J. Korean Math. Soc. **46** (2009), No. 1, pp. 187–213

ON THE MEAN VALUES OF DEDEKIND SUMS AND HARDY SUMS

HUANING LIU

ABSTRACT. For a positive integer k and an arbitrary integer h, the classical Dedekind sums $s(h,k)$ is defined by

$$s(h,k)=\sum_{j=1}^{k}\left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\dfrac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

J. B. Conrey et al proved that

$$\sum_{\substack{h=1\\(h,k)=1}}^{k} s^{2m}(h,k)=f_m(k)\left(\frac{k}{12}\right)^{2m}+O\left(\left(k^{\frac{9}{5}}+k^{2m-1+\frac{1}{m+1}}\right)\log^3 k\right).$$

For $m\geq 2$, C. Jia reduced the error terms to $O\left(k^{2m-1}\right)$. While for $m=1$, W. Zhang showed

$$\sum_{\substack{h=1\\(h,k)=1}}^{k} s^2(h,k)=\frac{5}{144}k\phi(k)\prod_{p^\alpha\|k}\left[\frac{\left(1+\frac{1}{p}\right)^2-\frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}\right]$$
$$+O\left(k\exp\left(\frac{4\log k}{\log\log k}\right)\right)$$

In this paper we give some formulae on the mean value of the Dedekind sums and and Hardy sums, and generalize the above results.

§ 1. Introduction

For a positive integer k and an arbitrary integer h, the classical Dedekind sums $s(h,k)$ is defined by

Received July 3, 2007.

2000 *Mathematics Subject Classification.* 11F20.

Key words and phrases. Dedekind sums, Hardy sums, mean value, asymptotic formula.

Research supported by the National Grand Fundamental Research 973 Program of China under Grant No.2007CB807902 and No.2007CB807903, the National Natural Science Foundation of China under Grant No.10671155; the Natural Science Foundation of Shaanxi province of China under Grant No.2006A04; and the funded projects of excellent Doctoral Thesis of Northwest University under Grant No.07YYB04.

$$s(h,k)=\sum_{j=1}^{k}\left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

The sum $s(h,k)$ plays an important role in the transformation theory of the Dedekind η function (see [17] and Chapter 3 of [1] for details).

Perhaps the most famous property of the Dedekind sums is the reciprocity formula

$$s(h,k)+s(k,h)=\frac{h^2+k^2+1}{12hk}-\frac{1}{4} \tag{1.1}$$

for positive coprime integers h and k. R. Dedekind [9], H. Rademacher [17], B. C. Berndt [2-4] and U. Dieter [10] gave different proofs for this famous reciprocity formula. Some three term versions of this formula were discovered by H. Rademacher [17], R. R. Hall [11], and J. Pommersheim [16].

Suppose that a, q, h, $k>0$ with $(a,q)=1$ and $(h,k)=1$. Suppose further that $z=qh-ak$ satisfies $|z|\le \frac{k}{q}$. J. B. Conrey, E. Fransen, R. Klein, and C. Scott [8] showed that

$$s(h,k)=\frac{k}{12qz}+O\left(|s\left(a,q\right)|+|z|+1\right). \tag{1.2}$$

Then they studied the $2m$-th power mean of Dedekind sums, and proved the following proposition by using the circle method.

Proposition 1.1. *Suppose that m is a given positive integer and k is any sufficiently large integer. Then*

$$\sum_{\substack{h=1\\(h,k)=1}}^{k} s^{2m}(h,k)=f_m(k)\left(\frac{k}{12}\right)^{2m}+O\left(\left(k^{\frac{9}{5}}+k^{2m-1+\frac{1}{m+1}}\right)\log^3 k\right),$$

where $f_m(k)$ is defined by the Dirichlet series

$$\sum_{k=1}^{+\infty}\frac{f_m(k)}{k^s}=2\cdot\frac{\zeta^2(2m)}{\zeta(4m)}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\cdot\zeta(s),$$

and $\zeta(s)$ is the Riemann zeta-function.

In [13], C. Jia improved the error terms in Proposition 1.1, and proved the following:

Proposition 1.2. *For every given integer $m\ge 2$ and any sufficiently large integer k, we have*

$$\sum_{\substack{h=1\\(h,k)=1}}^{k} s^{2m}(h,k)=f_m(k)\left(\frac{k}{12}\right)^{2m}+O\left(k^{2m-1}\right).$$

How to reduce the error terms in Proposition 1.1 for $m = 1$? Recall that H. Walum [19] established a relation between Dedekind sums and Dirichlet L-functions as following:

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1,\chi)|^4 = \frac{\pi^4(p-1)}{p^2} \sum_{h=1}^{p} |s(h,p)|^2,$$

where p is a prime number, $L(1,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n}$ is the Dirichlet L-functions, and χ denotes a Dirichlet character modulo p. In the spirit of [8] and [19], W. Zhang [20] showed that

$$s(h,k) = \frac{1}{\pi^2 k} \sum_{d|k} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(h)\, |L(1,\chi)|^2,$$

where $\phi(d)$ is the Euler function, and finally in [21] he proved the following:

Proposition 1.3. *For any sufficiently large integer k, we have*

(1.3)
$$\sum_{\substack{h=1 \\ (h,k)=1}}^{k} s^2(h,k) = \frac{5}{144} k\phi(k) \prod_{p^\alpha \| k} \left[\frac{\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}} \right] + O\left(k \exp\left(\frac{4\log k}{\log\log k}\right)\right),$$

where $\prod_{p^\alpha \| k}$ denotes the product over all prime divisors p of k with $p^\alpha \mid k$ and $p^{\alpha+1} \nmid k$, and $\exp(y) = e^y$.

Taking $m = 1$ in Proposition 1.1, we get

$$\sum_{\substack{h=1 \\ (h,k)=1}}^{k} s^2(h,k) = \frac{1}{144} f_1(k) k^2 + O\left(k^{\frac{9}{5}} \log^3 k\right). \tag{1.4}$$

Compare (1.3) and (1.4), we have

$$f_1(k) = 5 \cdot \frac{\phi(k)}{k} \prod_{p^\alpha \| k} \left[\frac{\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}} \right].$$

Does $f_m(k)$ have similar form for $m > 1$? If yes, then we can get some interesting result from Propositions 1.1, 1.2, and 1.3. In Section 2 we shall prove the following:

Theorem 1.1. *For any given integer $m \geq 1$ and any sufficiently large integer k, we have the identity*

$$f_m(k) = \frac{2\zeta^2(2m)}{\zeta(4m)} \prod_{p^\alpha \| k} \left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2 - \frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}} \right].$$

Taking $m=1$ in Theorem 1.1, we immediately get

$$\begin{aligned} f_1(k) &= 5\prod_{p^\alpha\|k}\left[\frac{\left(1-\frac{1}{p^2}\right)^2-\frac{1}{p^{3\alpha+1}}\left(1-\frac{1}{p}\right)^2}{1-\frac{1}{p^3}}\right] \\ &= 5\cdot\frac{\phi(k)}{k}\prod_{p^\alpha\|k}\left[\frac{\left(1-\frac{1}{p^2}\right)^2-\frac{1}{p^{3\alpha+1}}\left(1-\frac{1}{p}\right)^2}{\left(1-\frac{1}{p^3}\right)\left(1-\frac{1}{p}\right)}\right] \\ &= 5\cdot\frac{\phi(k)}{k}\prod_{p^\alpha\|k}\left[\frac{\left(1+\frac{1}{p}\right)^2-\frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}\right]. \end{aligned}$$

Then from Propositions 1.1-1.3 and Theorem 1.1 we have

Corollary 1.1. *Suppose that m is a given positive integer and k is any sufficiently large integer. Then*

$$\sum_{\substack{h=1\\(h,k)=1}}^{k} s^{2m}(h,k)$$

$$= \frac{2\zeta^2(2m)}{\zeta(4m)}\left(\frac{k}{12}\right)^{2m}\prod_{p^\alpha\|k}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right]+e_m(k),$$

where

$$e_m(k)\ll\begin{cases} k\exp\left(\frac{4\log k}{\log\log k}\right), & \text{if } m=1;\\ k^{2m-1}, & \text{if } m>1.\end{cases}$$

B. C. Berndt [5] defined the following Hardy sums:

$$S(h,k)=\sum_{j=1}^{k-1}(-1)^{j+1+\left[\frac{hj}{k}\right]}, \qquad s_1(h,k)=\sum_{j=1}^{k}(-1)^{\left[\frac{hj}{k}\right]}\left(\left(\frac{j}{k}\right)\right),$$

$$s_2(h,k)=\sum_{j=1}^{k}(-1)^{j}\left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{hj}{k}\right)\right), \qquad s_3(h,k)=\sum_{j=1}^{k}(-1)^{j}\left(\left(\frac{hj}{k}\right)\right),$$

$$s_4(h,k)=\sum_{j=1}^{k-1}(-1)^{\left[\frac{hj}{k}\right]}, \qquad s_5(h,k)=\sum_{j=1}^{k}(-1)^{j+\left[\frac{hj}{k}\right]}\left(\left(\frac{j}{k}\right)\right),$$

and studied their arithmetical properties in [6]. For $(h,k)=1$, R. Sitaramachandrarao [18] and M. R. Pettet [15] expressed Hardy sums in terms of Dedekind sum $s(h,k)$ as following:

$$S(h,k)=\begin{cases} 8s(h,2k)+8s(2h,k)-20s(h,k), & \text{if } h+k \text{ is odd};\\ 0, & \text{if } h+k \text{ is even},\end{cases}$$

$$s_1(h,k)=\begin{cases} 2s(h,k)-4s(h,2k), & \text{if } h \text{ is even;}\\ 0, & \text{if } h \text{ is odd,}\end{cases}$$

$$s_2(h,k)=\begin{cases} -s(h,k)+2s(2h,k), & \text{if } k \text{ is even;}\\ 0, & \text{if } k \text{ is odd,}\end{cases}$$

$$s_3(h,k)=\begin{cases} 2s(h,k)-4s(2h,k), & \text{if } k \text{ is odd;}\\ 0, & \text{if } k \text{ is even,}\end{cases}$$

$$s_4(h,k)=\begin{cases} -4s(h,k)+8s(h,2k), & \text{if } h \text{ is odd;}\\ 0, & \text{if } h \text{ is even,}\end{cases}$$

$$s_5(h,k)=\begin{cases} -10s(h,k)+4s(2h,k)+4s(h,2k), & \text{if } h+k \text{ is even;}\\ 0, & \text{if } h+k \text{ is odd.}\end{cases}$$

