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ISOMETRIES IN PROBABILISTIC 2-NORMED SPACES

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Abstract. The classical Mazur–Ulam theorem states that every surjective isometry between real normed spaces is affine. In this paper, we study 2-isometries in probabilistic 2-normed spaces.

1. Introduction

The theory of isometries mappings had its beginning in the classical paper [16] by Mazur and Ulam, who proved that every isometry of a real normed vector space onto another real normed vector space is a linear mapping up to translation. This property is not true for normed complex vector spaces, i.e., the conjugation on $\mathbb{C}$. The hypothesis of surjectiveness is essential. Without this assumption, Baker proved that every isometry from a real normed space into a strictly convex normed space is affine mapping [3]. Recently, Chu proved the Mazur–Ualm theorem on linear 2-normed spaces [6]. In this paper, we investigated the Mazur–Ualm theorem on probabilistic 2-normed spaces.

Let X and Y be metric spaces. A mapping $f : X \to Y$ is called an *isometry* if f satisfies

$$d_Y(f(x), f(y)) = d_X(x, y), \quad \forall x, y \in X,$$

where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the metric spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r, i.e., for all $x, y \in X$ with $d_X(x, y) = r$ implies

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$d_Y(f(x), f(y)) = r$. Then r is called a *conservative* (or *preserved*) *distance* for the mapping f.

Aleksandrov [1] posed the following problem:

Whether the existence of a single conservative distance for some mapping T implies that T is an isometry.

The isometric problems have been investigated in several papers (see [3, 7, 8, 10, 13, 15, 16, 18, 21, 22, 24, 28]).

Menger [17] introduced the notion of a probabilistic metric space in 1942. Since then, the theory of probabilistic metric spaces has been developed in many directions (see [2, 4, 25]). The idea of Menger was to use distribution functions as values of a meter instead of nonnegative real numbers. We shall adopt the usual terminology, notation and conventions of the theory of probabilistic Menger normed spaces as in [19, 25, 26].

Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) is denoted by Δ^+. A function $F : \mathbb{R} \to [0, 1]$ is called a *distribution function* if it is non-decreasing and left continuous with $\sup_{x\in\mathbb{R}} F(x) = 1$ and $\inf_{x\in\mathbb{R}} F(x) = 0$. The space Δ^+ is *partially ordered* by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The *maximal element* for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \le 0, \\ 1 & \text{if } t > 0. \end{cases}$$

The class of all distribution functions F with $F(0) = 0$ is denoted by D^+.

Definition 1.1. ([25]) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a *continuous triangular norm* (briefly, a t–norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a, 1) = a$ for all $a \in [0, 1]$;
(d) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Two typical example of continuous t–norm are $T(a, b) = ab$ and $T(a, b) = \min(a, b)$. A *triangle function* τ is a binary operation on D^+ which is commutative, associative and for which ε_0 is the identity, i.e,, $\tau(F, \varepsilon_0) = F$ for all $F \in D^+$.

DEFINITION 1.2. A *probabilistic 2-normed space* (briefly, P2N-space) is a triple $(\mathcal{X}, \mu, \tau)$, where $\mathcal{X}$ is a vector space of a dimension greater than one, τ is a triangle function and μ is a mapping from $\mathcal{X} \times \mathcal{X}$ into D^+ such that the following conditions hold: for all x, y in X,

(P2N1) $\mu_{x,y} = \varepsilon_0$ if x and y are linearly dependent;

(P2N2) $\mu_{x,y} \neq \varepsilon_0$ if x and y are linearly independent;

(P2N3) $\mu_{x,y} = \mu_{y,x}$;

(P2N4) $\mu_{\alpha x,y}(t) = \mu_{x,y}(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$;

(P2N5) $\mu_{x+y,z} \geq \tau(\mu_{x,z}, \mu_{y,z})$ for all $x, y, z \in X$.

