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AUTOMATIC CONTINUITY AND STABILITY OF APPROXIMATE HIGHER DERIVATIONS

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Abstract. We solve the automatic continuity problem of an approximate higher derivation on a semisimple Banach algebra and investigate Hyers-Ulam stability for higher derivations.

1. Introduction

Let A be a Banach algebra. A linear mapping $D : A \to A$ is an *ε-approximate derivation* if there is an $\varepsilon > 0$ such that

$$\|D(ab) - D(a)b - aD(b)\| \ \leq \varepsilon\|a\|\|b\|$$

for all $a, b \in A$. If $\varepsilon = 0$, we call D a derivation. Note that every continuous linear mapping T on A is a $3\|T\|$-approximate derivation. Now we define an approximate higher derivation. Let $\varepsilon_n > 0$ be given for each $n = 0, 1, \cdots, m$. A sequence $\{H_0, H_1, \cdots, H_m\}$ of linear mappings on a Banach algebra A is an (ε_n)-*approximate higher derivation of rank* m if H_0 is an identity mapping on A, $\varepsilon_n : A \times A \longrightarrow R^+$ is a mapping and

$$\left\| H_n(ab) - \sum_{i=0}^{n} H_i(a)H_{n-i}(b) \right\| \ \leq \varepsilon_n(a, b)$$

for each $n = 1, 2, \cdots, m$ and all $a,\ b \in A$. If $\varepsilon_n(a, b) = 0$ for all $n = 1, 2, \cdots, m$ and all $a,\ b \in A$, we call $\{H_0, H_1, \cdots, H_m\}$ a higher derivation of rank m. Note that if $\varepsilon_1(a, b) = \delta_1\|a\|\|b\|$ for all $a, b \in A$ then H_1 is a δ_1-approximate derivation.

In 1940, S. M. Ulam [see, 20] proposed the stability condition : *Give conditions in order to exist an additive mapping near an approximate additive mapping.* The case of approximate additive mappings between

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Banach spaces was solved by D. H. Hyers [7]. Th. M. Rassias [16] proved a substantial generalization of the result of Hyers. And many authors answered the Ulam's question for several cases [1, 3-5, 11-17].

Note that A. M. Sinclair [18] proved that every derivation on a semisimple Banach algebra is continuous. F. Gulick [6] introduced the concept of a higher derivation and N. P. Jewell [9] showed that the result of A. M. Sinclair can be extended to higher derivations.

In this paper we prove that the result of N. P. Jewell can be extended to approximate higher derivations. That is, we obtain the automatic continuity of (ε_n)-approximate higher derivations of any rank, as well as derivations and higher derivations. Also we investigate the stability of approximate higher derivations.

2. Automatic continuity of approximate higher derivations

If $S : X \to Y$ is a linear mapping from a Banach space into the Banach space then the separating space of S is defined as

$$\Omega(S) = \{y \in Y |\ \textit{there is a sequence } x_n \to 0 \textit{ in } X \textit{ with } Sx_n \to y\}.$$

By Closed Graph Theorem, we have that $\Omega(S) = 0$ if and only if S is continuous [19].

Definition 2.1. A closed ideal J of a Banach algebra A is a *separating ideal* if for every sequence $\{a_n\}$ in A there exists an integer N such that for all $n \geq N$,

$$(Ja_n \cdots a_1)^- = (Ja_N \cdots a_1)^-.$$

Note that every derivation on a Banach algebra and every epimorphism from a Banach algebra onto a Banach algebra has a separating space which is a separating ideal [9, 10]. First author [14] showed that every ε-approximate derivation also has a separating space which is a separating ideal.

The following Stability Lemma [10] for the separating space is used for proving the separating ideal. In fact, the original Stability Lemma [18] had the condition with "$R_nS = ST_n$" for all n, and then Thomas generalized this condition as "$R_nS - ST_n$ is continuous" for all n.