For $s_1(h,k)$, the author [14] proved the following:

Proposition 1.4. *For any fixed integer $m \geq 2$ and any sufficiently large odd number k, we have the asymptotic formula*

$$\sum_{\substack{h=1\\(h,k)=1\\2|h}}^{k} s_1^{2m}(h,k)=g_m(k)\left(\frac{k}{2}\right)^{2m}+O\left(k^{2m-1}\right),$$

where $g_m(k)$ is defined by the Dirichlet series

$$\sum_{\substack{k=1\\2\nmid k}}^{+\infty}\frac{g_m(k)}{k^s}=\frac{\left(2^{s+4m}-2\right)\left(2^s-1\right)}{\left(2^{s+2m}-1\right)^2\left(2^{2m}+1\right)}\cdot\frac{\zeta^2(2m)}{\zeta(4m)}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\cdot\zeta(s).$$

While X. Chen and W. Zhang [7] got the following:

Proposition 1.5. *For any odd number $k>2$, we have*

$$\sum_{\substack{h=1\\(h,k)=1\\2|h}}^{k} s_1^2(h,k)=\frac{1}{8}k\phi(k)\prod_{p^\alpha\|k}\left[\frac{\left(1+\frac{1}{p}\right)^2-\frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}\right]+O\left(k\exp\left(\frac{4\log k}{\log\log k}\right)\right).$$

In Section 3 we shall prove the following:

Theorem 1.2. *For any given integer $m\geq 2$ and any sufficiently large odd number k, we have the identity*

$$g_m(k)=\frac{\zeta^2(2m)}{\left(2^{2m}+1\right)\zeta(4m)}\prod_{p^\alpha\|k}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right].$$

Let $m = 1$, we immediately get

$$g_1(k) = \frac{1}{2}\prod_{p^\alpha \| k}\left[\frac{\left(1-\frac{1}{p^2}\right)^2 - \frac{1}{p^{3\alpha+1}}\left(1-\frac{1}{p}\right)^2}{1-\frac{1}{p^3}}\right] = \frac{1}{2}\cdot\frac{\phi(k)}{k}\prod_{p^\alpha \| k}\left[\frac{\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}\right].$$

Then from Propositions 1.4, 1.5, and Theorem 1.2 we have

Corollary 1.2. *Suppose that m is a given positive integer and k is any sufficiently large odd number. Then*

$$\sum_{\substack{h=1\\(h,k)=1\\2|h}}^{k} s_1^{2m}(h,k)$$

$$= \frac{\zeta^2(2m)}{(2^{2m}+1)\zeta(4m)}\left(\frac{k}{2}\right)^{2m}\prod_{p^\alpha \| k}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2 - \frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right] + e'_m(k),$$

where

$$e'_m(k) \ll \begin{cases} k\exp\left(\frac{4\log k}{\log\log k}\right), & \text{if } m = 1;\\ k^{2m-1}, & \text{if } m > 1.\end{cases}$$

In [22], W. Zhang obtained the following:

Proposition 1.6. *Let $k = 2^\beta M$ be an integer with $\beta \geq 1$ and $2 \nmid M$. Then we have*

$$\sum_{\substack{h=1\\(h,k)=1}}^{k} s_2^2(h,k) = \frac{5}{112}k\phi(k)\left(\frac{3}{5}-\frac{2}{2^{3\beta}}\right)\prod_{p^\alpha \| M}\left[\frac{\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}\right] + O\left(k\exp\left(\frac{4\log k}{\log\log k}\right)\right).$$

We shall study the $2m$-th power mean of $s_2(h,k)$ in Section 4, and prove the following:

Theorem 1.3. *For any fixed integer $m \geq 2$ and any sufficiently large even number k, we have the asymptotic formula*

$$\sum_{\substack{h=1\\(h,k)=1}}^{k} s_2^{2m}(h,k) = h_m(k)\left(\frac{k}{4}\right)^{2m} + O\left(k^{2m-1}\right),$$

where

$$h_m(k) = \sum_{\substack{q=1\\2|q}}^{+\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\(a,q)=1\\2\nmid a}}^{q}\sum_{\substack{h=-\infty\\(h,k)=1\\2\nmid h\\qh-ak\neq 0}}^{+\infty}\frac{1}{(qh-ak)^{2m}}.$$

In Section 5 we shall further show the following:

Theorem 1.4. *Let $k = 2^\beta M$ be an integer with $\beta \geq 1$ and $2 \nmid M$. Then for any integer $m \geq 2$, we have the identity*

$$h_m(k) = \frac{2^{2m}\zeta^2(2m)}{(2^{2m}+1)\zeta(4m)}\left(\frac{2^{2m}-1}{2^{2m}(2^{4m-1}-1)} - \frac{2^{6m-2}+1-2^{4m-1}-2^{4m-2}}{(2^{4m-1}-1)2^{(4m-1)\beta+2m-1}}\right) \times \prod_{p^\alpha \| M}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2 - \frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right].$$

Let $m = 1$, we immediately get

$$\begin{aligned} h_1(k) &= \frac{5}{14}\left(\frac{3}{5}-\frac{2}{2^{3\beta}}\right)\prod_{p^\alpha\|M}\left[\frac{\left(1-\frac{1}{p^2}\right)^2 - \frac{1}{p^{3\alpha+1}}\left(1-\frac{1}{p}\right)^2}{1-\frac{1}{p^3}}\right] \\ &= \frac{5}{14}\left(\frac{3}{5}-\frac{2}{2^{3\beta}}\right)\frac{\phi(M)}{M}\prod_{p^\alpha\|M}\left[\frac{\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}\right] \\ &= \frac{5}{14}\left(\frac{3}{5}-\frac{2}{2^{3\beta}}\right)\frac{2^\beta}{\phi(2^\beta)}\frac{\phi(k)}{k}\prod_{p^\alpha\|M}\left[\frac{\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}\right] \\ &= \frac{5}{7}\left(\frac{3}{5}-\frac{2}{2^{3\beta}}\right)\frac{\phi(k)}{k}\prod_{p^\alpha\|M}\left[\frac{\left(1+\frac{1}{p}\right)^2 - \frac{1}{p^{3\alpha+1}}}{1+\frac{1}{p}+\frac{1}{p^2}}\right]. \end{aligned}$$

Then from Proposition 1.6, Theorem 1.3 and Theorem 1.4 we have

Corollary 1.3. *Suppose that m is a given positive integer and $k = 2^\beta M$ is any sufficiently large even number with $\beta \geq 1$ and $2 \nmid M$. Then*

$$\begin{aligned} &\sum_{\substack{h=1 \\ (h,k)=1}}^{k} s_2^{2m}(h,k) \\ =&\frac{2^{2m}\zeta^2(2m)}{(2^{2m}+1)\zeta(4m)}\left(\frac{2^{2m}-1}{2^{2m}(2^{4m-1}-1)} - \frac{2^{6m-2}+1-2^{4m-1}-2^{4m-2}}{(2^{4m-1}-1)2^{(4m-1)\beta+2m-1}}\right)\left(\frac{k}{4}\right)^{2m} \\ &\times \prod_{p^\alpha\|M}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2 - \frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right] + e''_m(k), \end{aligned}$$

where

$$e''_m(k) \ll \begin{cases} k\exp\left(\frac{4\log k}{\log\log k}\right), & \text{if } m = 1; \\ k^{2m-1}, & \text{if } m > 1. \end{cases}$$

On the other hand, from Proposition 1.4 and Theorem 1.2 we have

$$\sum_{\substack{k=1\\2\nmid k}}^{+\infty}\frac{1}{k^s}\prod_{p^\alpha\|k}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right]$$

$$=\frac{\left(2^{s+4m}-2\right)\left(2^s-1\right)}{\left(2^{s+2m}-1\right)^2}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\cdot\zeta(s).$$

Therefore

$$\sum_{\substack{k=1\\2|k}}^{+\infty}\frac{h_m(k)}{k^s}=\sum_{\beta=1}^{+\infty}\sum_{\substack{k=1\\2^\beta\|k}}^{+\infty}\frac{h_m(k)}{k^s}=\sum_{\beta=1}^{+\infty}\sum_{\substack{M=1\\2\nmid M}}^{+\infty}\frac{h_m(2^\beta M)}{(2^\beta M)^s}$$

$$=\frac{2^{2m}\zeta^2(2m)}{(2^{2m}+1)\,\zeta(4m)}\sum_{\beta=1}^{+\infty}\frac{1}{2^{\beta s}}\left(\frac{2^{2m}-1}{2^{2m}\left(2^{4m-1}-1\right)}-\frac{2^{6m-2}+1-2^{4m-1}-2^{4m-2}}{\left(2^{4m-1}-1\right)2^{(4m-1)\beta+2m-1}}\right)$$

$$\times\sum_{\substack{M=1\\2\nmid M}}^{+\infty}\frac{1}{M^s}\prod_{p^\alpha\|M}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right]$$

$$=\frac{2^{s+2}+2^{2m+1}-6}{\left(2^{s+2m}-1\right)^2\left(2^{2m}+1\right)}\cdot\frac{\zeta^2(2m)}{\zeta(4m)}\cdot\frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\cdot\zeta(s).$$

Using the same methods, one can get similar interesting results for other Hardy sums.

§ 2. Proof of Theorem 1.1

First we prove the following lemma.

