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h-STABILITY FOR PERTURBED VOLTERRA DIFFERENCE SYSTEMS

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Abstract. We investigate $h-$stability of solutions of perturbed Volterra difference systems.

1. Introduction

The theory of difference equations is rapidly gaining attention because of its use in such fields as numerical analysis, control theory, finite mathematics, and computer science. Furthermore, difference equations occur in the investigation of discretization methods differential equations [9]. Medina and Pinto [11, 12] introduced the notion of $h-$stability which is an important extension of the notion of exponential asymptotic stability. In the study of the stability properties of difference systems, the notion of $h-$stability is very useful because, when we study the asymptotic stability, it is not easy to work with non-exponential types of stability. For the study of difference systems, we refer to Choi et al.[2], Medina [10], Medina and Pinto [11]. Also, Choi et al.[4], Medina and Pinto [12] studied the $h-$stability of Volterra difference systems. In [6], they studied $h-$stability of a solution of Volterra integro-differential equation

$$x' = A(t)x + \int_{t_0}^{t} K(t,s)x(s)ds, x(t_0) = x_0,$$

and its perturbation

$$y' = A(t)y + \int_{t_0}^{t} K(t,s)y(s)ds + g(t,y,Ty), \quad y(t_0) = y_0,$$

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and their relations to each other. In this paper, corresponding to these Volterra equations, we study $h-$stability of solutions of difference equation

$$x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n,s)x(s), \quad x(n_0) = x_0,$$

and its perturbation

$$y(n+1) = A(n)y(n) + \sum_{s=n_0}^{n} B(n,s)y(s) + g(n, y(n), Ty(n)), \quad y(n_0) = y_0,$$

Conditions on A, B, g and T are given later.

2. Preliminaries

We consider the linear Volterra difference system

$$x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n,s)x(s), \quad x(n_0) = x_0, \tag{2.1}$$

where $A(n)$ and $B(n,s)$ are $m \times m$ matrices for each $n, s \in N(n_0) = \{n_0, n_0+1, ..., n_0+k, ...\}, n_0$ is a nonnegative integer. Let $x(n) = x(n, n_0, x_0)$ be the unique solution of (2.1) satisfying the initial condition $x(n_0) = x_0$. Let $\mathbb{R}^m$ denote the Euclidean $m-$space. For $x \in \mathbb{R}^m$, let $|x| = (\sum_{j=1}^m x_j^2)^{1/2}$. For an $m \times m$ matrix A, define the norm $|A|$ of A by $|A| = sup_{|x| \le 1} |Ax|$. Let $\mathbb{R}^+$ be the half line $[0, \infty)$. We now give the main definitions [11] that we need in the sequel.

Definition 2.1. System (2.1) is called an *$h-$system around the null*, or more briefly an *$h-$system*, if there exist a positive function $h : N(n_0) \to \mathbb{R}$ and $c \ge 1$, such that

$$|x(n, n_0, x_0)| \le c\,|x_0|\,h(n)h^{-1}(n_0), \quad n \ge n_0,$$

for $|x_0|$ small enough (here, $h^{-1}(n) = 1/h(n)$).

If h is a bounded function, then an $h-$system permits the following type of stability:

Definition 2.2. The zero solution of system (2.1), or more briefly sysetem (2.1), is said to be *$h-$stable*(hS) if there exist $c \ge 1$, and $\delta > 0$ exist as well as a positive bounded function $h : N(n_0) \to \mathbb{R}$ such that

$$|x(n, n_0, x_0)| \le c\,|x_0|\,h(n)h^{-1}(n_0), \quad n \ge n_0 \quad \text{for} \quad |x_0| \le \delta.$$

For our discussion we need the following lemmas

LEMMA 2.3. [3] *Let $s(n,r)$ be a nonnegative nondecreasing function in r for any fixed $n \in N(n_0)$. Suppose that for any $n \geq n_0$, nonnegative functions $u(n)$ and $v(n)$ defined on $N(n_0)$ satisfy the following inequality:*

$$v(n) - \sum_{l=n_0}^{n-1} s(l, v(l), |T|\, v(l)) < u(n) - \sum_{l=n_0}^{n-1} s(l, u(l), |T|\, u(l)).$$

If $v(n_0) < u(n_0)$, then $v(n) < u(n)$ for all $n \geq n_0$.

Consider the linear Volterra difference system

$$x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n,s)x(s), \quad x(n_0) = x_0,$$

and its perturbation

$$\begin{aligned} y(n+1) &= A(n)y(n) + \sum_{s=n_0}^{n} B(n,s)y(s) + g(n, y(n)), \\ y(n_0) &= y_0, \end{aligned} \tag{2.2}$$

m *matricesforeach*n,s$\in N(n_0)$, $g : N(n_0) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $N(n_0) = \{n_0, n_0+1, ..., n_0+k, ...\}$, and $\mathbb{R}^m$ is the $m-$dimensional real Euclidean space.