Lemma 2.2. *(Stability Lemma) Let X and Y be Banach spaces and let $\{T_n\}$ and $\{R_n\}$ be sequences of continuous linear operators on X and Y, respectively. If S is a linear operator from X to Y such that*

$R_nS - ST_n$ is continuous for all n, then there is an integer N such that for all $n \geq N$,

$$(R_1 \cdots R_N \Omega(S))^- = (R_1 \cdots R_n \Omega(S))^-.$$

THEOREM 2.3. *Let A be a semisimple Banach algebra, $\delta_n \geq 0$ and $\varepsilon_n(a,b) = \delta_n \|a\| \|b\|$ for $n = 1, 2, \cdots, m$ and $a, b \in A$. Then every (ε_n)-approximate higher derivation of rank m on A is continuous.*

Proof. Let $\{H_0, H_1, \cdots, H_m\}$ be an (ε_n)-approximate higher derivation of rank m on a semisimple Banach algebra A. Note that H_0 is continuous. Suppose that H_i is continuous for all $1 \leq i \leq n$. Since H_{n+1} is linear mapping, $\Omega(H_{n+1})$ is a closed linear subspace of A. If $a \in A$ and $y \in \Omega(H_{n+1})$, then there is a sequence $\{x_k\}$ in A with $x_k \to 0$ such that $H_{n+1}(x_k) \to y$ as $k \to \infty$. Then $ax_k \to 0$ in A and

$$\begin{aligned}
&\|H_{n+1}(ax_k) - ay\| \\
&\quad \leq \left\| H_{n+1}(ax_k) - \sum_{i=0}^{n+1} H_i(a) H_{n+1-i}(x_k) \right\| \\
&\qquad + \|a\| \|H_{n+1}(x_k) - y\| + \sum_{i=1}^{n+1} \|H_i(a)\| \|H_{n+1-i}(x_k)\| \\
&\quad \leq \delta_{n+1} \|a\| \|x_k\| + \|a\| \|H_{n+1}(x_k) - y\| + \sum_{i=1}^{n+1} \|H_i(a)\| \|H_{n+1-i}(x_k)\| \\
&\quad \to 0
\end{aligned}$$

as $k \to \infty$. Thus we have $ay \in \Omega(H_{n+1})$, similarly $ya \in \Omega(H_{n+1})$ and so $\Omega(H_{n+1})$ is a closed ideal of A. Now let $\{b_k\}$ be any sequence in A and we define a linear mapping $R_k = T_k$ by $R_k(y) = T_k(y) = yb_k$ for each k and $y \in A$. Then for each k

$$\begin{aligned}
&\|(R_k H_{n+1} - H_{n+1} T_k)(y)\| \\
&\quad = \|H_{n+1}(y) b_k - H_{n+1}(y b_k)\| \\
&\quad \leq \left\| H_{n+1}(y b_k) - \sum_{i=0}^{n+1} H_i(y) H_{n+1-i}(b_k) \right\| + \sum_{i=0}^{n} \|H_i(y)\| \|H_{n+1-i}(b_k)\| \\
&\quad \leq \delta_{n+1} \|y\| \|b_k\| + \sum_{i=0}^{n} \|H_i(y)\| \|H_{n+1-i}(b_k)\|
\end{aligned}$$

Thus it is easily checked that $R_k H_{n+1} - H_{n+1} T_k$ is continuous for each k. By Lemma 2.2, $\Omega(H_{n+1})$ is a separating ideal. Since A is semisimple, A has the property that for each infinite dimensional closed two sided ideal J on A there is a sequence $\{a_i\}$ in A such that

$$(Ja_n \cdots a_1)^- \supsetneq J(a_{n+1} \cdots a_1)^-$$

for all positive integer n (see the part of the proof of Corollary 9 in [10]). Thus we have $\Omega(H_{n+1})$ is finite dimensional. Let $rad(\Omega(H_{n+1}))$ be the radical of $\Omega(H_{n+1})$. Since A is semisimple, it is known [2:Corollary 24.20] that

$$rad(\Omega(H_{n+1})) = \Omega(H_{n+1}) \cap rad(A) = (0).$$

This shows that $\Omega(H_{n+1})$ is a finite dimensional semisimple Banach algebra. By Wedderburn Structure Theorem [8, p40], $\Omega(H_{n+1})$ has an identity e. Then there is a sequence $\{x_k\}$ in A such that $x_k \to 0$ and $H_{n+1}(x_k) \to e$ as $k \to \infty$. Thus

