_1 = {} & 2\{\beta L(\beta^3 K + \gamma^2 T) + \alpha^2 K(\beta^2 O + \alpha^2\gamma^2 U)\} \\ & \{O(\beta^3 K + \gamma^2 T) + \beta K(\beta^2 O + \alpha^2\gamma^2 U)\} \\ & - 2\{\beta L(\beta^2 O + \alpha^2\gamma^2 U) + \alpha^4 K(\beta^3 K + \gamma^2 T)\} \\ & \{\gamma^2 OU + \beta K(\beta^3 K + \gamma^2 T)\} - 2\beta^2 O^2\{\beta\gamma^2 LU + \alpha^2 K(\beta^3 K \\ & + \gamma^2 T)\} - 2\beta^5 L_1 O^3 + 2\beta^{2h+5}(3\beta^{4h} O_1 + 3\alpha^2\beta^{2h} O_1^2 + \alpha^4 O_1^3), \end{aligned}$$

$$\begin{aligned} W_2 = {} & \{T(\beta^2 O + 2\alpha^4 K) + \alpha^2\beta U(\beta^2 K + 2L)\} \\ & \{\gamma^2 OU + \beta K(\beta^3 K + \gamma^2 T) + 2\beta^{2h+2} O_1 + \alpha^2\beta^2 O_1^2\} \\ & - \{\beta T(\beta^2 K + 2L) + U(\beta^2 O + 2\alpha^4 K)\}\{O(\beta^3 K + \gamma^2 T) \\ & + \beta K(\beta^2 O + \alpha^2\gamma^2 U)\} + \beta^{4h+2}\{2\alpha^2 KT + \beta U(\beta^2 K + 2L) \\ & + \beta^2(\alpha^2 O_1 T_1 + \beta^{2h} T_1 + 2\beta^{2h-1} O_1\}, \end{aligned}$$

$$O_1 = \sum_{k=1}^{h} (2k+1)\alpha^{2k-2}\beta^{2h-2k},$$

$$T_1 = \sum_{k=2}^{h} 2k(2k-1)\alpha^{2k-4}\beta^{2h-2k+1}.$$

If $V + W_1\gamma^2 = 0$ in (3.30), then the term of this equation which does not contain α^2 is included in $b^2W_1 + V_1 - \beta^2 W_2 = 0$. Therefore, the term of $b^2W_1 + V_1 - \beta^2W_2$ which does not contain α^2 is $2(b^2-1)\beta^{8h+3}$, where we assume $b^2 \neq 1$. Hence there exist $h_p(8h+1) : V_{8h+1}$ such that

$$2(b^2-1)\beta^{8h+3} = \alpha^2 V_{8h+1}.$$

Since $\alpha^2 \not\equiv 0$ (mod. β), we have $V_{8h+1} = 0$, which leads to a contradiction. Thus we get $V + W_1\gamma^2 \neq 0$. Hence we have $s_0 = 0$ from (3.29) and we obtain $r_{00} = 0$ easily. Substituting $s_0 = 0$ and $r_{00} = 0$ into the second equation of (3.26), we get

$$V(s^i{}_0 y^j - s^j{}_0 y^i) = 0. \tag{3.31}$$

If $V = 0$, then from the first equation of (3.26), we have

$$\begin{aligned} &2\alpha^2\{\beta L(\beta^3K + \gamma^2T) + \alpha^2K(\beta^2O + \alpha^2\gamma^2U)\}\\ &\{O(\beta^3K + \gamma^2T) + \beta K(\beta^2O + \alpha^2\gamma^2U)\}\\ &-2\{O(\beta^2O + \alpha^2\gamma^2U) + \alpha^2\beta K(\beta^3K + \gamma^2T)\}\\ &\{\beta L(\beta^2O + \alpha^2\gamma^2U) + \alpha^4K(\beta^3K + \gamma^2T)\} = 0. \end{aligned} \tag{3.32}$$

The term of (3.32) which does not contain α^2 is $2\beta^{8h+5}$. Thus there exists $h_p(8h+3) : V_{8h+3}$ such that

$$2\beta^{8h+5} = \alpha^2 V_{8h+3},$$

which leads to $V_{8h+3} = 0$. It is a contradiction. Hence we have $V \neq 0$, which leads to $s^i{}_0 y^j - s^j{}_0 y^i = 0$ in (3.31). Transvection of the above by y_j gives $s^i{}_0 = 0$, which implies $s_{ij} = 0$. Consequently, we obtain $r_{ij} = s_{ij} = 0$, that is, $b_{i:j} = 0$ is obtained.

