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ANALYTIC CONSTRUCTION OF VERSAL DEFORMATION OF RESOLUTION OF NORMAL ISOLATED SINGULARITIES

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ABSTRACT. We analytically construct the versal deformation space of resolution of normal isolated singularities based on the formalism in [9].

Introduction

Let V be a reduced normal Stein space locally embedded in $\mathbb{C}^N$. We established analytic construction of the versal deformation of the germ $(V,0)$ in [8] and also in [10] by an alternative way. However, in spite of its much interest from various viewpoints (e.g. the so-called Brieskorn resolution, hyper-Kähler structure, the MacKay-correspondence, etc.), the versal deformation of the resolution of $(V,0)$ was constructed only by algebraic method ([2], [7]). In this paper, we complete its analytic construction based on the formalism in [9]. We consider an arbitrary resolution of singularities $f_0 : \widetilde{V} \to V$ and formulate its deformation by a functor $\mathcal{R}es_{\widetilde{V}\to V} : \mathcal{A}n \to (\text{Sets})$ given by

$$\mathcal{R}es_{\widetilde{V}\to V}(T) := \left\{ F : \widetilde{\mathcal{V}} \to \mathcal{V} \;\middle|\; \begin{array}{l} \text{a family } \widetilde{\mathcal{V}} \to T \text{ of deformations of } \widetilde{V}, \\ \text{a family } \mathcal{V} \to T \text{ of deformations of } V \\ \text{and a holomorphic map } F : \widetilde{\mathcal{V}} \to \mathcal{V} \\ \text{satisfying the commutative diagram} \\ \qquad \begin{array}{ccc} \widetilde{\mathcal{V}} & \xrightarrow{F} & \mathcal{V} \\ \downarrow & & \downarrow \\ T & = & T \end{array} \\ \text{which induces } f_0 : \widetilde{V} \to V \text{ over } 0 \in T \end{array} \right\} \Big/ \sim$$

where $\mathcal{A}n$ denotes the category of germs of analytic spaces and $(F_1 : \widetilde{\mathcal{V}}_1 \to \mathcal{V}_1) \sim (F_2 : \widetilde{\mathcal{V}}_2 \to \mathcal{V}_2)$ if there exist isomorphisms of germs satisfying the

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following commutative diagram

$$\begin{array}{ccc} (\widetilde{\mathcal{V}}_1, f_0^{-1}(0)) & \xrightarrow{F_1} & (\mathcal{V}_1, 0) \\ \simeq\downarrow & & \downarrow\simeq \\ (\widetilde{\mathcal{V}}_2, f_0^{-1}(0)) & \xrightarrow{F_2} & (\mathcal{V}_2, 0). \end{array}$$

Let $\Omega = f_0^{-1}(B(c) \cap V)$ with denoting $B(c)$ a ball in $\mathbb{C}^N$ centered at 0 and with radius $c > 0$. We assume that $\partial B(c)$ meets V transversely for all $0 < c < b$. In the previous paper [9], we established the formalism of $\mathcal{R}es_{\widetilde{V}\to V}$ in terms of deformations of the pair of a complex structure on Ω and a holomorphic map from Ω to $\mathbb{C}^N$. In this paper, we concentrate on the analytical part of the construction of the versal family. We consider deformations of the pair and the main theorem of this paper is as follows.

Theorem 0.1 (Main Theorem). *There exists a family of pairs of a complex structure on Ω and a holomorphic map to $\mathbb{C}^N$, parametrized by an analytic space germ T such that*

(1) *the infinitesimal deformation map $T_0T \to H^1(K_\Omega^{\bullet,\bullet})$ is bijective,*

(2) *it is Kuranishi versal,*

where $H^1(K_\Omega^{\bullet,\bullet})$ denotes the first cohomology group of the deformation complex $K_\Omega^{\bullet,\bullet}$ (cf. §1) and we call the property listed in Proposition 4.1 "Kuranishi versal".

(3) *T is isomorphic to the parameter space of the versal family for $\mathcal{R}es_{\widetilde{V}\to V}$.*

The construction of the versal family will be done by constructing $(\phi(t), f(t)) \in A_{\overline{\Omega}}^{0,1}(T^{1,0}X)[[t_1, \ldots, t_d]] \oplus A_{\overline{\Omega}}^{0}(f_0^* T^{1,0}\mathbb{C}^N)[[t_1, \ldots, t_d]]$ and an ideal $\mathcal{J}_T \subset \mathbb{C}\{t_1, \ldots, t_d\}$ such that

$$(\phi(0), f(0)) = (0, 0), \tag{0.1}$$

$$\text{the linear term of } (\phi(t), f(t)) \text{ spans } H^1(K_\Omega^{\bullet,\bullet}), \tag{0.2}$$

$$\left(\bar{\partial}\phi(t) - \frac{1}{2}[\phi(t), \phi(t)],\ (\bar{\partial} - \phi(t))(f_0 + f(t))\right) \equiv (0,0) \mod \mathcal{J}_T, \tag{0.3}$$

$$\phi(t) \text{ and } f(t) \text{ are convergent with respect to any Sobolev-norm } \|\ \|_k, \tag{0.4}$$

$$\text{it is Kuranishi versal,} \tag{0.5}$$

where we denote $t := (t_1, \ldots, t_d)$ with $d := \dim_{\mathbb{C}} H^1(K_\Omega^{\bullet,\bullet})$.

By expanding $\phi(t)$ and $f(t)$ into power series, we will construct them inductively on the homogeneous degree by solving the linearized differential equation

$$\begin{aligned} D(\phi_\mu(t), f_\mu(t)) &:= (\bar{\partial}\phi_\mu(t), \bar{\partial} f_\mu(t) - df_0\phi_\mu(t)) \\ &= \text{the } \mu\text{-th order homogeneous term of} \end{aligned}$$

$$- \left(\bar{\partial}\phi^{(\mu-1)}(t) - \frac{1}{2}[\phi^{(\mu-1)}(t), \phi^{(\mu-1)}(t)], (\bar{\partial} - \phi^{(\mu-1)}(t))(f_0 + f^{(\mu-1)}(t)) \right)$$
$$(\mu = 2, 3, \dots)$$

with denoting $\phi_\mu(t)$ and $f_\mu(t)$ the μ-th order homogeneous terms of $\phi(t)$ and $f(t)$, respectively, and $\phi^{(\mu-1)}(t) := \sum_{\nu=1}^{\mu-1} \phi_\nu(t)$ and $f^{(\mu-1)}(t) := \sum_{\nu=1}^{\mu-1} f_\nu(t)$.