$$\lim H_{n+1}(x_k)e = e^2 = e.$$

Since $x_k e \to 0$ in $\Omega(H_{n+1})$ and $\Omega(H_{n+1})$ is finite dimensional, $H_{n+1}(x_k e)$ $\to 0$. From these facts, we have

$$\begin{aligned}
&\|H_{n+1}(x_k)e\| \\
&\quad \leq \left\| H_{n+1}(x_k e) - \sum_{i=0}^{n+1} H_i(x_k) H_{n+1-i}(e) \right\| \\
&\qquad + \|H_{n+1}(x_k e)\| + \sum_{i=0}^{n} \|H_i(x_k)\| \|H_{n+1-i}(e)\| \\
&\quad \leq \delta_{n+1} \|x_k\| \|e\| + \|H_{n+1}(x_k e)\| + \sum_{i=0}^{n} \|H_i(x_k)\| \|H_{n+1-i}(e)\| \\
&\quad \to 0
\end{aligned}$$

as $k \to \infty$. That is, $\lim H_{n+1}(x_k)e = e = 0$. For each x in $\Omega(H_{n+1})$, $xe = x = 0$. Therefore $\Omega(H_{n+1}) = (0)$. By Closed Graph Theorem, H_{n+1} is continuous. An Induction argument implies that H_n is continuous for all n. □

COROLLARY 2.4. *Every ε-approximate derivation on a semisimple Banach algebra is continuous.*

3. Stability of approximate higher derivations

Now we inveatigate the superstability of an approximate higher derivation as following :

THEOREM 3.1. *Let A be a commutative semisimple Banach algebra, $\delta_n \geq 0$ and $\varepsilon_n(a,b) = \delta_n$ for $n = 1, 2, \cdots, m$ and for all $a, b \in A$. Then every (ε_n)-approximate higher derivation of rank m on A is the zero higher derivation.*

Proof. Let $\{H_0, H_1, \cdots, H_m\}$ be an (ε_n)-approximate higher derivation on A. Note that for every $a, b \in A$

$$\|H_1(ab) - H_1(a)b - aH_1(b)\| \leq \delta_1.$$

For every $a, b, c \in A$ and any nonzero multiplicative linear functional ϕ on A,

$$\begin{aligned}
&|\phi(c)||\phi(H_1(ab) - H_1(a)b - aH_1(b))| \\
&\quad \leq |\phi(cH_1(ab) + abH_1(c) - H_1(abc))| \\
&\quad +|\phi(-bcH_1(a) - aH_1(bc) + H_1(abc))| \\
&\quad +|\phi(a(H_1(bc) - cH_1(b) - bH_1(c)))| \\
&\quad \leq 2\delta_1 + \delta_1\|a\|.
\end{aligned}$$

Dividing by $\phi(c)$ and as $|\phi(c)| \to \infty$, we have

$$\phi(H_1(ab) - H_1(a)b - aH_1(b)) = 0$$

for all $a, b \in A$ and any multiplicative linear functional ϕ on A. Since A is communicative semisimple,

$$H_1(ab) = H_1(a)b + aH_1(b)$$

for all $a, b \in A$. Thus H_1 is a derivation. By Thomas' theorem [19], Q_1 maps into the radical of A. Since A is semisimple, the radical of A is zero. Thus $H_1 = 0$. Suppose that $H_i = 0$ for all $1 \leq i \leq n$. For each $a, b \in A$ we have

$$\begin{aligned}
&\|H_{n+1}(ab) - H_{n+1}(a)b - aH_{n+1}(b)\| \\
&\quad = \|H_{n+1}(ab) - \sum_{i=0}^{n+1} H_i(a)H_{n+1-i}(b)\| \\
&\quad \leq \delta_{n+1}.
\end{aligned}$$

By the same method of the previous case for H_1, $H_{n+1} = 0$. An induction argument implies that

$$\{H_0, H_1, \cdots, H_m\} = \{H_0, 0, \cdots, 0\}.$$

Thus we complete the proof. □

Corollary 3.2. *Let D be a linear mapping on a semisimple Banach algebra A with*

$$\|D(ab) - aD(b) - D(a)b\| \leq \varepsilon$$

for all $a, b \in A$. Then $D = 0$.