Lemma 2.1. *For any positive integers k and m, we have*

$$\Lambda_1:=\sum_{d|k}\frac{\mu(d)}{d^{2m}}\sum_{g|\frac{k}{d}}\frac{1}{g^{2m-1}}\sum_{\substack{q=1\\(q,\frac{k}{d})=g}}^{+\infty}\frac{1}{q^{2m}}$$

$$=\zeta(2m)\prod_{p^\alpha\|k}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right].$$

Proof. By use of the Möbius relation

$$(2.1)\qquad \sum_{d|n}\mu(d)=\begin{cases}1, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases}$$

we have

$$\sum_{\substack{q=1\\(q,\frac{k}{d})=g}}^{+\infty}\frac{1}{q^{2m}} = \frac{1}{g^{2m}}\sum_{\substack{q=1\\(q,\frac{k}{dg})=1}}^{+\infty}\frac{1}{q^{2m}} = \frac{1}{g^{2m}}\sum_{q=1}^{+\infty}\frac{1}{q^{2m}}\sum_{\substack{t|q\\t|\frac{k}{dg}}}\mu(t)$$

$$= \frac{1}{g^{2m}}\sum_{t|\frac{k}{dg}}\frac{\mu(t)}{t^{2m}}\sum_{q=1}^{+\infty}\frac{1}{q^{2m}} = \zeta(2m)\cdot\frac{1}{g^{2m}}\sum_{t|\frac{k}{dg}}\frac{\mu(t)}{t^{2m}}.$$

This gives

$$\Lambda_1 = \zeta(2m)\sum_{d|k}\frac{\mu(d)}{d^{2m}}\sum_{g|\frac{k}{d}}\frac{1}{g^{4m-1}}\sum_{t|\frac{k}{dg}}\frac{\mu(t)}{t^{2m}}. \tag{2.2}$$

On the other hand, from the properties of multiplicative functions we have (2.3)

$$\sum_{d|k}\frac{\mu(d)}{d^{2m}}\sum_{g|\frac{k}{d}}\frac{1}{g^{4m-1}}\sum_{t|\frac{k}{dg}}\frac{\mu(t)}{t^{2m}} = \prod_{p^\alpha\|k}\left[\sum_{d|p^\alpha}\frac{\mu(d)}{d^{2m}}\sum_{g|\frac{p^\alpha}{d}}\frac{1}{g^{4m-1}}\sum_{t|\frac{p^\alpha}{dg}}\frac{\mu(t)}{t^{2m}}\right]$$

$$= \prod_{p^\alpha\|k}\left[\sum_{g|p^\alpha}\frac{1}{g^{4m-1}}\sum_{t|\frac{p^\alpha}{g}}\frac{\mu(t)}{t^{2m}} - \frac{1}{p^{2m}}\sum_{g|p^{\alpha-1}}\frac{1}{g^{4m-1}}\sum_{t|\frac{p^{\alpha-1}}{g}}\frac{\mu(t)}{t^{2m}}\right]$$

$$= \prod_{p^\alpha\|k}\left[\frac{1-\frac{1}{p^{2m}}+\frac{1}{p^{(4m-1)\alpha+2m}}-\frac{1}{p^{(4m-1)(\alpha+1)}}}{1-\frac{1}{p^{4m-1}}} - \frac{\frac{1}{p^{2m}}\left(1-\frac{1}{p^{2m}}+\frac{1}{p^{(4m-1)(\alpha-1)+2m}}-\frac{1}{p^{(4m-1)\alpha}}\right)}{1-\frac{1}{p^{4m-1}}}\right]$$

$$= \prod_{p^\alpha\|k}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right].$$

Then from (2.2) and (2.3) we have

$$\Lambda_1 = \zeta(2m)\prod_{p^\alpha\|k}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right].$$

□

Now we prove Theorem 1.1. From [8] or [13] we know that

$$f_m(k) = \sum_{q=1}^{+\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{\substack{h=-\infty\\(h,k)=1\\qh-ak\neq 0}}^{+\infty}\frac{1}{(qh-ak)^{2m}}.$$

By (2.1) we get

$$
\begin{aligned}
f_m(k) &= \sum_{q=1}^{+\infty}\frac{1}{q^{2m}}\sum_{a=1}^{q}\sum_{\substack{t|q\\ t|a}}\mu(t)\sum_{\substack{h=-\infty\\ qh-ak\neq 0}}^{+\infty}\frac{1}{(qh-ak)^{2m}}\sum_{\substack{d|h\\ d|k}}\mu(d)\\
&= \sum_{t=1}^{+\infty}\frac{\mu(t)}{t^{4m}}\sum_{q=1}^{+\infty}\frac{1}{q^{2m}}\sum_{a=1}^{q}\sum_{\substack{h=-\infty\\ qh-ak\neq 0}}^{+\infty}\frac{1}{(qh-ak)^{2m}}\sum_{\substack{d|h\\ d|k}}\mu(d)\\
&= \frac{1}{\zeta(4m)}\sum_{d|k}\frac{\mu(d)}{d^{2m}}\sum_{q=1}^{+\infty}\frac{1}{q^{2m}}\sum_{a=1}^{q}\sum_{\substack{h=-\infty\\ qh-\frac{ak}{d}\neq 0}}^{+\infty}\frac{1}{\left(qh-\frac{ak}{d}\right)^{2m}}.
\end{aligned}
\tag{2.4}
$$

Let $g=\left(q,\frac{k}{d}\right)$. Then

$$
\begin{aligned}
&\sum_{a=1}^{q}\sum_{\substack{h=-\infty\\ qh-\frac{ak}{d}\neq 0}}^{+\infty}\frac{1}{(qh-\frac{ak}{d})^{2m}}\\
&= \frac{1}{g^{2m}}\sum_{a=1}^{q}\sum_{\substack{h=-\infty\\ \frac{qh}{g}-\frac{ak}{dg}\neq 0}}^{+\infty}\frac{1}{\left(\frac{qh}{g}-\frac{ak}{dg}\right)^{2m}} = \frac{1}{g^{2m}}\sum_{a=1}^{q}\sum_{\substack{z=-\infty\\ z\neq 0\\ z\equiv-\frac{ak}{dg}\left(\bmod \frac{q}{g}\right)}}^{+\infty}\frac{1}{z^{2m}}\\
&= \frac{1}{g^{2m}}\sum_{a=1}^{q}\sum_{\substack{z=-\infty\\ z\neq 0\\ z\equiv a\left(\bmod \frac{q}{g}\right)}}^{+\infty}\frac{1}{z^{2m}} = \frac{1}{g^{2m-1}}\sum_{\substack{z=-\infty\\ z\neq 0}}^{+\infty}\frac{1}{z^{2m}} = \frac{2\zeta(2m)}{g^{2m-1}}.
\end{aligned}
\tag{2.5}
$$

Now from (2.4), (2.5), and Lemma 2.1 we have

$$
\begin{aligned}
f_m(k) &= \frac{2\zeta(2m)}{\zeta(4m)}\sum_{d|k}\frac{\mu(d)}{d^{2m}}\sum_{g|\frac{k}{d}}\frac{1}{g^{2m-1}}\sum_{\substack{q=1\\ \left(q,\frac{k}{d}\right)=g}}^{+\infty}\frac{1}{q^{2m}}\\
&= \frac{2\zeta^2(2m)}{\zeta(4m)}\prod_{p^\alpha\|k}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right].
\end{aligned}
$$

This proves Theorem 1.1.

§ 3. Proof of Theorem 1.2

First we have

Lemma 3.1. *For any integer $m \geq 1$ and positive odd number k, we have*

$$\Lambda_2 := \sum_{d|k} \frac{\mu(d)}{d^{2m}} \sum_{g|\frac{k}{d}} \frac{1}{g^{2m-1}} \sum_{\substack{q=1 \\ 2\nmid q \\ (q,\frac{k}{d})=g}}^{+\infty} \frac{1}{q^{2m}}$$

$$= \frac{(2^{2m}-1)}{2^{2m}} \cdot \zeta(2m) \prod_{p^\alpha \| k} \left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2 - \frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}} \right].$$

Proof. By (2.1) we get

$$\sum_{\substack{q=1 \\ 2\nmid q \\ (q,\frac{k}{d})=g}}^{+\infty} \frac{1}{q^{2m}} = \frac{1}{g^{2m}} \sum_{\substack{q=1 \\ 2\nmid q \\ (q,\frac{k}{dg})=1}}^{+\infty} \frac{1}{q^{2m}} = \frac{1}{g^{2m}} \sum_{\substack{q=1 \\ 2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{t|q \\ t|\frac{k}{dg}}} \mu(t)$$

$$= \frac{1}{g^{2m}} \sum_{t|\frac{k}{dg}} \frac{\mu(t)}{t^{2m}} \sum_{\substack{q=1 \\ 2\nmid q}}^{+\infty} \frac{1}{q^{2m}} = \frac{(2^{2m}-1)}{2^{2m}} \cdot \zeta(2m) \cdot \frac{1}{g^{2m}} \sum_{t|\frac{k}{dg}} \frac{\mu(t)}{t^{2m}}.$$

Then from (2.3) we have

$$\Lambda_2 = \frac{(2^{2m}-1)}{2^{2m}} \cdot \zeta(2m) \cdot \sum_{d|k} \frac{\mu(d)}{d^{2m}} \sum_{g|\frac{k}{d}} \frac{1}{g^{4m-1}} \sum_{t|\frac{k}{dg}} \frac{\mu(t)}{t^{2m}}$$

$$= \frac{(2^{2m}-1)}{2^{2m}} \cdot \zeta(2m) \prod_{p^\alpha \| k} \left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2 - \frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}} \right].$$

□

Now we prove Theorem 1.2. From [14] we know that

$$g_m(k) = \sum_{\substack{q=1 \\ 2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ (a,q)=1 \\ 2|a}}^{q} \sum_{\substack{h=-\infty \\ (h,k)=1 \\ 2|h \\ qh-ak\neq 0}}^{+\infty} \frac{1}{(qh-ak)^{2m}} + \sum_{\substack{h=1 \\ (h,k)=1 \\ 2|h}}^{+\infty} \frac{1}{h^{2m}} \tag{3.1}$$

$$:= g'_m(k) + g''_m(k).$$

Noting that k is odd. It is easy to show that

$$
\begin{aligned}
g''_m(k) &= \sum_{\substack{h=1\\(h,k)=1\\2|h}}^{+\infty} \frac{1}{h^{2m}} = \sum_{\substack{h=1\\2|h}}^{+\infty} \frac{1}{h^{2m}} \sum_{\substack{d|h\\d|k}} \mu(d) \\
&= \sum_{d|k} \frac{\mu(d)}{d^{2m}} \sum_{\substack{h=1\\2|h}}^{+\infty} \frac{1}{h^{2m}} = \frac{\zeta(2m)}{2^{2m}} \sum_{d|k} \frac{\mu(d)}{d^{2m}}.
\end{aligned} \tag{3.2}
$$