In order for the convergence, we need to estimate the norm $||a_1||_k + ||b_0||_k'$ of the solution of $D(a_1, b_0) = (a_2, b_1)$ by $||a_2||_{k-1} + ||b_1||_k$, where $||\ ||_k$ denotes the Sobolev norm of order k and $||\ ||_k'$ a norm intermediate between $||\ ||_k$ and $||\ ||_{k+1}$ (see §3). It is hard to apply the standard basic estimate technique to obtain the estimate with respect to this summing up norms of shifted orders. In the case of deformations of $V \setminus \{0\}$ ([10]), it is enough to control the norm $||f_\mu(t)||_k'$ since $\phi_\mu(t)$ is uniquely determined by $f_\mu(t)$ (in other words, the complex structure on $V \setminus \{0\}$ is uniquely determined by the embedding into $\mathbb{C}^N$). However, in the case of this paper, $\phi_\mu(t)$ is not uniquely determined by $f_\mu(t)$.

We will divide the argument into two steps. First, we will solve the D-equation near the boundary. Since this part is controlled by $||f_\mu||_k'$, we can manipulate it by the same technique as in [10]. Next, we reduce the problem to solving the D-equation with zero-boundary value condition. As is well known, this type of boundary value problem is coercive. Thus, we can control the norm $||\phi_\mu(t)||_k + ||f_\mu(t)||_k'$ of the solution for the D-equation. However, it causes a new difficulty other than in [8] and [10] in dealing with the obstruction classes.

In order to overcome this difficulty, we need to use a power series method rather than the Banach inverse mapping technique. Under the assumption of $H^2(K_\Omega^{\bullet,\bullet}) = 0$, we will do it by the method of K. Kodaira and D. C. Spencer. The general case is treated by modifying it using the Grauert division theorem.

The arrangement of this paper is as follows. In §1, we establish the analysis for the deformation complex, which is needed for our argument in the later section. In §2, we give a formal construction of the versal family, which is adjusted for the convergence-procedure. In §3, we prove its convergence. In §4, we show that it is Kuranishi versal. In order to clarify the convergence procedure, we will give the proof of (1) and (2) of Theorem 0.1 in §2-§4 under the assumption $H^2(K_\Omega^{\bullet,\bullet}) = 0$. The general case will be treated in §5. In §6, we compare our family to the versal family for $\mathcal{R}es_{\widetilde{V} \to V}$ and prove Theorem 0.1 (3).

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§ 1. Deformation complex and its analysis

Throughout this paper, we assume the followings: $V = \{w \in \mathbb{C}^N \mid h_1(w) = \dots = h_m(w) = 0\}$ and $\mathrm{Sing}(V) = \{0\}$, where $h_1(w), \dots, h_m(w)$ are holomorphic functions on a neighbourhood of $\overline{B(b)} = \{w \in \mathbb{C}^N \mid \sum_{\beta=1}^N |w^\beta|^2 \le b^2\}$. Let $f_0 : \widetilde{V} \to V$ be a resolution of singularities and denote $E := f_0^{-1}(0)$,

$U := \widetilde{V} \setminus E = V \setminus \{0\}$. We denote $r_N(w) := \sum_{\beta=1}^{N} |w^\beta|^2 \in C^\infty(\mathbb{C}^N)$, $r := r_N \circ f_0 \in C^\infty(X)$ and $\Omega_\epsilon := \{x \in \widetilde{V} \,|\, r(x) < \epsilon^2\}$. We fix $\Omega = \Omega_c$ $(0 < c < b)$. Then, the $\bar{\partial}$-theoretic description of the deformation complex for $\mathcal{R}es_{\widetilde{V}\to V}$ is the following double complex $K_\Omega^{\bullet,\bullet}$ (cf. [9]):

$$\begin{array}{ccccc}
0 & \longrightarrow & H^0(\Omega, f_0^* T^{1,0}\mathbb{C}^N) & \xrightarrow{H} & H^0(\Omega, \mathcal{O}_\Omega)^m \\
\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
K_\Omega^{0,0} := A_\Omega^0(T^{1,0}\Omega) & \xrightarrow{F} & A_\Omega^0(f_0^* T^{1,0}\mathbb{C}^N) & \xrightarrow{H} & (A_\Omega^0)^{\oplus m} \\
\downarrow{\scriptstyle \bar{\partial}} & & \downarrow{\scriptstyle \bar{\partial}} & & \downarrow{\scriptstyle \bar{\partial}} \\
A_\Omega^{0,1}(T^{1,0}\Omega) & \xrightarrow{F} & A_\Omega^{0,1}(f_0^* T^{1,0}\mathbb{C}^N) & \xrightarrow{H} & (A_\Omega^{0,1})^{\oplus m} \\
\downarrow{\scriptstyle \bar{\partial}} & & \downarrow{\scriptstyle \bar{\partial}} & & \downarrow{\scriptstyle \bar{\partial}} \\
A_\Omega^{0,2}(T^{1,0}\Omega) & \xrightarrow{F} & A_\Omega^{0,2}(f_0^* T^{1,0}\mathbb{C}^N) & \xrightarrow{H} & (A_\Omega^{0,2})^{\oplus m} \\
\downarrow{\scriptstyle \bar{\partial}} & & \downarrow{\scriptstyle \bar{\partial}} & & \downarrow{\scriptstyle \bar{\partial}} \\
\vdots & & \vdots & & \vdots
\end{array}$$

where $F := df_0$ and H denotes the homomorphism given by $H(v) = (v(h_1), \dots, v(h_m))$ for $v \in T^{1,0}\mathbb{C}^N$, and i denote the natural inclusion map.