LEMMA 2.4. [3] *The unique solution $y(n, n_0, y_0)$ of (2.2) satisfying $y(n_0) = y_0$ is given by*

$$y(n, n_0, y_0) = R(n, n_0)y_0 + \sum_{s=n_0}^{n-1} R(n, s+1)g(s, y(s)),$$

where $\mathbb{R}(n,m)$ is the unique solution of the matrix difference equation

$$\begin{aligned} R(n,m) &= R(n, m+1)A(n) + \sum_{r=m}^{n-1} R(n, r+1)B(r,m), \\ & n-1 \geq m \geq n_0, \end{aligned} \tag{2.3}$$

with $R(m,m) = I$.

THEOREM 2.5. [10] *The Volterra difference system (2.1) is hS if and only if there exist a constant $c \geq 1$ and a positive bounded funtion $h : N(n_0) \to \mathbb{R}$ such that $|R(n, n_0)| \leq ch(n)h(n_0)^{-1}$, for $n \geq n_0$, where $R(n, n_0)$ is the resolvent matrix of system (2.3).*

Remark 2.6. We note that the resolvent matrix $R(n,m)$ for equation (2.3) is closely related to the fundemental matrix $\Phi(n,n_0)$. By the uniquence of solution, it is easy to see that $R(n,n_0)=\Phi(n,n_0)$.

To illustrate Remark 2.6, we give the following example.

Example 2.7. [4] *We consider the linear Volterra difference equation*

$$\begin{aligned} x(n+1) &= A(n)x(n) + \sum_{s=n_0}^{n} B(n,s)x(s) \\ &= 2x(n) + \sum_{s=n_0}^{n} 2^{n-s}x(s), \end{aligned} \tag{2.4}$$

where $A(n)=2$ *and* $B(n,s)=2^{n-s}$. *Then any solution of (2.4) through the initial point* $x(n_0,n_0,x_0)=x_0$ *is given by*

$$x(n,n_0,x_0)=\frac{x_0}{3}[1+2\cdot 4^{n-n_0}], \quad n\in N(n_0).$$

Thus the resolvent matrix solution $R(n,m)$ *with* $R(m,m)=1$ *of the difference equation*

$$\begin{aligned} R(n,m) &= R(n,m+1)A(n) + \sum_{r=m}^{n-1} R(n,r+1)B(r,m) \\ &= R(n,m+1)2 + \sum_{r=m}^{n-1} R(n,r+1)2^{r-m}, \\ & \qquad n-1\ge m\ge n_0, \end{aligned}$$

is given by

$$R(n,m)=\frac{1}{3}[1+2\cdot 4^{n-m}].$$

In fact, we note that the fundamental matrix of (2.4) is given by

$$\frac{\partial}{\partial x_0}x(n,n_0,x_0)=\Phi(n,n_0)=\frac{1}{3}[1+2\cdot 4^{n-n_0}]$$

and

$$\Phi(n,n_0)=R(n,n_0).$$

We need the following difference inequality to obtain hS between the solutions of (2.1) and (3.1).

LEMMA 2.8. [2] *Let $a(n), b(n)$ and $c(n)$ be non-negative functions defined on $N(n_0)$ and d be a positive number. If, for $n \geq n_0$, the following inequality hold:*

$$u(n) \leq d + \sum_{s=n_0}^{n-1} a(s)u(s) + \sum_{s=n_0}^{n-1} b(s) \sum_{l=n_0}^{s-1} c(l)u(l),$$

then

$$u(n) \leq d \exp[\sum_{s=n_0}^{n-1} (a(s) + b(s) \sum_{l=n_0}^{s-1} c(l))], \quad n \geq n_0.$$

3. Main results

In this section, we examine the property of hS for the perturbed difference system of linear difference system (2.1) using the comparison principle and discrete Bihari-type inequality. We consider the linear Volterra difference system

$$x(n+1) = A(n)x(n) + \sum_{s=n_0}^{n} B(n,s)x(s), \quad x(n_0) = x_0,$$

and its perturbed system

$$y(n+1) = A(n)y(n) + \sum_{s=n_0}^{n} B(n,s)y(s) + g(n, y(n), Ty(n)), \qquad (3.1)$$
$$y(n_0) = y_0$$

where $A(n)$ and $B(n,s)$ are $m \times m$ matrices for each $n, s \in N(n_0)$, $g : N(n_0) \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, and $T : F(N(n_0), \mathbb{R}^m) \to \mathbb{R}^m$ is an operator on

$$F(N(n_0), \mathbb{R}^m) = \{y \mid y : N(n_0) \to \mathbb{R}^m \quad \text{is a sequence}\},$$

and $g(n, 0, 0) = 0$.

THEOREM 3.1. *Assume that*

(i) The zero solution $x = 0$ of (2.1) is hS with the non-increasing fumction $h(n)$,

(ii) $|g(n, y, Ty)| \leq s(n, |y|, |Ty|)$, where $s : N(n_0) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing in u,v for each fixed $n \in N(n_0)$ with $s(n, 0, 0) = 0$. Consider the scalar difference equation

$$u(n+1) = u(n) + cs(n, u(n), |T| u(n)), \qquad (3.2)$$
$$u(n_0) = u_0 = c |y_0|, c > 1.$$

If the zero solution $u = 0$ *of (3.2) is* hS, *then the zero solution* $y = 0$ *of (3.1) is also* hS *wherever* $u_0 = c\,|y_0|$.