On the other hand, by (2.1) we get

$$
\begin{aligned}
g'_m(k) &= \sum_{\substack{q=1\\2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\(a,q)=1\\2|a}}^{q} \sum_{\substack{h=-\infty\\(h,k)=1\\2|h\\qh-ak\neq 0}}^{+\infty} \frac{1}{(qh-ak)^{2m}} \\
&= \sum_{\substack{q=1\\2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\2|a}}^{q} \sum_{\substack{t|q\\t|a}} \mu(t) \sum_{\substack{h=-\infty\\2|h\\qh-ak\neq 0}}^{+\infty} \frac{1}{(qh-ak)^{2m}} \sum_{\substack{d|h\\d|k}} \mu(d) \\
&= \sum_{\substack{t=1\\2\nmid t}}^{+\infty} \frac{\mu(t)}{t^{4m}} \sum_{\substack{q=1\\2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\2|a}}^{q} \sum_{\substack{h=-\infty\\2|h\\qh-ak\neq 0}}^{+\infty} \frac{1}{(qh-ak)^{2m}} \sum_{\substack{d|h\\d|k}} \mu(d) \\
&= \frac{2^{4m}}{(2^{4m}-1)\zeta(4m)} \sum_{d|k} \frac{\mu(d)}{d^{2m}} \sum_{\substack{q=1\\2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\2|a}}^{q} \sum_{\substack{h=-\infty\\2|h\\qh-\frac{ak}{d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{d}\right)^{2m}}.
\end{aligned} \tag{3.3}
$$

Let $g = \left(q, \frac{k}{d}\right)$. Then

$$
\begin{aligned}
\sum_{\substack{a=1\\2|a}}^{q} \sum_{\substack{h=-\infty\\2|h\\qh-\frac{ak}{d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{d}\right)^{2m}} &= \frac{1}{g^{2m}} \sum_{\substack{a=1\\2|a}}^{q} \sum_{\substack{h=-\infty\\2|h\\\frac{qh}{g}-\frac{ak}{dg}\neq 0}}^{+\infty} \frac{1}{\left(\frac{qh}{g}-\frac{ak}{dg}\right)^{2m}} \\
&= \frac{1}{2^{2m} g^{2m}} \sum_{a=1}^{(q-1)/2} \sum_{\substack{h=-\infty\\\frac{qh}{g}-\frac{ak}{dg}\neq 0}}^{+\infty} \frac{1}{\left(\frac{qh}{g}-\frac{ak}{dg}\right)^{2m}}
\end{aligned} \tag{3.4}
$$

$$= \frac{1}{2^{2m}g^{2m}} \sum_{a=1}^{(q-1)/2} \sum_{\substack{z=-\infty \\ z\neq 0 \\ z\equiv -\frac{ak}{dg}(\bmod \frac{q}{g})}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{1}{2^{2m}g^{2m}} \sum_{a=1}^{(q-1)/2} \sum_{\substack{z=1 \\ z\equiv -\frac{ak}{dg}(\bmod \frac{q}{g})}}^{+\infty} \frac{1}{z^{2m}} + \frac{1}{2^{2m}g^{2m}} \sum_{a=1}^{(q-1)/2} \sum_{\substack{z=-\infty \\ z\equiv -\frac{ak}{dg}(\bmod \frac{q}{g})}}^{-1} \frac{1}{z^{2m}}$$

$$= \frac{1}{2^{2m}g^{2m}} \sum_{a=1}^{q-1} \sum_{\substack{z=1 \\ z\equiv -\frac{ak}{dg}(\bmod \frac{q}{g})}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{1}{2^{2m}g^{2m}} \sum_{a=1}^{q} \sum_{\substack{z=1 \\ z\equiv -\frac{ak}{dg}(\bmod \frac{q}{g})}}^{+\infty} \frac{1}{z^{2m}} - \frac{1}{2^{2m}g^{2m}} \sum_{\substack{z=1 \\ z\equiv 0(\bmod \frac{q}{g})}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{1}{2^{2m}g^{2m-1}} \sum_{z=1}^{+\infty} \frac{1}{z^{2m}} - \frac{1}{2^{2m}q^{2m}} \sum_{z=1}^{+\infty} \frac{1}{z^{2m}} = \frac{\zeta(2m)}{2^{2m}g^{2m-1}} - \frac{\zeta(2m)}{2^{2m}q^{2m}}.$$

Now from (3.3) and (3.4) we have

$$\begin{aligned} g'_m(k) \quad = \quad & \frac{2^{2m}}{(2^{4m}-1)} \cdot \frac{\zeta(2m)}{\zeta(4m)} \cdot \sum_{d|k} \frac{\mu(d)}{d^{2m}} \sum_{g|\frac{k}{d}} \frac{1}{g^{2m-1}} \sum_{\substack{q=1 \\ 2\nmid q \\ (q,\frac{k}{d})=g}}^{+\infty} \frac{1}{q^{2m}} \\ & - \frac{2^{2m}}{(2^{4m}-1)} \cdot \frac{\zeta(2m)}{\zeta(4m)} \cdot \sum_{d|k} \frac{\mu(d)}{d^{2m}} \sum_{\substack{q=1 \\ 2\nmid q}}^{+\infty} \frac{1}{q^{4m}}. \end{aligned} \tag{3.5}$$

Noting that

$$- \frac{2^{2m}}{(2^{4m}-1)} \cdot \frac{\zeta(2m)}{\zeta(4m)} \cdot \sum_{d|k} \frac{\mu(d)}{d^{2m}} \sum_{\substack{q=1 \\ 2\nmid q}}^{+\infty} \frac{1}{q^{4m}} = - \frac{\zeta(2m)}{2^{2m}} \sum_{d|k} \frac{\mu(d)}{d^{2m}}, \tag{3.6}$$

then combining (3.1), (3.2), (3.5), (3.6), and Lemma 3.1 we have

$$g_m(k) = \frac{\zeta^2(2m)}{(2^{2m}+1)\,\zeta(4m)} \prod_{p^\alpha \| k} \left[\frac{\left(1 - \frac{1}{p^{2m}}\right)^2 - \frac{1}{p^{(4m-1)\alpha+1}} \left(1 - \frac{1}{p^{2m-1}}\right)^2}{1 - \frac{1}{p^{4m-1}}} \right].$$

This completes the proof of Theorem 1.2.

§ 4. Proof of Theorem 1.3

We need the following lemmas.

Lemma 4.1. *For any given positive integer k and any integer h with $(h,k)=1$ and any $P>1$, there exist a positive integer $q \le P$ and an integer a with $(a,q)=1$ such that*

$$\left|\frac{h}{k}-\frac{a}{q}\right|<\frac{1}{qP}.$$

Proof. This is a well-known result; See Theorem 36 of [12]. □

Lemma 4.2. *Let a, b, c, d, h and k be positive integers with $ad-bc=1$ and $(h,k)=1$. Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then we have

$$s(a,c)+s(h,k)-s(x,y)=\frac{c^2+k^2+y^2}{12cky}-\frac{1}{4}.$$

Proof. This is equation (26) of [11]. □

Lemma 4.3. *Let k be a positive even number, h be an odd number with $(k,h)=1$. Then*

$$s_2(h,k)=2s\left(h,\frac{k}{2}\right)-s(h,k).$$

Proof. See [18] or [15]. □

Lemma 4.4. *For any positive integer q, we have*

$$\sum_{\substack{a=1\\(a,q)=1}}^{q}|s(a,q)| \ll q\log^2 q.$$

Proof. This is Lemma 6 of [8]. □

Lemma 4.5. *Let k be a positive even number, a, q and h be positive integers with $(h,k)=(a,q)=1$, $z=qh-ak$. If $1\le |z|\le \frac{k}{q}$, then*

$$s_2(h,k)=\begin{cases}\dfrac{k}{4qz}+O\left(\left|s\left(a,\dfrac{q}{2}\right)\right|+|s(a,q)|+|z|\right), & \text{if } q \text{ is an even number;}\\ O\left(q+|z|\right), & \text{if } q \text{ is an odd number.}\end{cases}$$

Proof. Suppose that q is even. Since $(a,q)=1$, a must be odd number.

First we consider the case that $z<0$. Since $(a,q)=1$, there exist positive integers b and d such that

$$ad-bq=1, \qquad 1\le d<q.$$

Since q is even, d must be odd. Let $f = dh - bk$. Then we have

$$\begin{pmatrix} d & -b \\ -q & a \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} f \\ -z \end{pmatrix}$$

and

$$\begin{pmatrix} a & 2b \\ \frac{q}{2} & d \end{pmatrix} \begin{pmatrix} f \\ -\frac{z}{2} \end{pmatrix} = \begin{pmatrix} h \\ \frac{k}{2} \end{pmatrix}.$$

The fact that $d < q$ and $z \geq -\dfrac{k}{q}$ yields

$$f = kd\left(\frac{h}{k} - \frac{b}{d}\right) = kd\left(\frac{h}{k} - \frac{a}{q} + \frac{1}{qd}\right) = kd\left(\frac{z}{qk} + \frac{1}{qd}\right) = \frac{kd}{q}\left(\frac{z}{k} + \frac{1}{d}\right) > 0.$$

On the other hand, since $(h,k) = 1$ and f is odd, we get $(f,-z) = 1$. Then by Lemma 4.2,

$$s\left(a, \frac{q}{2}\right) + s(f,-z) - s\left(h, \frac{k}{2}\right) = -\frac{q^2 + k^2 + z^2}{6qkz} - \frac{1}{4}.$$

That is,

$$s\left(h, \frac{k}{2}\right) = \frac{k}{6qz} + O\left(\left|s\left(a, \frac{q}{2}\right)\right| + |z|\right).$$

From Lemma 8 of [8] we also have

$$s(h,k) = \frac{k}{12qz} + O\left(|s(a,q)| + |z|\right).$$

Therefore by Lemma 4.3 we immediately get

$$s_2(h,k) = 2s\left(h, \frac{k}{2}\right) - s(h,k) = \frac{k}{4qz} + O\left(\left|s\left(a, \frac{q}{2}\right)\right| + |s(a,q)| + |z|\right) \text{ if } z < 0.$$

For $z > 0$, we can find positive integers b and d satisfying

$$ad - bq = -1, \qquad 1 \leq d < q.$$

Since q is even, d must be odd. Let $f = bk - dh$. Then we have

$$\begin{pmatrix} b & -d \\ -a & q \end{pmatrix} \begin{pmatrix} k \\ h \end{pmatrix} = \begin{pmatrix} f \\ z \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{q}{2} & d \\ a & 2b \end{pmatrix} \begin{pmatrix} f \\ \frac{z}{2} \end{pmatrix} = \begin{pmatrix} \frac{k}{2} \\ h \end{pmatrix}.$$