We denote $(K_\Omega^\bullet, d)$ its total simple complex. That is

$$K_\Omega^q := K_\Omega^{0,q} \oplus K_\Omega^{1,q-1} \oplus K_\Omega^{2,q-2},$$
$$d(a_q, b_{q-1}, c_{q-2}) := (\bar{\partial} a_q, \bar{\partial} b_{q-1} + (-1)^q F a_q, \bar{\partial} c_{q-2} + (-1)^{q-1} H b_{q-1}),$$

where we denote $\bar{\partial} b_{-1} := i b_{-1}$, $\bar{\partial} c_{-1} := i c_{-1}$.

In [9], we obtained

Proposition 1.1 ([9, Proposition 1.4]). *For $q = 1, 2$,*

$$H^q(K_\Omega^{\bullet,\bullet}) \simeq H^q(K_{\overline{\Omega}}^{\bullet,\bullet}).$$

Theorem 1.2 ([9, Theorem 1 and Theorem 2]). *For $q = 1, 2$, $H^q(K_{\overline{\Omega}}^{\bullet,\bullet})$ is finite dimensional and there exist operators $Z_q : K_{\overline{\Omega}}^q \to \operatorname{Ker} d \cap K_{\overline{\Omega}}^q$ and $Q_q : \operatorname{Ker} d \cap K_{\overline{\Omega}}^q \to K_{\overline{\Omega}}^{q-1}$ satisfying*

(1) $Z_{q|\operatorname{Ker} d} = \mathrm{id}_{|\operatorname{Ker} d}$,
(2) $d \circ Q_q \circ d = d$.

The operators Z_2 and Q_2 will play basic roles in the construction of the versal family and Z_1 and Q_1 are needed for the proof of the Kuranishi-versality. In this section, we study the norms of these operators which will be needed for the argument in the convergence procedure.

The followings are the basic estimates on which our argument will rely. Let $X = \Omega$ or $\Omega \setminus \overline{\Omega}_\epsilon$, and E denotes a holomorphic vector bundle over a neighbourhood of $\overline{X}$.

Then, the $\bar{\partial}$-Neumann operator has the property

$$||Y\bar{\partial}^* Nu||_k + ||\bar{Z}\bar{\partial}^* Nu||_k \le C||u||_k \tag{1.1}$$

for $u \in A^{0,1}_{\overline{X}}(E)$ if $\bar{Z}$ is a (0,1)-vector field and Y is *an admissible vector field* in the sense that $Y_x \in T^{1,0}_x X \cap \mathbb{C}T_x \partial X$ holds for all $x \in \partial X$ (cf. [3], [6]).

The Green's operator satisfies the estimate

$$||\bar{\partial}^* Gu||_k \le C||u||_{k-1} \tag{1.2}$$

for $u \in A^{0,q}_{\overline{\Omega}}(E)$ with $u_{|\partial\Omega} = 0$ (cf. [4]).

1.1. Estimates for Z_1 and Q_1

Z_1 and Q_1 are constructed by the same way as in [8], relying on the $\bar{\partial}$-analysis of $A^{0,1}_{\overline{\Omega}}(T^{1,0})$ and $A^{0,1}_{\overline{\Omega}}(f_0^* T^{1,0}\mathbb{C}^N)$. Hence, we infer the following estimate from (1.1).

Theorem 1.3. *Let* $Z_1(a_1, b_0, c_{-1}) = (a_1', b_0', c_{-1}')$ *and* $Q_1(a_1', b_0', c_{-1}') = (a_0'', b_{-1}'')$ *for* $(a_1, b_0, c_{-1}) \in K^1_{\overline{\Omega}}$. *Then,*

$$||a_0''||_k + ||Ya_0''||_k + ||\bar{Z}a_0''||_k \le C_1||a_1'||_k \le C_2||a_1||_k \tag{1.3}$$

holds for a $(0,1)$*-vector field* $\bar{Z}$ *and an admissible* $(1,0)$*-vector field* Y.

1.2. Estimates for Z_2 and Q_2

Main analytical task in the construction of the versal family is to construct the solution operator for the d-equation at $K^2_{\overline{\Omega}}$

$$d(a_1, b_0, c_{-1}) = (a_2, b_1, c_0)$$

under the condition $d(a_2, b_1, c_0) = (0,0,0)$.

The essential part of this equation is

$$D(a_1, b_0) := (\bar{\partial}a_1, \bar{\partial}b_0 - Fa_1) = (a_2, b_1)$$

since c_{-1} is determined by $c_{-1} := c_0 - Hb_0$.

We consider it in the following double-complex. Let $X := \Omega$ or $\Omega \setminus \overline{\Omega}_\epsilon$ $(0 < \epsilon < c)$ and denote $L^{\bullet,\bullet}_X$ the following double-complex.

$$
\begin{array}{ccc}
0 & \longrightarrow & H^0(X, f_0^* T^{1,0}\mathbb{C}^N) \\
\downarrow & & \downarrow i \\
L_X^{0,0} := A_X^0(T^{1,0}X) & \xrightarrow{F} & A_X^0(f_0^* T^{1,0}\mathbb{C}^N) \\
\downarrow \bar\partial & & \downarrow \bar\partial \\
A_X^{0,1}(T^{1,0}X) & \xrightarrow{F} & A_X^{0,1}(f_0^* T^{1,0}\mathbb{C}^N) \\
\downarrow \bar\partial & & \downarrow \bar\partial \\
A_X^{0,2}(T^{1,0}X) & \xrightarrow{F} & A_X^{0,2}(f_0^* T^{1,0}\mathbb{C}^N) \\
\downarrow \bar\partial & & \downarrow \bar\partial \\
\vdots & & \vdots
\end{array}
$$

We denote

$$L_X^q := L_X^{0,q} \oplus L_X^{0,q-1},$$
$$D(\phi, g) := (\bar\partial\phi, \bar\partial g + (-1)^q F\phi) \text{ for } (\phi, g) \in L_X^q.$$

We consider the D-equation

$$D(a_1, b_0) = (a_2, b_1) \in L_X^2$$

under the condition $D(a_2, b_1) = (0, 0)$.