Now, we will give an enlargement of the notion of probabilistic 2-normed space by generalizing the axiom which given a connection between the distribution functions of vector and its product by a real number [12].

Let φ be a function defined on the real field $\mathbb{R}$ into itself with the following properties:

(1) $\varphi(-t) = \varphi(t)$ for all $t \in \mathbb{R}$;

(2) $\varphi(1) = 1$;

(3) φ is strict increasing and continuous on $[0, \infty)$, $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Examples of such function are $\varphi(t) = |t|$, $\varphi(t) = |t|^p$ for all $p \in (0, \infty)$ and $\varphi(t) = \frac{2t^{2n}}{|t|+1}$ for all $n \in \mathbb{N}$.

DEFINITION 1.3. Let $\mathcal{X}$ be a vector space of a dimension greater than one, τ is a triangle function and μ is a mapping from X into D^+. If the condition (P2N1), (P2N2), (P2N3) and (P2N5) are satisfied and the condition (P2N4) is replaced by

(P2N4′) $\mu_{\alpha x,y}(t) = \mu_{x,y}(\frac{t}{\varphi(\alpha)})$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$,

then the triple $(\mathcal{X}, \mu, \tau)$ is called a *probabilistic φ-2-normed space.*

We recall that a *2-normed space* is a pair $(\mathcal{X}, \|\cdot, \cdot\|)$, where $\mathcal{X}$ is a linear space of a dimension greater than one and $(\|\cdot, \cdot\|)$is a real valued mapping on $\mathcal{X} \times \mathcal{X}$ such that

(N1) $\|x, y\| = 0$ if and only if x, y are linearly dependent;

(N2) $\|x, y\| = \|y, x\|$;

(N3) $\|rx, y\| = |r|\|x, y\|$ for all $r \in \mathbb{R}$ and $x, y \in \mathcal{X}$);

(N4) the triangle inequality

$$\|x, y + z\| \leq \|x, y\| + \|x, z\|, \quad \forall x, y, z \in X.$$

If the conditions (N1), (N2) and (N4) are satisfied and the condition (N3) is replaced by

(N3′) $\|rx, y\| = \varphi(r)\|x, y\|, \quad \forall r \in \mathbb{R},\ x, y \in \mathcal{X}$,

then the pair $(\mathcal{X}, \|.,\|)$ is called a *φ-2-normed space*.

For more details on 2-normed spaces, see the books ([5, 9]).

Remark 1.4. It is easy to check that every φ-2-normed space $(\mathcal{X}, \|\cdot,\cdot\|)$ can be made a probabilistic φ-2-normed space, in a natural way, by setting $\mu_{x,y}(t) = \varepsilon_0(t - \|x, y\|)$ for all $x, y \in \mathcal{X}$ and $t \in \mathbb{R}$ and $\tau_T((F, G)(t) = \sup_{t_1+t_2<t} T(F(t_1), G(t_2))$ for all $t > 0$ with $T = Min$.

Definition 1.5. ([6]) Let $\mathcal{X}$ and $\mathcal{Y}$ be probabilistic φ-2-normed spaces and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. Then f is called a 2-*isometry* if

$$\mu_{x-z,y-z} = \mu_{f(x)-f(z),f(y)-f(z)}, \quad \forall x, y, z \in \mathcal{X}.$$

In this paper, we investigate 2-isometries in probabilistic φ 2-normed spaces.

2. Main results

First, we give some lemmas for our main results in this paper.