Similarly we can get $(f,z) = 1$ and

$$f = kd\left(\frac{b}{d} - \frac{h}{k}\right) = kd\left(\frac{1}{qd} + \frac{a}{q} - \frac{h}{k}\right) = kd\left(\frac{1}{qd} - \frac{z}{qk}\right) = \frac{kd}{q}\left(\frac{1}{d} - \frac{z}{k}\right) > 0.$$

Then by Lemma 4.2,

$$s\left(\frac{q}{2}, a\right) + s\left(f, \frac{z}{2}\right) - s\left(\frac{k}{2}, h\right) = \frac{a^2 + \frac{z^2}{4} + h^2}{6azh} - \frac{1}{4}.$$

Noting that (see (1.1))

$$s\left(\frac{q}{2},a\right)+s\left(a,\frac{q}{2}\right)=\frac{\frac{q^2}{4}+a^2+1}{6aq}-\frac{1}{4}$$

and

$$s\left(\frac{k}{2},h\right)+s\left(h,\frac{k}{2}\right)=\frac{\frac{k^2}{4}+h^2+1}{6kh}-\frac{1}{4},$$

we have

$$s\left(h,\frac{k}{2}\right)=\frac{k}{6qz}+O\left(\left|s\left(a,\frac{q}{2}\right)\right|+|z|\right).$$

So from Lemma 8 of [8] and Lemma 4.3 we immediately get

$$s_2(h,k)=2s\left(h,\frac{k}{2}\right)-s(h,k)=\frac{k}{4qz}+O\left(\left|s\left(a,\frac{q}{2}\right)\right|+|s(a,q)|+|z|\right)\ \text{ for } z>0.$$

This proves that

$$s_2(h,k)=\frac{k}{4qz}+O\left(\left|s\left(a,\frac{q}{2}\right)\right|+|s(a,q)|+|z|\right),\qquad \text{if } q \text{ is an even number.}$$

On the other hand, if q is an odd number, using similar methods we can get

$$s_2(h,k)\ll q+|z|. \qquad \square$$

Now we prove Theorem 1.3. We suppose that $m\geq 2$, a sufficiently large even number k are given and we set

$$Q=\left[k^{1/2}\right],\qquad P=2Q.$$

Let

$$I(a,q)=\left(\frac{a}{q}-\frac{1}{qP},\frac{a}{q}+\frac{1}{qP}\right).$$

When $\frac{a_1}{q_1}\neq\frac{a_2}{q_2}$ and $q_1,q_2\leq Q$, one has

$$\left|\frac{a_1}{q_1}-\frac{a_2}{q_2}\right|\geq\frac{1}{q_1q_2}\geq\left(\frac{1}{q_1P}+\frac{1}{q_2P}\right).$$

Thus the intervals $I(a,q)$ are pairwise disjoint.

If $1\leq h\leq k$ and $(h,k)=1$, then by Lemma 4.1, $\frac{h}{k}$ falls into an interval $I(a,q)$ with $1\leq q\leq P$, $0\leq a\leq q$ and $(a,q)=1$.

Let $z=qh-ak$. It is easy to see that $z\neq 0$ and

$$|z|=qk\left|\frac{h}{k}-\frac{a}{q}\right|\leq\frac{k}{P}\leq\frac{k}{q}.$$

If $\frac{h}{k}$ falls into an interval $I(a,q)$ with $1\leq q\leq P$, q is an odd number, $0\leq a\leq q$ and $(a,q)=1$, by Lemma 4.5, we have

$$s_2(h,k)=O\left(q+|z|\right)\ll P+\frac{k}{P}\ll k^{\frac{1}{2}}.$$

Thus,

$$\sum\nolimits^{*} s_2^{2m}(h,k) \ll k^{m+1} \ll k^{2m-1}, \tag{4.1}$$

where the asterisk indicates summation over those integers h, $1 \le h \le k$, $(h,k)=1$ for which $\frac{h}{k}$ falls into an interval $I(a,q)$ with $1 \le q \le P$, q is an odd number, $0 \le a \le q$ and $(a,q)=1$.

If $\frac{h}{k}$ falls into an interval $I(a,q)$ with $Q \le q \le P$, q is an even number, $1 \le a \le q$ and $(a,q)=1$, by Lemma 4.5, we have

$$s_2(h,k) = \frac{k}{4qz} + O\left(\left|s\left(a,\frac{q}{2}\right)\right| + |s(a,q)| + |z|\right) \ll \frac{k}{q} + q + \frac{k}{P} \ll \frac{k}{Q} + P + \frac{k}{P} \ll k^{\frac{1}{2}}.$$

Thus,

$$\sum\nolimits^{*} s_2^{2m}(h,k) \ll k^{m+1} \ll k^{2m-1}, \tag{4.2}$$

where the asterisk indicates summation over those integers h, $1 \le h \le k$, $(h,k)=1$ for which $\frac{h}{k}$ falls into an interval $I(a,q)$ with $Q \le q \le P$, q is an even number, $1 \le a \le q$ and $(a,q)=1$.

Therefore

$$\sum_{\substack{h=1\\(h,k)=1}}^{k} s_2^{2m}(h,k) = \sum_{\substack{q=1\\2|q}}^{Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{h/k\in I(a,q)}^{*} s_2^{2m}(h,k) + O\left(k^{2m-1}\right), \tag{4.3}$$

where the asterisk means that $1 \le h \le k$, $(h,k)=1$.

Lemma 4.5 produces

$$s_2(h,k) = \frac{k}{4qz} + O\left(\left|s\left(a,\frac{q}{2}\right)\right| + |s(a,q)| + |z|\right).$$

Using the estimate

$$(A+B+C)^{2m} = A^{2m} + O\left(|A|^{2m-1}(|B|+|C|)\right) + O\left(B^{2m}+C^{2m}\right),$$

we obtain

$$s_2^{2m}(h,k) = \left(\frac{k}{4qz}\right)^{2m} + O\left(\left(\frac{k}{q\,|z|}\right)^{2m-1}\left(\left|s\left(a,\frac{q}{2}\right)\right| + |s(a,q)| + |z|\right)\right)$$
$$+O\left(\left(\left|s\left(a,\frac{q}{2}\right)\right| + |s(a,q)|\right)^{2m} + z^{2m}\right).$$

Therefore

$$\sum_{\substack{q=1\\2|q}}^{Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \sum_{h/k\in I(a,q)}^{*} s_2^{2m}(h,k) = \Omega_1 + O(\Omega_2) + O(\Omega_3), \tag{4.4}$$

where

$$(4.5)\quad \Omega_1 = \sum_{\substack{q=1\\ 2|q}}^{Q} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \sideset{}{^*}\sum_{h/k\in I(a,q)} \left(\frac{k}{4qz}\right)^{2m},$$

$$(4.6)\quad \Omega_2 = \sum_{\substack{q=1\\ 2|q}}^{Q} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \sideset{}{^*}\sum_{h/k\in I(a,q)} \left(\frac{k}{q\,|z|}\right)^{2m-1} \left(\left|s\left(a,\frac{q}{2}\right)\right| + |s(a,q)| + |z|\right),$$

$$(4.7)\quad \Omega_3 = \sum_{\substack{q=1\\ 2|q}}^{Q} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \sideset{}{^*}\sum_{h/k\in I(a,q)} \left(\left(\left|s\left(a,\frac{q}{2}\right)\right| + |s(a,q)|\right)^{2m} + z^{2m}\right).$$

Note that for the fixed a, q, k and z, the equation $z = qh - ak$ has at most one solution h. By Lemma 4.4, we have

(4.8)

$$\begin{aligned}
\Omega_2 &\ll k^{2m-1} \sum_{\substack{q=1\\ 2|q}}^{Q} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \sideset{}{^*}\sum_{h/k\in I(a,q)} \frac{1}{q^{2m-1}} \cdot \frac{1}{z^{2m-2}} \left(\left|s\left(a,\frac{q}{2}\right)\right| + |s(a,q)| + 1\right) \\
&\ll k^{2m-1} \sum_{\substack{q=1\\ 2|q}}^{Q} \frac{1}{q^{2m-1}} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \left(\left|s\left(a,\frac{q}{2}\right)\right| + |s(a,q)| + 1\right) \sum_{z\neq 0} \frac{1}{z^2} \\
&\ll k^{2m-1} \sum_{\substack{q=1\\ 2|q}}^{Q} \frac{1}{q^{2m-1}} \cdot q \cdot \log^2(q+1) \ll k^{2m-1} \sum_{\substack{q=1\\ 2|q}}^{Q} \frac{\log^2(q+1)}{q^2} \ll k^{2m-1}.
\end{aligned}$$

Moreover,

(4.9)

$$\Omega_3 \ll \sum_{\substack{q=1\\ 2|q}}^{Q} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \sideset{}{^*}\sum_{h/k\in I(a,q)} \left(q^{2m} + \left(\frac{k}{P}\right)^{2m}\right) \ll k^m \sum_{\substack{h=1\\ (h,k)=1}}^{k} 1 \ll k^{m+1} \ll k^{2m-1}.$$

Combining (4.3)-(4.9), we obtain

$$(4.10)\qquad \sum_{\substack{h=1\\ (h,k)=1}}^{k} s_2^{2m}(h,k) = \Omega_1 + O(k^{2m-1}),$$

where

$$\Omega_1 = \left(\frac{k}{4}\right)^{2m} \sum_{\substack{q=1\\ 2|q}}^{Q} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ (a,q)=1}}^{q} \sideset{}{^*}\sum_{h/k\in I(a,q)} \frac{1}{z^{2m}}.$$

It remains to obtain an asymptotic formula for Ω_1. Note that if $1 \leq h \leq k$, then $\frac{h}{k} \notin I(a,q)$ if and only if $|z| \geq \frac{k}{P}$. Hence

$$\left(\frac{k}{4}\right)^{2m} \sum_{\substack{q=1 \\ 2|q}}^{Q} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sideset{}{^*}\sum_{h/k \notin I(a,q)} \frac{1}{z^{2m}} \leq \left(\frac{k}{4}\right)^{2m} \sum_{\substack{q=1 \\ 2|q}}^{Q} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{|z| \geq k/P} \frac{1}{z^{2m}}$$

$$\ll k^{2m} \left(\frac{P}{k}\right)^{2m-1} \sum_{\substack{q=1 \\ 2|q}}^{Q} \frac{1}{q^{2m-1}} \ll kP^{2m-1} \ll k^{m+\frac{1}{2}} \ll k^{2m-1}.$$