Since F is injective, we have a long exact sequence of cohomology groups:

$$\text{(1.4)} \quad \cdots \to H^1(X, T^{1,0}X) \xrightarrow{F} H^1(X, f_0^* T^{1,0}\mathbb{C}^N) \to H^2(L_X^{\bullet,\bullet}) \to H^2(X, T^{1,0}X) \xrightarrow{F} \cdots.$$

If X is a neighbourhood of the boundary (i.e., $X = \Omega \setminus \overline{\Omega}_\epsilon$), then

$$H^2(L_X^{\bullet,\bullet}) \simeq H^1(X, N_{U/\mathbb{C}^N})$$

holds. Hence, solving a D-equation in L_X^2 near the boundary is reduced to solving a $\bar\partial$-equation in $A_X^{0,1}(N_{U/\mathbb{C}^N})$.

If X is a neighbourhood of the exceptional set E (i.e., $X = \Omega_\epsilon$), we infer from (1.4) that the solution of a D-equation near the exceptional set can be obtained by solving $\bar\partial$-equations in $A_X^{0,2}(T^{1,0}X)$ and in $A_X^{0,1}(f_0^* T^{1,0}\mathbb{C}^N)$.

In the next subsections, we construct the solution operators in each cases.

1.3. $\bar\partial$-Neumann boundary condition for $L^2_{\overline{\Omega}\setminus\Omega_\epsilon}$

In this subsection, we recall the solution operator of the D-equation obtained in [10].

Let

$$\text{(1.5)} \quad {}^\circ\tilde{T}^{1,0}U := \{u \in T^{1,0}U \mid u(r) = 0\},$$

$$(1.6) \qquad {}^{\circ}\tilde{T}^{1,0}\mathbb{C}^N := \{v \in T^{1,0}\mathbb{C}^N{}_{|\mathbb{C}^N\setminus\{0\}} \mid v(r_N) = 0\}$$

be differentiable subbundles of $T^{1,0}U$ and $T^{1,0}\mathbb{C}^N{}_{|\mathbb{C}^N\setminus\{0\}}$, respectively.

We fix a splitting of differentiable vector bundles over U.

$$(1.7) \qquad 0 \to T^{1,0}U \underset{p}{\overset{F}{\rightleftarrows}} T^{1,0}\mathbb{C}^N{}_{|U} \underset{q}{\overset{R}{\rightleftarrows}} N_{U/\mathbb{C}^N} \to 0$$

such that

$$p\left({}^{\circ}\tilde{T}^{1,0}\mathbb{C}^N{}_{|U}\right) \subset {}^{\circ}\tilde{T}^{1,0}U \text{ and } q\left(N_{U/\mathbb{C}^N}\right) \subset {}^{\circ}\tilde{T}^{1,0}\mathbb{C}^N{}_{|U}.$$

Using this splitting together with the $\bar{\partial}$-Neumann operator N at $A^{0,1}_{\overline{\Omega}\setminus\Omega_\epsilon}(N_{U/\mathbb{C}^N})$, we infer the following theorem from the argument of [8] with the estimate (1.1).

Theorem 1.4 ([10]). *There exist operators $Z_2' : L^2_{\overline{\Omega}\setminus\Omega_\epsilon} \to \text{Ker } D \cap L^2_{\overline{\Omega}\setminus\Omega_\epsilon}$ and $Q_2' : \text{Ker } D \cap L^2_{\overline{\Omega}\setminus\Omega_\epsilon} \to L^1_{\overline{\Omega}\setminus\Omega_\epsilon}$ satisfying*

(1) $Z'_{2|\text{Ker } D} = \text{id}_{|\text{Ker } D}$,

(2) $D \circ Q_2' \circ D = D$.

(3) *Let $Z_2'(a_2, b_1) = (a_2', b_1')$ and $Q_2'(a_2', b_1') = (a_1'', b_0'')$ for $(a_2, b_1) \in L^2_{\overline{\Omega}\setminus\Omega_\epsilon}$. Then,*

$$(1.8) \qquad ||b_1'||_k \leq C_1||b_1||_k,$$

$$(1.9) \qquad ||a_1''||_k + ||Yb_0''||_k + ||\bar{Z}b_0''||_k \leq C_1||b_1'||_k$$

hold for a $(0,1)$-vector field $\bar{Z}$ and an admissible $(1,0)$-vector field Y.

Remark 1.5. As remarked in [10], the isomorphism

$$\text{Ker } \Box_{\bar{\partial}} \simeq H^1(\Omega \setminus \overline{\Omega}_\epsilon, N_{U/\mathbb{C}^N}) \simeq H^2(L^{\bullet,\bullet}_{\Omega\setminus\overline{\Omega}_\epsilon})$$

does not necessarily hold unless $n \geq 3$. However, the argument of this paper will not require this property.

1.4. Dirichlet boundary condition for $L^2_{\overline{\Omega}}$

In §2, we will need a solution of the D-equation supported in a prescribed subdomain.

We fix $\Omega' := \Omega_{c'}$ $(0 < c' < c)$. Let E be a holomorphic vector bundle over a neighbourhood of $\overline{\Omega}$. Set $\mathcal{D}_0^q := \left\{u \in A^{0,q}_{\overline{\Omega}}(E) \mid \text{Supp } u \subset \Omega'\right\}$ and denote $[\mathcal{D}_0^q]_{L^2}$ its L^2-completion.

Then, a Gårding-type inequality

$$||u||_1 \leq C\left(||\bar{\partial}u||^2 + ||\vartheta u||^2 + ||u||^2\right) \text{ for } u \in \mathcal{D}_0^q$$

asserts the following.

(1) $\mathcal{H}_0^q := \text{Ker } (\Box)$ is finite dimensional.

(2) There exists the Green operator $G : [\mathcal{D}_0^q]_{L^2} \to \mathrm{Dom}\,(\square)$ such that

$$u = \rho_0 u + \square G u \ (u \in [\mathcal{D}_0^q]_{L^2}),$$

where ρ_0 denotes the orthogonal projection onto Ker $(\square)$.

(3) $G\rho_0 = \rho_0 G = 0$, $\square G = G\square$, $\bar\partial G = G\bar\partial$ holds on $\mathrm{Dom}(\bar\partial)$, $\vartheta G = G\vartheta$ holds on $\mathrm{Dom}(\bar\partial^*)$.