Lemma 2.1. *Let $(\mathcal{X}, \mu, \tau)$ be a probabilistic φ-2-normed space. Then $\mu_{x,y} = \mu_{x,y+\gamma x}$ for all $x, y \in \mathcal{X}$ and $\gamma \in \mathbb{R}$.*

Proof. Let $x, y \in \mathcal{X}$. Then we have

$$\mu_{x,y+\gamma x} \geq \tau(\mu_{x,y}, \mu_{x,\gamma x}) = \tau(\mu_{x,y}, \varepsilon_0) = \mu_{x,y} \tag{2.1}$$

and

$$\begin{aligned} \mu_{x,y} = \mu_{x,y+\gamma x-\gamma x} &\geq \tau(\mu_{x,y+\gamma x}, \mu_{x,-\gamma x}) \\ &= \mu_{x,y+\gamma x}. \end{aligned} \tag{2.2}$$

Thus conclusion follows from inequalities (2.1) and (2.2). □

Definition 2.2. Let $\mathcal{X}$ be a real linear space and x, y, z be mutually disjoint elements of $\mathcal{X}$. Then the elements x, y, z are said to be *collinear* if $x - y = \alpha(x - z)$ for some $\alpha \in \mathbb{R}$.

Lemma 2.3. *Let $(\mathcal{X}, \mu, \tau)$ be a real probabilistic φ-2-normed space and let $x, y \in \mathcal{X}$. Then $\frac{x+y}{2}$ is the unique member $u \in \mathcal{X}$ satisfying*

$$\mu_{x-z,x-u}(t) = \mu_{y-u,y-z}(t) = \mu_{x-z,y-z}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big), \quad \forall t \in \mathbb{R}$$

for some $z \in \mathcal{X}$ with $\mu_{x-z,y-z} \neq \varepsilon_0$ and collinear elements $u, x, y \in \mathcal{X}$.

Proof. Set $u = \frac{x+y}{2}$. Then the elements $u, x, y \in \mathcal{X}$ are collinear. By Lemma 2.1, we have

$$\begin{aligned}\mu_{x-z,x-u}(t) &= \mu_{x-z,x-\frac{x+y}{2}}(t) = \mu_{x-z,\frac{x-y}{2}}(t)\\ &= \mu_{x-z,x-y}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big) = \mu_{x-z,y-z}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mu_{y-z,y-u}(t) &= \mu_{y-z,y-\frac{x+y}{2}}(t) = \mu_{y-z,\frac{y-x}{2}}(t)\\ &= \mu_{y-z,y-x}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big) = \mu_{x-z,y-z}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big).\end{aligned}$$

Now, we show the uniqueness $u \in \mathcal{X}$. Assume that $v \in \mathcal{X}$ is another element satisfying

$$\mu_{x-z,x-v} = \mu_{y-v,y-z} = \mu_{x-z,y-z}$$

for some $z \in \mathcal{X}$ with $\mu_{x-z,y-z} \neq \varepsilon_0$ and collinear elements $v, x, y \in \mathcal{X}$. Since v, x, y are collinear, there exists a real number r such that $v = rx + (1-r)y$. In view of Lemma 2.1, we obtain

$$\begin{aligned}\mu_{x-z,y-z}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big) &= \mu_{x-z,x-v}(t) = \mu_{x-z,(1-r)(x-y)}(t)\\ &= \mu_{x-z,x-y}\Big(\frac{t}{\varphi(1-r)}\Big)\\ &= \mu_{x-z,y-z}\Big(\frac{t}{\varphi(1-r)}\Big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mu_{x-z,y-z}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big) &= \mu_{y-v,y-z}(t) = \mu_{y-(rx+(1-r)y),y-z}(t)\\ &= \mu_{r(y-x),y-z}(t) = \mu_{y-x,y-z}\Big(\frac{t}{\varphi(r)}\Big)\\ &= \mu_{x-z,y-z}\Big(\frac{t}{\varphi(r)}\Big).\end{aligned}$$

Hence it follows that

$$\mu_{x-z,y-z}\Big(\frac{t}{\varphi(1-r)}\Big) = \mu_{x-z,y-z}\Big(\frac{t}{\varphi(r)}\Big).$$

Since $\mu_{x-z,y-z}(t) \neq \varepsilon_0$, we have $\frac{t}{\varphi(1-r)} = \frac{t}{\varphi(r)}$ and so $r = \frac{1}{2}$. Therefore, $u = v$. This completes the proof. □