Secondly, we treat the case (B^0). In the case of (B^0) we investigate $V_{8h+4} \neq 0$ and $\alpha^2 \not\equiv 0$ (mod. β). The equation (3.28) shows that there exists a function $f = f(x)$ satisfying

$$\beta s_0 - r_{00} = f(x)\alpha^2. \tag{3.33}$$

Substituting (3.33) into (3.27) and using (3.26), we have

$$
\begin{aligned}
s_0V+\gamma^2W &= s_0V+b^2\alpha^2W_3-\beta^2W_3\\
&= s_0(\alpha^2V_1+2\beta^{8h+5})+b^2\alpha^2W_3-\beta^2W_3\\
&= \alpha^2(s_0V_1+b^2W_3)+2\beta^{8h+5}s_0-\beta^2W_3\\
&= \alpha^2(s_0V_1+b^2W_3)+2\beta^{8h+5}s_0-\beta^2(2\beta^{8h+3}s_0+\alpha^2W_4)\\
&= \alpha^2(s_0V_1+b^2W_3-\beta^2W_4)\\
&= 0,
\end{aligned}
\tag{3.34}
$$

where

$$
\begin{aligned}
W_3 =\ & 2\beta^{8h+3}s_0+\alpha^2(C+\beta^4s_0D),\\
C =\ & \beta^2s_0OT\{\gamma^2OU+\beta K(\beta^3K+\gamma^2T)\}+\beta^2O^2\big[s_0\beta U(\beta^2K+2L)\\
& -f(x)\beta TO+\alpha^2(2s_0KT-f(x)\beta^2KU)\big]-\beta^3s_0O\{T(\beta^2K\\
& +2L)+\beta OU\}(2\beta K-T)-\alpha^2\big[\beta^2O\{2\alpha^2s_0K-f(x)\beta(\beta KT\\
& +OU)\}(2\beta K-T)+s_0\beta\{T(\beta^2K+2L)+\beta OU\}(b^2OT\\
& +\beta\gamma^2KU)-\{s_0\beta U(\beta^2K+2L)-f(x)\beta TO+\alpha^2(2s_0KT\\
& -f(x)\beta^2KU)\}\{\gamma^2OU+\beta K(\beta^3K+\gamma^2T)\}+\alpha^2\{2\alpha^2s_0K\\
& -f(x)\beta(\beta KT+OU)\}(b^2OT+\beta\gamma^2KU)\big],\\
D =\ & 6\beta^{6h-1}O_1+\beta^{6h}T_1+\alpha^2\big[3\beta^{2h}O_1^2+\alpha^2O_1^3+T_1(3\beta^{4h}O_1\\
& +3\beta^{2h}\alpha^2O_1^2+\alpha^4O_1^3)\big],\\
W_4 =\ & C+\beta^4s_0D,
\end{aligned}
$$

which take the follow of form

$$
s_0V_1+b^2W_3-\beta^2W_4=0. \tag{3.35}
$$

Thus the term of (3.35) which seemingly does not contain α^2 is included in the term: $2\{(b^2-10)s_0+f(x)\beta\}\beta^{8h+3}$. Thus there exists $hp(8h+2)$: V_{8h+2} such that

$$
2\{(b^2-10)s_0+f(x)\beta\}\beta^{8h+3}=\alpha^2V_{8h+2}.
$$

From $\alpha^2 \not\equiv 0$ (mod. β), if follows that V_{8h+2} must vanish and hence we have

$$s_0 = \frac{f(x)}{10 - b^2}\beta, \tag{3.36}$$

where we assume $b^2 \neq 10$. Therefore, we have $s_i = f(x)b_i/(10 - b^2)$ by virture of (3.36). Transvection of the above by b^i leads to $f(x)b^2 = 0$. Hence we have $f(x) = 0$. Substituting $f(x) = 0$ into (3.33) and (3.36), we obtain $s_0 = 0$ and $r_{00} = 0$. From (3.25), we have $V(s^i{}_0 y^j - s^j{}_0 y^i) = 0$. If $V = 0$, then it is a contradiction. Thus $V \neq 0$ is obtained. Therefore, we have $s^i{}_0 y^j - s^j{}_0 y^i = 0$. Transvection of this equation by y_j gives $s^i{}_0 = 0$. Hence both the case (A^0) and (B^0) lead to $r_{ij} = 0$ and $s_{ij} = 0$, that is, $b_{i;j} = 0$ is concluded.

Conversely, if $b_{i:j} = 0$, then we have $B^{ij} = 0$ by virture of (2.6). Hence the Finsler space F^n $(n > 2)$ with (2.1) is a Douglas space.

Thus we have the following

Theorem 3.1. *An n-dimensional Finsler space F^n $(n > 2)$ with an approximate infinite series (α, β)-metric (2.1) provided $b^2 \neq 1, 7, 10$ is a Douglas space if and only if $b_{i:j} = 0$ is obtained.*

By Theorem 2.1 and Theorem 3.1, we have

Theorem 3.2. *If an n-dimensional Finsler space F^n $(n > 2)$ with an approximate infinite series (α, β)-metric (2.1) provided $b^2 \neq 1, 7, 10$ is a Douglas space, then it is a Berwald space.*

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