Secondly we prove that every approximate higher derivation is near higher derivation. That is, we have Hyers-Ulam stability for a higher derivation as following :

Theorem 3.3. *Let A be a semisimple Banach algebra with the multiplicative norm, $\delta_n \geq 0$ and $\varepsilon_n(a,b) = \delta_n\|a\|\|b\|$ for $n = 1, 2, \cdots, m$ and for each $a, b \in A$. If $\{H_0, H_1, \cdots, H_m\}$ is an (ε_n)-approximate higher derivation such that $H_n(a)a = aH_n(a)$ for $n = 1, 2, \cdots, m$ and for each $a \in A$, then*

$$\|H_n\| \leq c_n$$

where $c_0 = 1, c_1 = \delta_1, \cdots, c_n = \sum_{i=1}^{n-1} c_i c_{n-i} + \delta_n$ for $n = 1, 2, \cdots, m$. That is, there is the zero higher derivation $\{D_0, D_1, \cdots, D_m\}$ such that for each $n = 1, 2, \cdots, m$

$$\|H_n - D_n\| \leq c_n.$$

Proof. By Theorem 2.3, H_i is continuous for each $i = 1, 2, \cdots, m$. If $\|H_1\| > \delta_1$, we can choose $a \in A$ with $\|a\| = 1$ such that $\|H_1(a)\| > \delta_1$. Let $p = \|H_1(a)\| - \delta_1 > 0$. Then we have

$$\begin{aligned}\|H_1(a^2)\| &= \|2aH_1(a) - (2aH_1(a) - H_1(a^2))\| \\ &> 2\|H_1(a)\| - \delta_1 = \delta_1 + 2p.\end{aligned}$$

By induction, we get

$$\|H_1(a^{2^k})\| > \delta_1 + 2^k p.$$

Since $\|a^{2^k}\| = 1$, it contradicts to the continuity of H_1. Therefore $\|H_1\| \leq \delta_1$. Suppose that

$$\|H_i\| \leq c_i$$

for all $1 \leq i \leq n$, where $c_0 = 1$, $c_1 = \delta_1$ and $c_n = \sum_{i=1}^{n-1} c_i c_{n-i} + \delta_n$. If $\|H_{n+1}\| > c_{n+1}$, we can choose $a \in A$ with $\|a\| = 1$ such that

$\|H_{n+1}(a)\| > c_{n+1}$. Let $p = \|H_{n+1}(a)\| - c_{n+1} > 0$. Then we have

$$\begin{aligned}
&\|H_{n+1}(a^2)\| \\
&\quad = \|2aH_{n+1}(a)\| - \left\| \left(\sum_{i=1}^{n} H_i(a)H_{n+1-i}(a) \right) \right\| \\
&\qquad - \left\| \left(\sum_{i=0}^{n+1} H_i(a)H_{n+1-i}(a) - H_{n+1}(a^2) \right) \right\| \\
&\quad \geq 2\|H_{n+1}(a)\| - \sum_{i=1}^{n} c_i c_{n+1-i} - \delta_{n+1} \\
&\quad = c_{n+1} + 2p.
\end{aligned}$$

By induction, we get

$$\|H_{n+1}(a^{2^k})\| > c_{n+1} + 2^k p.$$

Since $\|a^{2^k}\| = 1$, it contradicts to the continuity of H_{n+1}. Thus we have $\|H_{n+1}\| \leq c_{n+1}$. By induction, we complete the proof. □

Corollary 3.4. *Let A be a semisimple Banach algebra with the multiplicative norm. If D is an ε-approximate derivation such that $D(a)a = aD(a)$ for all $a \in A$, then $\|D\| \leq \varepsilon$.*

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