Thus

$$\Omega_1 = \left(\frac{k}{4}\right)^{2m} \sum_{\substack{q=1 \\ 2|q}}^{Q} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \frac{1}{(qh-ak)^{2m}} + O(k^{2m-1}).$$

Using the estimate

$$\sum_{h \geq k+1} \frac{1}{(qh-ak)^{2m}} \leq \int_k^{+\infty} \frac{dx}{(qx-ak)^{2m}} = \int_{(q-a)k}^{+\infty} \frac{dy}{qy^{2m}} \ll \frac{1}{qk^{2m-1}},$$

we get

$$\left(\frac{k}{4}\right)^{2m} \sum_{\substack{q=1 \\ 2|q}}^{Q} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{h \geq k+1} \frac{1}{(qh-ak)^{2m}} \ll k^{2m} \sum_{\substack{q=1 \\ 2|q}}^{Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \frac{1}{qk^{2m-1}} \ll k.$$

Since

$$\sum_{h \leq 0} \frac{1}{(qh-ak)^{2m}} \leq \frac{1}{k^{2m}} + \sum_{r \geq 1} \frac{1}{(qr+ak)^{2m}} \leq \frac{1}{k^{2m}} + \int_0^{+\infty} \frac{dx}{(qx+ak)^{2m}}$$

$$= \frac{1}{k^{2m}} + \int_{ak}^{+\infty} \frac{dy}{qy^{2m}} \ll \frac{1}{k^{2m}} + \frac{1}{qk^{2m-1}} \ll \frac{1}{k^{2m-1}},$$

we have

$$\left(\frac{k}{4}\right)^{2m} \sum_{\substack{q=1 \\ 2|q}}^{Q} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{h \leq 0} \frac{1}{(qh-ak)^{2m}} \ll k^{2m} \sum_{\substack{q=1 \\ 2|q}}^{Q} \frac{1}{q^{2m}} \sum_{a=1}^{q} \frac{1}{k^{2m-1}} \ll k.$$

Therefore

$$\Omega_1 = \left(\frac{k}{4}\right)^{2m} \sum_{\substack{q=1 \\ 2|q}}^{Q} \frac{1}{q^{2m}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\substack{h=-\infty \\ (h,k)=1}}^{+\infty} \frac{1}{(qh-ak)^{2m}} + O(k^{2m-1}).$$

Since

$$\left(\frac{k}{4}\right)^{2m}\sum_{\substack{q>Q\\2|q}}\frac{1}{q^{2m}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{\substack{h=-\infty\\(h,k)=1\\qh-ak\neq0}}^{+\infty}\frac{1}{(qh-ak)^{2m}}$$

$$\ll k^{2m}\sum_{\substack{q>Q\\2|q}}\frac{1}{q^{2m}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{\substack{z=-\infty\\z\neq0}}^{+\infty}\frac{1}{z^{2m}}$$

$$\ll k^{2m}\sum_{\substack{q>Q\\2|q}}\frac{1}{q^{2m}}\sum_{\substack{a=1\\(a,q)=1}}^{q}1\ll k^{2m}\sum_{\substack{q>Q\\2|q}}\frac{1}{q^{2m-1}}\ll\frac{k^{2m}}{Q^{2m-2}}\ll k^{m+1}\ll k^{2m-1},$$

we have

$$\Omega_1=\left(\frac{k}{4}\right)^{2m}\sum_{\substack{q=1\\2|q}}^{+\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{\substack{h=-\infty\\(h,k)=1\\qh-ak\neq0}}^{+\infty}\frac{1}{(qh-ak)^{2m}}+O(k^{2m-1}).\tag{4.11}$$

Therefore by (4.10) and (4.11)

$$\sum_{\substack{h=1\\(h,k)=1}}^{k}s_2^{2m}(h,k)=h_m(k)\left(\frac{k}{4}\right)^{2m}+O\left(k^{2m-1}\right),$$

where

$$h_m(k)=\sum_{\substack{q=1\\2|q}}^{+\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{\substack{h=-\infty\\(h,k)=1\\qh-ak\neq0}}^{+\infty}\frac{1}{(qh-ak)^{2m}}$$

$$=\sum_{\substack{q=1\\2|q}}^{+\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\(a,q)=1\\2\nmid a}}^{q}\sum_{\substack{h=-\infty\\(h,k)=1\\2\nmid h\\qh-ak\neq0}}^{+\infty}\frac{1}{(qh-ak)^{2m}}.$$

This proves Theorem 1.3.

§ 5. Proof of Theorem 1.4

Let $k=2^\beta M$ be an integer with $\beta\geq1$ and $2\nmid M$. By (2.1) we get

$$h_m(k)=\sum_{\substack{q=1\\2|q}}^{+\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\(a,q)=1\\2\nmid a}}^{q}\sum_{\substack{h=-\infty\\(h,k)=1\\2\nmid h\\qh-ak\neq0}}^{+\infty}\frac{1}{(qh-ak)^{2m}}$$

$$= \sum_{\substack{q=1\\ 2\mid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{q} \sum_{\substack{t\mid q\\ t\mid a}} \mu(t) \sum_{\substack{h=-\infty\\ 2\nmid h\\ qh-ak\neq 0}}^{+\infty} \frac{1}{(qh-ak)^{2m}} \sum_{\substack{d\mid h\\ d\mid k}} \mu(d)$$

$$= \sum_{\substack{t=1\\ 2\nmid t}}^{+\infty} \frac{\mu(t)}{t^{4m}} \sum_{\substack{q=1\\ 2\mid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{q} \sum_{\substack{h=-\infty\\ 2\nmid h\\ qh-ak\neq 0}}^{+\infty} \frac{1}{(qh-ak)^{2m}} \sum_{\substack{d\mid h\\ d\mid k}} \mu(d)$$

$$= \frac{2^{4m}}{(2^{4m}-1)\,\zeta(4m)} \sum_{d\mid M} \frac{\mu(d)}{d^{2m}} \sum_{\substack{q=1\\ 2\mid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{q} \sum_{\substack{h=-\infty\\ 2\nmid h\\ qh-\frac{ak}{d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{d}\right)^{2m}}.$$

Then we have
(5.1)

$$h_m(k) = \frac{2^{4m}}{(2^{4m}-1)\,\zeta(4m)} \sum_{d\mid M} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=1}^{\beta-1} \sum_{\substack{q=1\\ 2^{\gamma}\| q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{q} \sum_{\substack{h=-\infty\\ 2\nmid h\\ qh-\frac{ak}{d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{d}\right)^{2m}}$$

$$+ \frac{2^{4m}}{(2^{4m}-1)\,\zeta(4m)} \sum_{d\mid M} \frac{\mu(d)}{d^{2m}} \sum_{\substack{q=1\\ 2^{\beta}\| q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{q} \sum_{\substack{h=-\infty\\ 2\nmid h\\ qh-\frac{ak}{d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{d}\right)^{2m}}$$

$$+ \frac{2^{4m}}{(2^{4m}-1)\,\zeta(4m)} \sum_{d\mid M} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{+\infty} \sum_{\substack{q=1\\ 2^{\gamma}\| q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{q} \sum_{\substack{h=-\infty\\ 2\nmid h\\ qh-\frac{ak}{d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{d}\right)^{2m}}$$

$$= \frac{2^{4m}}{(2^{4m}-1)\,\zeta(4m)} \sum_{d\mid M} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=1}^{\beta-1} \frac{1}{2^{4m\gamma}} \sum_{\substack{q=1\\ 2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{2^{\gamma}q} \sum_{\substack{h=-\infty\\ 2\nmid h\\ qh-\frac{ak}{2^{\gamma}d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{2^{\gamma}d}\right)^{2m}}$$

$$+ \frac{2^{4m}}{(2^{4m}-1)\,\zeta(4m)} \sum_{d\mid M} \frac{\mu(d)}{d^{2m}} \frac{1}{2^{4m\beta}} \sum_{\substack{q=1\\ 2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{2^{\beta}q} \sum_{\substack{h=-\infty\\ 2\nmid h\\ qh-\frac{ak}{2^{\beta}d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{2^{\beta}d}\right)^{2m}}$$

$$+ \frac{2^{4m}}{(2^{4m}-1)\,\zeta(4m)} \sum_{d\mid M} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} \sum_{\substack{q=1\\ 2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{2^{\gamma}q} \sum_{\substack{h=-\infty\\ 2\nmid h\\ 2^{\gamma}qh-\frac{ak}{d}\neq 0}}^{+\infty} \frac{1}{\left(2^{\gamma}qh-\frac{ak}{d}\right)^{2m}}$$

$$:= \Psi_1 + \Psi_2 + \Psi_3.$$

First we consider Ψ_1. Let $g = \left(q, \frac{M}{d}\right)$. Then

$$\sum_{\substack{a=1\\2\nmid a}}^{2^\gamma q} \sum_{\substack{h=-\infty\\2\nmid h\\qh-\frac{ak}{2^\gamma d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{2^\gamma d}\right)^{2m}} = \frac{1}{g^{2m}} \sum_{\substack{a=1\\2\nmid a}}^{2^\gamma q} \sum_{\substack{h=-\infty\\2\nmid h\\\frac{qh}{g}-\frac{ak}{2^\gamma dg}\neq 0}}^{+\infty} \frac{1}{\left(\frac{qh}{g}-\frac{ak}{2^\gamma dg}\right)^{2m}}$$

$$= \frac{1}{g^{2m}} \sum_{\substack{a=1\\2\nmid a}}^{2^\gamma q} \sum_{\substack{z=-\infty\\2\nmid z\\z\equiv -\frac{ak}{2^\gamma dg}\left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}} = \frac{1}{g^{2m}} \sum_{a=1}^{2^{\gamma-1}q} \sum_{\substack{z=-\infty\\2\nmid z\\z\equiv -\frac{ak}{2^{\gamma-1}dg}+\frac{k}{2^\gamma dg}\left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{1}{g^{2m}} \sum_{b=0}^{2^{\gamma-1}-1} \sum_{c=1}^{q} \sum_{\substack{z=-\infty\\2\nmid z\\z\equiv -\frac{(c+bq)k}{2^{\gamma-1}dg}+\frac{k}{2^\gamma dg}\left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{1}{g^{2m}} \sum_{b=0}^{2^{\gamma-1}-1} \sum_{c=1}^{q} \sum_{\substack{z=-\infty\\2\nmid z\\z\equiv -\frac{ck}{2^{\gamma-1}dg}-\frac{bqk}{2^{\gamma-1}dg}+\frac{k}{2^\gamma dg}\left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{1}{g^{2m}} \sum_{b=0}^{2^{\gamma-1}-1} \sum_{c=1}^{q} \sum_{\substack{z=-\infty\\2\nmid z\\z\equiv c\left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{2^{\gamma-1}}{g^{2m-1}} \sum_{\substack{z=-\infty\\2\nmid z}}^{+\infty} \frac{1}{z^{2m}} = \frac{\left(2^{2m}-1\right)}{2^{2m}} \cdot \frac{2^\gamma}{g^{2m-1}} \cdot \zeta(2m).$$