Proof. By Lemma 2.4, the solution $y(n)$ of (3.1) is given by

$$y(n) = R(n, n_0)y_0 + \sum_{j=n_0}^{n-1} R(n, j+1)g(j, y(j), Ty(j)),$$

where $R(n, m)$ is the resolvent solution of the matrix difference equation (2.3). Let $u = u(n, n_0, u_0)$ be the solution of (3.2) with the initial value $u(n_0) = u_0$. Then, in view of the assumptions and Theorem 2.5, we have

$$\begin{aligned}
|y(n)| &\le |R(n, n_0)|\,|y_0| + \sum_{j=n_0}^{n-1} |R(n, j+1)|\,|g(j, y(j), Ty(j))| \\
&\le c\,|y_0|\,h(n)h^{-1}(n_0) + c\sum_{j=n_0}^{n-1} h(n)h^{-1}(j+1)s(j, |y(j)|, |Ty(j)|) \\
&\le c\,|y_0| + c\sum_{j=n_0}^{n-1} s(j, |y(j)|, |T|\,|y(j)|)
\end{aligned}$$

since $h(n)$ is non-increasing. Thus we obtain

$$\begin{aligned}
&|y(n)| - c\sum_{j=n_0}^{n-1} s(j, |y(j)|, |T|\,|y(j)|) \le c\,|y_0| = u_0 \\
&= u(n) - c\sum_{j=n_0}^{n-1} s(j, u(j), |T|\,u(j)).
\end{aligned}$$

By Lemma 2.3, we have $|y(n)| < u(n)$ for all $n \ge n_0 \ge 0$. Also we have

$$\begin{aligned}
|y(n)| < u(n) &\le c_1 u_0 h(n)h^{-1}(n_0) \\
&= c_1 c\,|y_0|\,h(n)h^{-1}(n_0) \\
&= d\,|y_0|\,h(n)h^{-1}(n_0), \quad d = c_1 c > 1
\end{aligned}$$

since $u = 0$ of (3.2) is hS. This completes the proof.

We improve the result in [8], which is the hS property for the equation (3.1)

THEOREM 3.2. *Suppose that the zero solution $x = 0$ of (2.1) is hS with the positive function $h(n)$ and for any $n \geq n_0$*

$$|g(n, y, Ty)| \leq a(n)\,|y| + b(n) \sum_{j=n_0}^{n-1} c(j)\,|y(j)|$$

where $a, b, c \in F(N(n_0), \mathbb{R}^+)$ and

$$M(n) = \exp[c_1 \sum_{j=n_0}^{n-1} [h(j)h^{-1}(j+1)a(j) + b(j)h^{-1}(j+1) \sum_{k=n_0}^{j-1} h(k)c(k)]] < \infty.$$

Then the zero solution $y = 0$ of (3.1) is hS.

Proof. By Lemma 2.4, the solution $y(n, n_0, y_0)$ of (3.1) is given by

$$y(n, n_0, y_0) = R(n, n_0)y_0 + \sum_{j=n_0}^{n-1} R(n, j+1)g(j, y(j), Ty(j)),$$

where $R(n, m)$ is the resolvent solution of the matrix difference equation (2.3). Then, by assumptions, we have

$$\begin{aligned}
|y(n, n_0, y_0)| &\leq |R(n, n_0)|\,|y_0| + \sum_{j=n_0}^{n-1} |R(n, j+1)|\,|g(j, y(j), Ty(j))| \\
&\leq c_1 h(n)h^{-1}(n_0)\,|y_0| + \sum_{j=n_0}^{n-1} c_1 h(n)h^{-1}(j+1)\,|g(j, y(j), Ty(j))| \\
&\leq c_1 h(n)h^{-1}(n_0)\,|y_0| \\
&\quad + c_1 \sum_{j=n_0}^{n-1} h(n)h^{-1}(j+1)[a(j)|y(j)| + b(j) \sum_{k=n_0}^{j-1} c(k)\,|y(k)|].
\end{aligned}$$

Putting $u(n) = |y(n)|\,h^{-1}(n)$, we obtain the following inequality from Lemma 2.8

$$\begin{aligned}
u(n) \leq c_1 u(n_0) + c_1 \sum_{j=n_0}^{n-1} [h(j)h^{-1}(j+1)a(j)u(j) \\
+ b(j)h^{-1}(j+1) \sum_{k=n_0}^{j-1} h(k)c(k)u(k)]
\end{aligned}$$

$$\leq c_1 u(n_0) \exp[c_1 \sum_{j=n_0}^{n-1} [h(j)h^{-1}(j+1)a(j)$$

$$+ b(j)h^{-1}(j+1) \sum_{k=n_0}^{j-1} h(k)c(k)]]$$

$$\leq c_1 u(n_0) M(n).$$

Hence we obtain $|y(n)| \leq M\,|y(n_0)|\,h(n)h^{-1}(n_0)$, where $M = c_1 M(n) \geq 1$, for all $n \geq n_0$, and the proof is complete.

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