(4) $Gu \in A_{\overline{\Omega}}^{0,q}(E)$ $(u \in A_{\overline{\Omega}}^{0,q}(E))$ and Supp $Gu \subset \overline{\Omega'}$.

(5) $||Gu||_{k+2} \leq C||u||_k$ holds for $u \in \mathcal{D}_0^{0,q}$.

Using this Green's operator G, we construct a solution operator of the D-equation under the $\mathcal{D}_0^q$-type boundary condition. Let $\mathcal{L}_0^q := \{(\theta,\xi) \in L_{\overline{\Omega}}^q \mid \mathrm{Supp}\,(\theta,\xi) \subset \overline{\Omega'}\}$.

For $(a_2, b_1) \in \mathcal{L}_0^2$, we set

$$h_2 = \rho_0 a_2 \in \mathcal{H}_0^2(T^{1,0}\Omega), \tag{1.10}$$

$$h_1 = \rho_0(b_1 + F\vartheta G a_2 + \vartheta G F h_2) \in \mathcal{H}_0^1(f_0^* T^{1,0}\mathbb{C}^N). \tag{1.11}$$

Proposition 1.6. *If $D(a_2, b_1) = (0,0)$, then*

$$h_2 \in \mathrm{Ker}\{\mathcal{H}_0^2(T^{1,0}\Omega) \overset{\rho_0\circ F}{\to} \mathcal{H}_0^2(f_0^* T^{1,0}\mathbb{C}^N)\}$$

holds and we have the following decomposition

$$a_2 = h_2 + \bar\partial\vartheta G a_2,$$

$$b_1 = h_1 - \vartheta G F h_2 + \bar\partial\vartheta G(b_1 + F\vartheta G a_2) - F\vartheta G a_2.$$

Proof. Since $\bar\partial a_2 = 0$, we have

$$a_2 = \rho_0 a_2 + \bar\partial\vartheta G a_2.$$

Since $F\rho_0 a_2 = Fa_2 - F\bar\partial\vartheta G a_2 = -\bar\partial b_1 - \bar\partial F\vartheta G a_2$, we have

$$\rho_0 a_2 \in \mathrm{Ker}\{\mathcal{H}_0^2(T^{1,0}\Omega) \to \mathcal{H}_0^2(f_0^* T^{1,0}\mathbb{C}^N)\}.$$

$\bar\partial(b_1 + F\vartheta G a_2 + \vartheta G F h_2) = F(-a_2 + \bar\partial\vartheta G a_2) + \bar\partial\vartheta G F h_2 = -Fh_2 + \bar\partial\vartheta G F h_2 = 0$ since Fh_2 is $\bar\partial$-exact.

Hence, $b_1 + F\vartheta G a_2 + \vartheta G F h_2 = h_1 + \bar\partial\vartheta G(b_1 + F\vartheta G a_2 + \vartheta G F h_2)$. □

Let $(a_2, b_1) \in \mathcal{L}_0^2$ satisfy $D(a_2, b_1) = (0,0)$ and denote h_2, h_1 as (1.10) and (1.11). Then, we define

$$\rho_{\mathcal{L}_0}(a_2, b_1) := (h_2, h_1 - \vartheta G F h_2), \tag{1.12}$$

$$Q_{\mathcal{L}_0}(a_2, b_1) := (\vartheta G a_2, \vartheta G(b_1 + F\vartheta G a_2)). \tag{1.13}$$

Obviously, $\rho_{\mathcal{L}_0}(a_2, b_1)$ and $Q_{\mathcal{L}_0}(a_2, b_1)$ are supported in $\overline{\Omega'}$, and we have

$$(a_2, b_1) = \rho_{\mathcal{L}_0}(a_2, b_1) + DQ_{\mathcal{L}_0}(a_2, b_1). \tag{1.14}$$

Proposition 1.7. *If there exists $c_0 \in K_{\overline{\Omega}}^{2,0}$ such that $\bar\partial c_0 = Hb_1$ and $(a_2, b_1, c_0) = (0,0,0)$ in $H^2(K_{\overline{\Omega}}^{\bullet,\bullet})$, then $\rho_{\mathcal{L}_0}(a_2, b_1) = (0,0)$ in $H^2(L_{\overline{\Omega}}^{\bullet,\bullet})$.*

Proof. It is clear that $(a_2, b_1) = (0,0)$ in $H^2(L^{\bullet,\bullet}_{\overline{\Omega}})$. Hence, it follows from (1.14) that $\rho_{\mathcal{L}_0}(a_2, b_1)$ is D-exact. □

From the estimate (1.2), we infer

Proposition 1.8. *Let* $(a_1', b_0') = Q_{\mathcal{L}_0}(a_2, b_1)$. *Then*

$$||a_1'||_k + ||b_0'||_{k+1} \leq C\left(||a_2||_{k-1} + ||b_1||_k\right).$$

Proof.

$$||a_1'||_k = ||\vartheta G a_2||_k \leq C||a_2||_{k-1}.$$
$$||b_0'||_{k+1} = ||\vartheta G(b_1 + F\vartheta G a_2)||_{k+1} \leq C\left(||b_1||_k + ||a_2||_{k-1}\right).$$ □

§ 2. Analytic construction of the versal family

In this section, we construct $(\phi(t), f(t))$ satisfying (0.1)–(0.5) in the introduction, under the assumption $H^2(K^{\bullet,\bullet}_{\Omega}) = 0$. General case will be treated in §5.