Therefore

$$\text{(5.2)}\quad \Psi_1 = \frac{2^{2m}}{(2^{2m}+1)} \cdot \frac{\zeta(2m)}{\zeta(4m)} \cdot \sum_{\gamma=1}^{\beta-1} \frac{1}{2^{(4m-1)\gamma}} \sum_{d\mid M} \frac{\mu(d)}{d^{2m}} \sum_{g\mid \frac{M}{d}} \frac{1}{g^{2m-1}} \sum_{\substack{q=1\\2\nmid q\\(q,\frac{M}{d})=g}}^{+\infty} \frac{1}{q^{2m}}.$$

Now we consider Ψ_2. Let $g = \left(q, \frac{M}{d}\right)$. Then

$$\sum_{\substack{a=1\\2\nmid a}}^{2^\beta q} \sum_{\substack{h=-\infty\\2\nmid h\\qh-\frac{ak}{2^\beta d}\neq 0}}^{+\infty} \frac{1}{\left(qh-\frac{ak}{2^\beta d}\right)^{2m}} = \frac{1}{g^{2m}} \sum_{\substack{a=1\\2\nmid a}}^{2^\beta q} \sum_{\substack{h=-\infty\\2\nmid h\\\frac{qh}{g}-\frac{ak}{2^\beta dg}\neq 0}}^{+\infty} \frac{1}{\left(\frac{qh}{g}-\frac{ak}{2^\beta dg}\right)^{2m}}$$

$$= \frac{1}{g^{2m}} \sum_{\substack{a=1\\ 2\nmid a}}^{2^\beta q} \sum_{\substack{z=-\infty\\ z\neq 0\\ 2|z\\ z\equiv -\frac{ak}{2^\beta dg} \left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}} = \frac{1}{g^{2m}} \sum_{a=1}^{2^{\beta-1} q} \sum_{\substack{z=-\infty\\ z\neq 0\\ 2|z\\ z\equiv -\frac{ak}{2^{\beta-1} dg}+\frac{k}{2^\beta dg} \left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{1}{g^{2m}} \sum_{b=0}^{2^{\beta-1}-1} \sum_{c=1}^{q} \sum_{\substack{z=-\infty\\ z\neq 0\\ 2|z\\ z\equiv -\frac{(c+bq)k}{2^{\beta-1} dg}+\frac{k}{2^\beta dg} \left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{1}{g^{2m}} \sum_{b=0}^{2^{\beta-1}-1} \sum_{c=1}^{q} \sum_{\substack{z=-\infty\\ z\neq 0\\ 2|z\\ z\equiv -\frac{ck}{2^{\beta-1} dg}-\frac{bqk}{2^{\beta-1} dg}+\frac{k}{2^\beta dg} \left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{1}{g^{2m}} \sum_{b=0}^{2^{\beta-1}-1} \sum_{c=1}^{q} \sum_{\substack{z=-\infty\\ z\neq 0\\ 2|z\\ z\equiv c \left(\bmod \frac{q}{g}\right)}}^{+\infty} \frac{1}{z^{2m}}$$

$$= \frac{2^{\beta-1}}{g^{2m-1}} \sum_{\substack{z=-\infty\\ z\neq 0\\ 2|z}}^{+\infty} \frac{1}{z^{2m}} = \frac{2^\beta}{2^{2m} g^{2m-1}} \cdot \zeta(2m).$$

Therefore

$$\Psi_2 = \frac{2^{2m}}{(2^{4m}-1)} \cdot \frac{\zeta(2m)}{\zeta(4m)} \cdot \frac{1}{2^{(4m-1)\beta}} \sum_{d|M} \frac{\mu(d)}{d^{2m}} \sum_{g|\frac{M}{d}} \frac{1}{g^{2m-1}} \sum_{\substack{q=1\\ 2\nmid q\\ \left(q, \frac{M}{d}\right)=g}}^{+\infty} \frac{1}{q^{2m}}. \tag{5.3}$$

For Ψ_3, we have

Ψ_3

$$= \frac{2^{4m}}{(2^{4m}-1)\,\zeta(4m)} \sum_{d|M} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} \sum_{\substack{q=1\\ 2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty\\ 2^\gamma qh-\frac{ak}{d}\neq 0}}^{+\infty} \frac{1}{\left(2^\gamma qh - \frac{ak}{d}\right)^{2m}}$$

$$- \frac{2^{4m}}{(2^{4m}-1)\,\zeta(4m)} \sum_{d|M} \frac{\mu(d)}{d^{2m}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} \sum_{\substack{q=1\\ 2\nmid q}}^{+\infty} \frac{1}{q^{2m}} \sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty\\ 2|h\\ 2^\gamma qh-\frac{ak}{d}\neq 0}}^{+\infty} \frac{1}{\left(2^\gamma qh - \frac{ak}{d}\right)^{2m}}$$

$$-\frac{2^{4m}}{(2^{4m}-1)\zeta(4m)}\sum_{d|M}\frac{\mu(d)}{d^{2m}}\sum_{\gamma=\beta+1}^{+\infty}\frac{1}{2^{2m\gamma}}\sum_{\substack{q=1\\2\nmid q}}^{+\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2|a}}^{2^\gamma q}\sum_{\substack{h=-\infty\\2^\gamma qh-\frac{ak}{d}\neq 0}}^{+\infty}\frac{1}{\left(2^\gamma qh-\frac{ak}{d}\right)^{2m}}$$

$$+\frac{2^{4m}}{(2^{4m}-1)\zeta(4m)}\sum_{d|M}\frac{\mu(d)}{d^{2m}}\sum_{\gamma=\beta+1}^{+\infty}\frac{1}{2^{2m\gamma}}\sum_{\substack{q=1\\2\nmid q}}^{+\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2|a}}^{2^\gamma q}\sum_{\substack{h=-\infty\\2|h\\2^\gamma qh-\frac{ak}{d}\neq 0}}^{+\infty}\frac{1}{\left(2^\gamma qh-\frac{ak}{d}\right)^{2m}}$$

$$=\frac{2^{4m}}{(2^{4m}-1)\zeta(4m)}\sum_{d|M}\frac{\mu(d)}{d^{2m}}\sum_{\gamma=\beta+1}^{+\infty}\frac{1}{2^{2m(\gamma+\beta)}}\sum_{\substack{q=1\\2\nmid q}}^{+\infty}\frac{1}{q^{2m}}\sum_{a=1}^{2^\gamma q}\sum_{\substack{h=-\infty\\2^{\gamma-\beta}qh-\frac{ak}{2^\beta d}\neq 0}}^{+\infty}\frac{1}{\left(2^{\gamma-\beta}qh-\frac{ak}{2^\beta d}\right)^{2m}}$$

$$-\frac{2^{4m}}{(2^{4m}-1)\zeta(4m)}\sum_{d|M}\frac{\mu(d)}{d^{2m}}\sum_{\gamma=\beta+1}^{+\infty}\frac{1}{2^{2m(\gamma+\beta)}}\sum_{\substack{q=1\\2\nmid q}}^{+\infty}\frac{1}{q^{2m}}\sum_{a=1}^{2^\gamma q}\sum_{\substack{h=-\infty\\2^{\gamma-\beta+1}qh-\frac{ak}{2^\beta d}\neq 0}}^{+\infty}\frac{1}{\left(2^{\gamma-\beta+1}qh-\frac{ak}{2^\beta d}\right)^{2m}}$$

$$-\frac{2^{4m}}{(2^{4m}-1)\zeta(4m)}\sum_{d|M}\frac{\mu(d)}{d^{2m}}\sum_{\gamma=\beta+1}^{+\infty}\frac{1}{2^{2m(\gamma+\beta+1)}}\sum_{\substack{q=1\\2\nmid q}}^{+\infty}\frac{1}{q^{2m}}\sum_{a=1}^{2^{\gamma-1} q}\sum_{\substack{h=-\infty\\2^{\gamma-\beta-1}qh-\frac{ak}{2^\beta d}\neq 0}}^{+\infty}\frac{1}{\left(2^{\gamma-\beta-1}qh-\frac{ak}{2^\beta d}\right)^{2m}}$$

$$+\frac{2^{4m}}{(2^{4m}-1)\zeta(4m)}\sum_{d|M}\frac{\mu(d)}{d^{2m}}\sum_{\gamma=\beta+1}^{+\infty}\frac{1}{2^{2m(\gamma+\beta+1)}}\sum_{\substack{q=1\\2\nmid q}}^{+\infty}\frac{1}{q^{2m}}\sum_{a=1}^{2^{\gamma-1} q}\sum_{\substack{h=-\infty\\2^{\gamma-\beta}qh-\frac{ak}{2^\beta d}\neq 0}}^{+\infty}\frac{1}{\left(2^{\gamma-\beta}qh-\frac{ak}{2^\beta d}\right)^{2m}}$$

$$:=\Omega_1-\Omega_2-\Omega_3+\Omega_4.$$

Let $g=\left(q,\frac{M}{d}\right)$. Then

$$\sum_{a=1}^{2^\gamma q}\sum_{\substack{h=-\infty\\2^{\gamma-\beta}qh-\frac{ak}{2^\beta d}\neq 0}}^{+\infty}\frac{1}{\left(2^{\gamma-\beta}qh-\frac{ak}{2^\beta d}\right)^{2m}}$$

$$=\frac{1}{g^{2m}}\sum_{a=1}^{2^\gamma q}\sum_{\substack{h=-\infty\\\frac{2^{\gamma-\beta}qh}{g}-\frac{ak}{2^\beta dg}\neq 0}}^{+\infty}\frac{1}{\left(\frac{2^{\gamma-\beta}qh}{g}-\frac{ak}{2^\beta dg}\right)^{2m}}$$

$$=\frac{1}{g^{2m}}\sum_{a=1}^{2^\gamma q}\sum_{\substack{z=-\infty\\z\neq 0\\z\equiv-\frac{ak}{2^\beta dg}\left(\bmod\frac{2^{\gamma-\beta}q}{g}\right)}}^{+\infty}\frac{1}{z^{2m}}=\frac{1}{g^{2m}}\sum_{a=1}^{2^\gamma q}\sum_{\substack{z=-\infty\\z\neq 0\\z\equiv a\left(\bmod\frac{2^{\gamma-\beta}q}{g}\right)}}^{+\infty}\frac{1}{z^{2m}}$$

$$=\frac{2^\beta}{g^{2m-1}}\sum_{\substack{z=-\infty\\z\neq 0}}^{+\infty}\frac{1}{z^{2m}}=\frac{2^{\beta+1}\zeta(2m)}{g^{2m-1}}.$$