Theorem 2.4. *Let $(\mathcal{X}, \mu, \tau)$ and $(\mathcal{Y}, \mu, \tau)$ be real probabilistic φ-2-normed spaces. Assume that a mapping $f : \mathcal{X} \to \mathcal{Y}$ is such that $f(x)$, $f(y)$ and $f(z)$ are collinear in $\mathcal{Y}$ when x, y and z are collinear in $\mathcal{X}$. If a mapping $f : \mathcal{X} \to \mathcal{Y}$ is a 2-isometry, then $f - f(0)$ is linear.*

Proof. Let $g(x) = f(x) - f(0)$. Then g is a 2-isometry and $g(0) = 0$. Let $x, y \in \mathcal{X}$ with $x \neq y$. Since $dim\,\mathcal{X} > 1$, there exists an element $z \in \mathcal{X}$ such that $\mu_{x-z,y-z} \neq 0$. Since g is a 2-isometry, we have

$$\begin{aligned}
\mu_{g(x)-g(z),g(x)-g\left(\frac{x+y}{2}\right)}(t) &= \mu_{x-z,x-\frac{x+y}{2}}(t) \\
&= \mu_{x-z,\frac{x-y}{2}}(t) \\
&= \mu_{x-z,x-y}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big) = \mu_{x-z,y-z}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big) \\
&= \mu_{g(x)-g(z),g(y)-g(z)}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big), \quad \forall x, y \in \mathcal{X}.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
\mu_{g(x)-g\left(\frac{x+y}{2}\right),g(y)-g(z))}(t) &= \mu_{y-\frac{x+y}{2},y-z}(t) \\
&= \mu_{\frac{x-y}{2},y-z}(t) \\
&= \mu_{y-x,y-z}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big) = \mu_{x-z,y-z}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big) \\
&= \mu_{g(x)-g(z),g(y)-g(z)}\Big(\frac{t}{\varphi(\frac{1}{2})}\Big), \quad \forall x, y \in \mathcal{X}.
\end{aligned}$$

Since $\frac{x+y}{2}$, x and y collinear, $g\left(\frac{x+y}{2}\right)$, $g(x)$ and $g(y)$ are also collinear. It follows from Lemma 2.3 that

$$g\left(\frac{x+y}{2}\right) = \frac{g(x) + g(y)}{2}, \quad \forall x, y \in \mathcal{X}.$$

Hence $g = f - f(0)$ is additive since $g(0) = 0$.

Letting $r \in \mathbb{R}^+$ with $r \neq 1$ and $x \in \mathcal{X}$, $0, x$ and rx become collinear, then $g(0)$, $g(x)$ and $g(rx)$ are collinear. Since $g(0) = 0$, there exists a real number k such that $g(rx) = kg(x)$. Because $dim\,\mathcal{X} > 1$, there exists an element $y \in \mathcal{X}$ such that $\mu_{x,y} \neq \varepsilon_0$. Thus we have

$$\begin{aligned}
\mu_{x,y}\Big(\frac{t}{\varphi(r)}\Big) &= \mu_{rx,y}(t) \\
&= \mu_{rx-0,y-0}(t) \\
&= \mu_{g(rx)-g(0),g(y)-g(0)}(t)
\end{aligned}$$

$$\begin{aligned} &= \mu_{g(rx),g(y)}(t) = \mu_{kg(x),g(y)}(t) \\ &= \mu_{g(x),g(y)}\Big(\frac{t}{\varphi(k)}\Big) = \mu_{g(x)-g(0),g(y)-g(0)}\Big(\frac{t}{\varphi(k)}\Big) \\ &= \mu_{x,y}\Big(\frac{t}{\varphi(k)}\Big) \end{aligned}$$

and so $\varphi(r) = \varphi(k)$. If $k > 0$, then $r = k$. Assume that $k < 0$. Since $\varphi(k) = \varphi(-k)$, then $r = -k$ and so $g(rx) = -rg(x)$. Since r is a positive real number, there are positive rational numbers p, q such that $p < r < q$. Since $dim\mathcal{X} > 1$, there exists an element $z \in \mathcal{X}$ such that $\mu_{rx-qx,z-qx} \neq \varepsilon_0$.