We fix the following notations. Let $U := V\backslash\{0\}$ and let ${}^\circ\tilde{T}^{1,0}U$ and ${}^\circ\tilde{T}^{1,0}\mathbb{C}^N$ be the differentiable vector subbundles of $T^{1,0}U$ and $T^{1,0}\mathbb{C}^N{}_{|\mathbb{C}^N\backslash\{0\}}$, respectively, introduced in §1.1. We fix a (1,0)-type vector field $\eta^{1,0}$ on U complementary to ${}^\circ\tilde{T}^{1,0}U$ and $\xi^{1,0} := \sum_{\beta=1}^N w^\beta \frac{\partial}{\partial w^\beta}$. Then we have splittings of differentiable vector bundles;

$$T^{1,0}\mathbb{C}^N{}_{|\mathbb{C}^N\backslash\{0\}} = \mathbb{C}\xi^{1,0} \oplus {}^\circ\tilde{T}^{1,0}\mathbb{C}^N{}_{|\mathbb{C}^N\backslash\{0\}},$$
$$T^{1,0}U = \mathbb{C}\eta^{1,0} \oplus {}^\circ\tilde{T}^{1,0}U.$$

We denote the projections with respect to these splittings by the same symbols p_0, p_1. This decomposition of $T^{1,0}U$ together with the decomposition of $T^{1,0}\mathbb{C}^N{}_{|\mathbb{C}^N\backslash\{0\}}$ induces the decomposition of $(T^{0,1}U)^*$ and further the following decompositions

(2.1)
$$f_0^*T^{1,0}\mathbb{C}^N{}_{|U} \otimes \wedge^q(T^{0,1}U)^* = {}^\circ\tilde{T}^{1,0}\mathbb{C}^N{}_{|U} \otimes \wedge^q(T^{0,1}U)^* + \mathbb{C}\xi^{1,0} \otimes \wedge^q({}^\circ\tilde{T}^{0,1}U)^* + \mathbb{C}\xi^{1,0} \otimes \wedge^{q-1}({}^\circ\tilde{T}^{0,1}U)^* \otimes (\mathbb{C}\overline{\eta^{1,0}})^*,$$

(2.2)
$$T^{1,0}U \otimes \wedge^q(T^{0,1}U)^* = {}^\circ\tilde{T}^{1,0}U \otimes \wedge^q(T^{0,1}U)^* + \mathbb{C}\eta^{1,0} \otimes \wedge^q({}^\circ\tilde{T}^{0,1}U)^* + \mathbb{C}\eta^{1,0} \otimes \wedge^{q-1}({}^\circ\tilde{T}^{0,1}U)^* \otimes (\mathbb{C}\overline{\eta^{1,0}})^*.$$

We denote the projections with respect to these splittings by the same symbols p_0, p_{10} and p_{11}.

We fix $0 < \epsilon_0 < \epsilon_1 < c$ and denote $\Omega_j = \{x \in X \mid r(x) < \epsilon_j\}$ and $\Omega_j^* := \Omega\backslash\overline{\Omega}_j$ for $j = 0, 1$.

We assume that there exists $\theta \in \Gamma(\overline{\Omega},\, T^{1,0}\Omega)$ such that the following conditions are satisfied;

(2.3)
$$\theta_{|\partial\Omega} = 0,$$

$$(2.4) \qquad (p_{10}+p_{11})\big(\eta^{1,0}\otimes(\overline{\eta^{1,0}})^{*}-\bar{\partial}\theta\big)_{|\partial\Omega}=0.$$

(The existence of θ is proved in [1] pp. 346–347.) From now on, we fix such θ and $\theta_1 \in A^{0,1}_{\overline{\Omega}}(T^{1,0}X)$ such that

$$\theta_1 := \eta^{1,0}\otimes(\overline{\eta^{1,0}})^{*}-\bar{\partial}\theta$$

holds on a neighbourhood of $\overline{\Omega}^{*}_{0}$.

By a similar calculation as in Proposition 3.7 of [10], we have

Proposition 2.1. *Any cohomology class of $H^1(K^{\bullet,\bullet}_{\overline{\Omega}})$ has a representative $(a_1, b_0, c_{-1}) \in K^{1}_{\overline{\Omega}}$ having the following form;*

$$a_1 = \phi_0 + \psi\theta_1 - \theta\otimes\bar{\partial}\psi,$$
$$b_0 = b_{0,0} - \psi F\theta$$

with $\phi_0 \in A^{0,1}_{\overline{\Omega}}(T^{1,0}X)$, $\psi \in C^{\infty}(\overline{\Omega})$ and $b_{0,0} \in A^{0}_{\overline{\Omega}}(f_0^{*}T^{1,0}\mathbb{C}^N)$ such that ϕ_0 (resp. $b_{0,0}$) is ${}^{\circ}\tilde{T}^{1,0}U$-valued (resp. ${}^{\circ}\tilde{T}^{1,0}\mathbb{C}^{N}{}_{|U}$-valued) over $\overline{\Omega}^{*}_{0}$.

We denote by $(\phi_\mu(t),\ f_\mu(t),\ k_\mu(t))$ the homogeneous part of $(\phi(t),\ f(t),\ k(t))$ of degree μ and construct it inductively on μ.

Let

$$(2.5) \qquad (\phi_1(t),\ f_1(t),\ k_1(t)) = \sum_{\sigma=1}^{d}(\phi_\sigma,\ f_\sigma,\ k_\sigma)t_\sigma,$$

where $\{(\phi_\sigma,\ f_\sigma,\ k_\sigma)\}_{1\le\sigma\le d}$ is a basis of $H^1(K^{\bullet,\bullet}_{\overline{\Omega}})$ having the properties in Proposition 2.1.

Assume that $(\phi_1(t),\ f_1(t),\ k_1(t)),\dots,(\phi_{\mu-1}(t),\ f_{\mu-1}(t),\ k_{\mu-1}(t))$ are determined for $\mu\ge 2$ such that

(2.6)

$$\begin{aligned} &P(\phi^{(\nu)}(t), f^{(\nu)}(t), k^{(\nu)}(t))\\ &:= \Big(\bar{\partial}\phi^{(\nu)}(t)-\frac{1}{2}[\phi^{(\nu)}(t),\phi^{(\nu)}(t)],\ (\bar{\partial}-\phi^{(\nu)}(t))(f_0+f^{(\nu)}(t)),\ (h+k^{(\nu)}(t))\circ(f_0+f^{(\nu)}(t))\Big)\\ &\equiv 0 \qquad \mathrm{mod}\ \mathfrak{m}^{\nu+1}\end{aligned}$$

holds for $\nu\le\mu-1$.