Similarly, we have

$$\sum_{a=1}^{2^\gamma q} \sum_{\substack{h=-\infty \\ 2^{\gamma-\beta+1}qh-\frac{ak}{2^\beta d}\neq 0}}^{+\infty} \frac{1}{\left(2^{\gamma-\beta+1}qh-\frac{ak}{2^\beta d}\right)^{2m}} = \frac{2^\beta \zeta(2m)}{g^{2m-1}},$$

$$\sum_{a=1}^{2^{\gamma-1} q} \sum_{\substack{h=-\infty \\ 2^{\gamma-\beta-1}qh-\frac{ak}{2^\beta d}\neq 0}}^{+\infty} \frac{1}{\left(2^{\gamma-\beta-1}qh-\frac{ak}{2^\beta d}\right)^{2m}} = \frac{2^{\beta+1} \zeta(2m)}{g^{2m-1}},$$

$$\sum_{a=1}^{2^{\gamma-1} q} \sum_{\substack{h=-\infty \\ 2^{\gamma-\beta}qh-\frac{ak}{2^\beta d}\neq 0}}^{+\infty} \frac{1}{\left(2^{\gamma-\beta}qh-\frac{ak}{2^\beta d}\right)^{2m}} = \frac{2^\beta \zeta(2m)}{g^{2m-1}}.$$

Therefore
(5.4)

$$\begin{aligned}
\Psi_3
&= \frac{2^{4m}\zeta(2m)}{(2^{4m}-1)\,\zeta(4m)} \cdot \frac{1}{2^{(2m-1)\beta-1}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} \sum_{d|M} \frac{\mu(d)}{d^{2m}} \sum_{g|\frac{M}{d}} \frac{1}{g^{2m-1}} \sum_{\substack{q=1 \\ 2\nmid q \\ (q,\frac{M}{d})=g}}^{+\infty} \frac{1}{q^{2m}} \\
&- \frac{2^{4m}\zeta(2m)}{(2^{4m}-1)\,\zeta(4m)} \cdot \frac{1}{2^{(2m-1)\beta}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} \sum_{d|M} \frac{\mu(d)}{d^{2m}} \sum_{g|\frac{M}{d}} \frac{1}{g^{2m-1}} \sum_{\substack{q=1 \\ 2\nmid q \\ (q,\frac{M}{d})=g}}^{+\infty} \frac{1}{q^{2m}} \\
&- \frac{2^{4m}\zeta(2m)}{(2^{4m}-1)\,\zeta(4m)} \cdot \frac{1}{2^{(2m-1)(\beta+1)}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} \sum_{d|M} \frac{\mu(d)}{d^{2m}} \sum_{g|\frac{M}{d}} \frac{1}{g^{2m-1}} \sum_{\substack{q=1 \\ 2\nmid q \\ (q,\frac{M}{d})=g}}^{+\infty} \frac{1}{q^{2m}} \\
&+ \frac{2^{4m}\zeta(2m)}{(2^{4m}-1)\,\zeta(4m)} \cdot \frac{1}{2^{(2m-1)\beta+2m}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} \sum_{d|M} \frac{\mu(d)}{d^{2m}} \sum_{g|\frac{M}{d}} \frac{1}{g^{2m-1}} \sum_{\substack{q=1 \\ 2\nmid q \\ (q,\frac{M}{d})=g}}^{+\infty} \frac{1}{q^{2m}}.
\end{aligned}$$

Now from (5.1)-(5.4) and Lemma 3.1 we have

$$\begin{aligned}
h_m(k)
&= \frac{2^{4m}\zeta(2m)}{(2^{4m}-1)\,\zeta(4m)} \times \left[\frac{(2^{2m}-1)}{2^{2m}} \sum_{\gamma=1}^{\beta-1} \frac{1}{2^{(4m-1)\gamma}} + \frac{1}{2^{(4m-1)\beta+2m}} + \frac{1}{2^{(2m-1)\beta-1}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} \right. \\
&\left. - \frac{1}{2^{(2m-1)\beta}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} - \frac{1}{2^{(2m-1)(\beta+1)}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} + \frac{1}{2^{(2m-1)\beta+2m}} \sum_{\gamma=\beta+1}^{+\infty} \frac{1}{2^{2m\gamma}} \right]
\end{aligned}$$

$$\times\left[\sum_{d\mid M}\frac{\mu(d)}{d^{2m}}\sum_{g\mid\frac{M}{d}}\frac{1}{g^{2m-1}}\sum_{\substack{q=1\\2\nmid q\\(q,\frac{M}{d})=g}}^{+\infty}\frac{1}{q^{2m}}\right]$$

$$=\frac{2^{2m}\zeta(2m)}{(2^{2m}+1)\zeta(4m)}\times\left[\frac{(2^{2m}-1)}{2^{2m}}\cdot\frac{\left(1-\frac{1}{2^{(4m-1)(\beta-1)}}\right)}{(2^{4m-1}-1)}+\frac{1}{2^{(4m-1)\beta+2m}}+\frac{1}{(2^{2m}-1)\,2^{(4m-1)\beta-1}}\right.$$

$$\left.-\frac{1}{(2^{2m}-1)\,2^{(4m-1)\beta}}-\frac{1}{(2^{2m}-1)\,2^{(4m-1)\beta+2m-1}}+\frac{1}{(2^{2m}-1)\,2^{(4m-1)\beta+2m}}\right]$$

$$\times\prod_{p^\alpha\|M}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right]$$

$$=\frac{2^{2m}\zeta^2(2m)}{(2^{2m}+1)\zeta(4m)}\left(\frac{2^{2m}-1}{2^{2m}(2^{4m-1}-1)}-\frac{2^{6m-2}+1-2^{4m-1}-2^{4m-2}}{(2^{4m-1}-1)\,2^{(4m-1)\beta+2m-1}}\right)$$

$$\times\prod_{p^\alpha\|M}\left[\frac{\left(1-\frac{1}{p^{2m}}\right)^2-\frac{1}{p^{(4m-1)\alpha+1}}\left(1-\frac{1}{p^{2m-1}}\right)^2}{1-\frac{1}{p^{4m-1}}}\right].$$

This completes the proof of Theorem 1.4.

References

[1] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Graduate Texts in Mathematics, No. 41. Springer-Verlag, New York-Heidelberg, 1976.

[2] B. C. Berndt, *A new proof of the reciprocity theorem for Dedekind sums*, Elem. Math. **29** (1974), 93–94.

[3] ———, *Dedekind sums and a paper of G. H. Hardy*, J. London Math. Soc. (2) **13** (1976), no. 1, 129–137.

[4] ———, *Reciprocity theorems for Dedekind sums and generalizations*, Advances in Math. **23** (1977), no. 3, 285–316.

[5] ———, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **303/304** (1978), 332–365.

[6] B. C. Berndt and L. A. Goldberg, *Analytic properties of arithmetic sums arising in the theory of the classical theta functions*, SIAM J. Math. Anal. **15** (1984), no. 1, 143–150.

[7] X. Chen and W. Zhang, *A sum analogous to Dedekind sums and its mean value formula*, Chinese Ann. Math. Ser. A **21** (2000), no. 6, 715–722.

[8] J. B. Conrey, E. Fransen, R. Klein, and C. Scott, *Mean values of Dedekind sums*, J. Number Theory **56** (1996), no. 2, 214–226.

[9] R. Dedekind, *Erläuterungen zu der Riemannschen Fragmenten über die Grenzfalle der elliptischen Funktionen*, Gesammelte Math. Werke 1, Braunschweig, 1930, 159–173.

[10] U. Dieter, *Cotangent sums, a further generalization of Dedekind sums*, J. Number Theory **18** (1984), no. 3, 289–305.

[11] R. R. Hall and M. N. Huxley, *Dedekind sums and continued fractions*, Acta Arith. **63** (1993), no. 1, 79–90.

[12] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fifth edition. The Clarendon Press, Oxford University Press, New York, 1979.

[13] C. Jia, *On the mean value of Dedekind sums*, J. Number Theory **87** (2001), no. 2, 173–188.

[14] H. Liu and W. Zhang, *On the even power mean of a sum analogous to Dedekind sums*, Acta Math. Hungar. **106** (2005), no. 1-2, 67–81.

[15] M. R. Pettet and R. Sitaramachandrarao, *Three-term relations for Hardy sums*, J. Number Theory **25** (1987), no. 3, 328–339.

[16] J. Pommersheim, *Toric varieties, lattice points and Dedekind sums*, Math. Ann. **295** (1993), no. 1, 1–24.

[17] H. Rademacher and E. Grosswald, *Dedekind sums*, The Carus Mathematical Monographs, No. 16. The Mathematical Association of America, Washington, D.C., 1972.

[18] R. Sitaramachandrarao, *Dedekind and Hardy sums*, Acta Arith. **48** (1987), no. 4, 325–340.

[19] H. Walum, *An exact formula for an average of L-series*, Illinois J. Math. **26** (1982), no. 1, 1–3.

[20] W. Zhang, *On the mean values of Dedekind sums*, J. Theor. Nombres Bordeaux **8** (1996), no. 2, 429–442.

[21] ———, *A note on the mean square value of the Dedekind sums*, Acta Math. Hungar. **86** (2000), no. 4, 275–289.

[22] ———, *A sum analogous to the Dedekind sum and its mean value formula* J. Number Theory **89** (2001), no. 1, 1–13.

Department of Mathematics
Northwest University
Xi'an, Shaanxi, P. R. China
E-mail address: hnliu@nwu.edu.cn