Now, we have

$$\begin{aligned} \mu_{g(x),g(z)-g(qx)}\Big(\frac{t}{\varphi(r+q)}\Big) &= \mu_{rg(x)+qg(x),g(z)-g(qx)}(t) \\ &= \mu_{-g(rx)+g(qx),g(z)-g(qx)}(t) \\ &= \mu_{rx-qx,z-qx}(t) \\ &= \mu_{x,z-qx}(\frac{t}{\varphi(q-r)}) \\ &\geq \mu_{x,z-qx}\Big(\frac{t}{\varphi(q-p)}\Big) \\ &= \mu_{qx-px,z-qx}(t) \\ &= \mu_{g(qx)-g(px),g(z)-g(qx)}(t) \\ &= \mu_{qg(x)-pg(x),g(z)-g(qx)}(t) \\ &= \mu_{g(x),g(z)-g(qx)}\Big(\frac{t}{\varphi(q-p)}\Big). \end{aligned}$$

Since $\mu_{rx-qx,z-qx} \neq \varepsilon_0$, we have

$$\mu_{g(rx)-g(qx),g(z)-g(qx)} = \mu_{(r-q)g(x),g(z)-g(qx)} \neq \varepsilon_0$$

and

$$\mu_{g(x),g(z)-g(qx)} \neq \varepsilon_0.$$

Therefore, we have $\frac{t}{\varphi(r+q)} \geq \frac{t}{\varphi(q-p)}$, which is a contradiction. So $k = r$, that is, $g(rx) = rg(x)$ for all positive real number r.

Now, we show that $g(-x) = -g(x)$. For all $x \in \mathcal{X}$, since we have

$$\mu_{g(x),g(-x)} = \mu_{g(-x)-g(0),g(0)-g(x)} = \mu_{-x,-x} = \varepsilon_0,$$

then there exists a real number k such that $g(x) = kg(-x)$. And also we have

$$\begin{aligned}\mu_{x,y}\Big(\frac{t}{\varphi(k)}\Big) &= \mu_{g(x),g(y)}\Big(\frac{t}{\varphi(k)}\Big) = \mu_{kg(-x),g(y)}\Big(\frac{t}{\varphi(k)}\Big)\\ &= \mu_{g(-x),g(y)}\Big(\frac{t}{\varphi(k)^2}\Big) = \mu_{x,y}\Big(\frac{t}{\varphi(k)^2}\Big).\end{aligned}$$

So, we have $\varphi(k) = \varphi(k)^2$ and thus $k = \pm 1$. If $k = 1$, then, for all $x \in \mathcal{X}$, we obtain $0 = g(0) = g(x - x) = 2g(x)$ since g is an additive mapping, which is a contradiction. So, $k = 1$, that is, $g(-x) = -g(x)$. Hence, for all real number r, $g(rx) = rg(x)$. This completes the proof. □

REMARK 2.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be real φ-2-normed spaces and $f : \mathcal{X} \to \mathcal{Y}$ be a 2-isometry, i.e., $\|f(x) - f(z), f(y) - f(z)\| = \|x - z, y - z\|$ for all $x, y, z \in \mathcal{X}$. Then, by Remark 1.4 and Theorem 2.4, $f - f(0)$ is linear.