Let

$$(2.7) \qquad (p_\mu(t),\ q_\mu(t),\ r_\mu(t)) = -\ \text{the } \mu\text{-th homogeneous term of } P(\phi^{(\mu-1)}(t), f^{(\mu-1)}(t), k^{(\mu-1)}(t)).$$

Then, $(\phi_\mu(t),\ f_\mu(t),\ k_\mu(t))$ is determined by solving the following equation

$$(2.8) \qquad d(\phi_\mu(t),\ f_\mu(t),\ k_\mu(t)) = -(p_\mu(t),\ q_\mu(t),\ r_\mu(t)).$$

By a similar calculation as [8, Proposition 8.4], we have

Proposition 2.2.

$$d(p_\mu(t),\, q_\mu(t),\, r_\mu(t)) = (0,0,0).$$

Hence, (2.8) is solvable under the assumption $H^2(K^{\bullet,\bullet}_{\overline{\Omega}}) = 0$. We note that the essential part of this equation is

$$D(\phi_\mu(t),\, f_\mu(t)) = -(p_\mu(t),\, q_\mu(t)) \tag{2.9}$$

since $k_\mu(t)$ is uniquely determined by

$$k_\mu(t) = -r_\mu(t) - Hf_\mu(t).$$

2.1. Solution of (2.9) near the boundary

First, we solve the equation (2.9) over $\overline{\Omega}_0^*$ by using the solution operator Q_2' for the D-equation and making the adjustment as in [10].

We recall the following two operators A and B which were introduced there.

A is an operator which was used to modify $(a_1'', b_0'') \in L^1_{\overline{\Omega}_0^*}$ with $b_0'' \in {}^\circ\widetilde{T}^{1,0}\mathbb{C}^N$ over $\overline{\Omega}_0^*$ so that $p_{10}(\bar\partial b_0'') = 0$ holds. Since $\widetilde{L}_U := p_{10} \circ F \circ \bar\partial : A^0_U({}^\circ\widetilde{T}^{1,0}U) \to A^{0,1}_U(\mathbb{C}\xi^{1,0})$ is a $C^\infty(U)$-isomorphism, we define

$$A(a_1'', b_0'') := (a_1'', b_0'') - D(\widetilde{L}_U^{-1} p_{10}\bar\partial b_0'',\, 0). \tag{2.10}$$

If we denote

$$(a_1''', b_0''') = A(a_1'', b_0''),$$

then $p_1 b_0''' = 0$ and $p_{10}\bar\partial b_0''' = 0$ hold.

Next, B is an operator which was used to modify (a_1''', b_0''') as above so that $(p_{10} + p_{11})\bar\partial b'''_{0\,|\partial\Omega} = 0$ holds. Let $p_{11}\bar\partial b_0''' = \psi\xi^{1,0} \otimes \overline{\eta^{1,0}}^*$ with $\psi \in C^\infty(\overline{\Omega}_0^*)$. Then B is defined by

$$B(a_1''', b_0''') := (a_1''', b_0''') - D(\psi\theta,\, 0). \tag{2.11}$$

If we set

$$(a_1'''', b_0'''') = B(a_1''', b_0'''),$$

then $p_1 b''''_{0\,|\partial\Omega} = 0$ and $(p_{10} + p_{11})\bar\partial b''''_{0\,|\partial\Omega} = 0$ hold.

We note that $DB = DA = D$ holds.

Let

$$(a''_{1\,\mu}(t),\, b''_{0\,\mu}(t)) = -Q_2' Z_2' (p_\mu(t),\, q_\mu(t))_{|\overline{\Omega}_0^*},$$

where Z_2' and Q_2' are operators obtained in Theorem 1.3. We denote

$$(a'''_{1\mu}(t),\, b'''_{0\mu}(t)) = A(a''_{1\mu}(t),\, b''_{0\mu}(t))$$

and

$$(a''''_{1\mu}(t),\, b''''_{0\mu}(t)) = B(a'''_{1\mu}(t),\, b'''_{0\mu}(t)).$$

2.2. Solution of (2.9) near the exceptional set

Next, we solve the remaining part of the equation (2.9) using the operators $Q_{\mathcal{L}_0}$ and $\rho_{\mathcal{L}_0}$ for $\Omega' := \Omega_1$.

In the followings, we fix $\rho_1 \in C^\infty(X)$ such that $0 \le \rho_1 \le 1$, Supp $\rho_1 \subset \Omega_1$ and $\rho_1 \equiv 1$ on Ω_0. Let

$$(\phi'_\mu(t), f'_\mu(t)) := (1-\rho_1)\left(a_{1\,\mu}^{''''}(t), b_{0\,\mu}^{''''}(t)\right), \tag{2.12}$$

$$(p'_\mu(t), q'_\mu(t)) := (p_\mu(t), q_\mu(t)) + D(\phi'_\mu(t), f'_\mu(t)). \tag{2.13}$$

Then $(\phi'_\mu(t), f'_\mu(t)) \in L^1_{\overline{\Omega}}$ and $(p'_\mu(t), q'_\mu(t)) \in \mathcal{L}^2_0$ hold. By Proposition 2.2 and (1.14), we have

$$(p'_\mu(t), q'_\mu(t)) = \rho_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t)) + DQ_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t)).$$

We remark that $\rho_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t))$ belongs to a finite dimensional subspace $\mathbf{H}^2 := \{(h_2, h_1 - \vartheta GFh_2) \,|\, h_2 \in \mathrm{Ker}\{\mathcal{H}^2_0(T^{1,0}X) \to \mathcal{H}^2_0(f_0^*T^{1,0}\mathbb{C}^N)\}, h_1 \in \mathcal{H}^1_0(f_0^*T^{1,0}\mathbb{C}^N)\}$ of $\mathcal{L}^2_0$ and

$$\rho_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t)) = 0 \text{ in } H^2(L^{\bullet,\bullet}_{\overline{\Omega}}).$$

Then, let

$$(\phi''_\mu(t), f''_\mu(t)) := -D^*N\rho_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t)),$$
$$(\phi'''_\mu(t), f'''_\mu(t)) := -Q_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t)),$$

where N denotes the Neumann operator for $L^2_{\overline{\Omega}}$ ([9]) and set

$$(\phi_\mu(t), f_\mu(t)) := (\phi'_\mu(t), f'_\mu(t)) + (\phi''_\mu(t), f''_\mu(t)) + (\phi'''_\mu(t), f'''_\mu(t)). \tag{2.14}$$