3. Aleksandrov problems

Let $(\mathcal{X}, \mu, \tau)$ and $(\mathcal{Y}, \mu, \tau)$ be P2N-spaces and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. We say that f has the *area one preserving property* (AOPP) if

$$\mu_{x-z,y-z} = \varepsilon_1, \quad \forall x, y, z \in \mathcal{X},$$

implies that

$$\mu_{f(x)-f(z),f(y)-f(z)} = \varepsilon_1.$$

PROPOSITION 3.1. *Let $(\mathcal{X}, \mu, \tau)$ and $(\mathcal{Y}, \mu, \tau)$ be P2N-spaces. Assume that, if x, y and z are collinear in $\mathcal{X}$, then $f(x), f(y)$ and $f(z)$ are collinear in $\mathcal{Y}$, where $f : \mathcal{X} \to \mathcal{Y}$ is a mapping. Let f be injective and satisfies (AOPP). Then f preserves the area n for all $n \in \mathbb{N}$, (i.e., if $\mu_{x-z,y-z} = \nu_n$, then $\mu_{f(x)-f(z),f(y)-f(z)} = \nu_n$ in which $(\mathbb{R}, \nu, \tau)$ is a real PN-space).*

Proof. Let $x, y, z \in \mathcal{X}$ and let $n \in \mathbb{N}$. Let $\mu_{x-z,y-z} = \nu_n$ and $x_i = x + \frac{i}{n}(y - x)$ for all $i = 0, 1, \cdots, n$. Then we have

$$\begin{aligned}\mu_{x-z,x_{i+1}-x_i}(t) &= \mu_{x-z,x+\frac{i+1}{n}(y-x)-(x+\frac{i}{n}(y-x)}(t)\\ &= \mu_{x-z,\frac{1}{n}(y-x)}(t) = \mu_{x-z,y-x}(\frac{t}{n})\\ &= \nu_n(\frac{t}{n}) = \nu_1(t), \quad \forall i = 0, 1, 2, \cdots, n.\end{aligned}$$

Thus we have

$$\mu_{f(x)-f(z),f(x_{i+1})-f(x_i)} = \nu_1, \quad \forall i = 0, 1, 2, \cdots, n.$$

Since x_0, x_1 and x_2 are collinear, $f(x_0)$, $f(x_1)$ and $f(x_2)$ are also collinear. Hence there is a real number r such that $f(x_2)-f(x_1) = r(f(x_1)-f(x_0))$.

On the other hand, we have

$$\begin{aligned}\mu_{f(x)-f(z),f(x_1)-f(x_0)}(t) &= \mu_{f(x)-f(z),f(x_2)-f(x_1)}(t)\\ &= \mu_{f(x)-f(z),r(f(x_1)-f(x_0))}(t)\\ &= \mu_{f(x)-f(z),f(x_1)-f(x_0)}\Big(\frac{t}{|r|}\Big)\end{aligned}$$

and so $r = \pm 1$. If $r = -1$, then $f(x_2) - f(x_1) = -f(x_1) + f(x_0)$, that is, $f(x_2) = f(x_0)$. Since f is injective, $x_2 = x_0$, which is a contradiction. Thus $r = 1$. Hence we have

$$f(x_2) - f(x_1) = f(x_1) - f(x_0).$$

Similarly, one can obtain

$$f(x_{i+1}) - f(x_i) = f(x_i) - f(x_{i-1}), \quad \forall i = 0, 1, 2, \cdots, n.$$

Therefore, it follows that

$$f(x_{i+1}) - f(x_i) = f(x_1) - f(x_0), \quad \forall i = 0, 1, 2, \cdots, n$$

and hence

$$\begin{aligned}f(y) \ &- \ f(x) = f(x_n) - f(x_0)\\ &= \ f(x_n) - f(x_{n-1}) + f(x_{n-1}) - f(x_{n-2}) + \cdots + f(x_1) - f(x_0)\\ &= \ n(f(x_1) - f(x_0)).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mu_{f(x)-f(z),f(y)-f(x)}(t) &= \mu_{f(x)-f(z),n(f(x_1)-f(x_0))}(t)\\ &= \mu_{f(x)-f(z),f(x_1)-f(x_0)}(\frac{t}{n}) = \nu_1(\frac{t}{n})\\ &= \nu_n(t).\end{aligned}$$

This completes the proof. □

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