Proposition 2.3.

$$D(\phi_\mu(t), f_\mu(t)) = -(p_\mu(t), q_\mu(t)).$$

Proof.

$$\begin{aligned}
&D(\phi_\mu(t), f_\mu(t))\\
&= D(\phi'_\mu(t), f'_\mu(t)) - DD^*N\rho_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t)) - DQ_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t))\\
&= -(p_\mu(t), q_\mu(t)) + (p'_\mu(t), q'_\mu(t)) - \rho_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t)) - DQ_{\mathcal{L}_0}(p'_\mu(t), q'_\mu(t))\\
&= -(p_\mu(t), q_\mu(t)).
\end{aligned}$$

□

Finally if we set

$$k_\mu(t) := -r_\mu(t) - Hf_\mu(t)$$

then we have

$$\bar\partial k_\mu(t) = -\bar\partial r_\mu(t) - H\bar\partial f_\mu(t) = 0.$$

It implies that $k_\mu(t) \in K^{2,-1}_{\overline{\Omega}}[t]$ and (2.8) holds.

Thus we complete the formal construction of $(\phi^{(\mu)}(t), f^{(\mu)}(t), k^{(\mu)}(t))$ ($\mu = 1, 2, \dots$)

§ 3. Proof of convergence

In this section, we prove the convergence of $\phi(t)$ and $f(t)$.

3.1. Boundary behavior of $\phi(t)$ and $f(t)$

First, we shall analyze the boundary behavior of $(\phi_\mu(t), f_\mu(t))$. Let $\chi_\mu(t) \in A^{0,1}_{\overline{\Omega_0^*}}$ and $\psi_\mu(t) \in C^\infty(\overline{\Omega_0^*})$ be defined by

$$(3.1) \qquad (p_{10}+p_{11})q_\mu(t)_{|\overline{\Omega_0^*}} = \chi_\mu(t)\otimes\xi^{1,0},$$

$$(3.2) \qquad p_{11}\bar\partial b'''_{0\mu}(t)_{|\overline{\Omega_0^*}} = \psi_\mu(t)\xi^{1,0}\otimes\overline{\eta^{1,0}}^*,$$

respectively.

Proposition 3.1. *Let*

$$\begin{aligned}\phi_{\mu\,0}(t) &:= \phi_\mu(t)_{|\overline{\Omega_0^*}} + \bar\partial\big((1-\rho_1)\psi_\mu(t)\theta\big)\\ &\quad - (1-\rho_1)\psi_\mu(t)\eta^{1,0}\otimes(\overline{\eta^{1,0}})^* - (1-\rho_1)\chi_\mu(t)\otimes\eta^{1,0},\\ f_{\mu 0}(t) &:= f_\mu(t)_{|\overline{\Omega_0^*}} + F\big((1-\rho_1)\psi_\mu(t)\theta\big).\end{aligned}$$

Then we have

(1) $(p_{10}+p_{11})\phi_{\mu\,0}(t)_{|\overline{\Omega}\setminus\Omega_1} = 0,$
(2) $p_1 f_{\mu\,0}(t)_{|\overline{\Omega}\setminus\Omega_1} = 0.$

Since $1-\rho_1 = 1$, $\phi_\mu(t) = \phi'_\mu(t)$ and $f_\mu(t) = f'_\mu(t)$ hold on $\Omega\setminus\Omega_1$, it is proved by the same argument as Proposition 3.9 (1) of [10].

Using this decompositions of $\phi_\mu(t)$ and $f_\mu(t)$, we shall describe their boundary behavior. Denote $\phi_0^{(\mu-1)}(t) = \sum_{\nu=1}^{\mu-1}\phi_{\mu\,0}(t)$ and $f_0^{(\mu-1)}(t) = \sum_{\nu=1}^{\mu-1} f_{\mu\,0}(t)$.

We fix a finite open covering $\overline{\Omega}\subset\cup_{i\in I}U_i$. Let $I_0 = \{i\in I\,|\,U_i\cap\partial\Omega\neq\emptyset\}$ and assume $\overline{\Omega}\setminus\Omega_0\subset\cup_{i\in I_0}U_i$ and ${}^\circ\tilde{T}'U$ is smoothly trivialized on U_i. From now on, we fix a local frame $\{e_{i,\lambda}\}_{1\le\lambda\le n-1}$ of ${}^\circ\tilde{T}'U_{|U_i}$.

We define (0,1)-forms $\chi_{\mu\,0}(t)$, $\chi_{\mu\,I}(t)$, $\chi_{\mu\,III}(t)$, $\chi_{\mu\,IV}(t)$ on $\overline{\Omega_0^*}$, and (0,1)-forms $\chi^\lambda_{\mu\,II\,i}(t)$ $(1\le\lambda\le n)$ on $U_i\cap\overline{\Omega_0^*}$, inductively as follows: Let $\chi_{1\,0}(t) = \chi_{1\,I}(t) = \chi_{1\,III}(t) = \chi_{1\,IV}(t) = 0$ and $\chi^\lambda_{1\,II\,i}(t) = 0$.

After determining $\chi_{\nu\,0}(t)$, $\chi_{\nu\,I}(t)$, $\chi_{\nu\,III}(t)$, $\chi_{\nu\,IV}(t)$ and $\chi^\lambda_{\nu\,II\,i}(t)$ for $1\le\nu\le\mu-1$, $\chi_{\mu\,0}(t)$, $\chi_{\mu\,I}(t)$, $\chi_{\mu\,III}(t)$, $\chi_{\mu\,IV}(t)$ and $\chi^\lambda_{\mu\,II\,i}(t)$ are defined by

$$\begin{aligned} r\chi_{\mu\,0}(t) = {}&\text{the } \mu\text{-th homogeneous term of } -\phi^{(\mu-1)}(t)\big(f_0^{(\mu-1)}(t)(r_N)\big)\\ &-(1-\rho_1)^2\psi^{(\mu-1)}(t)\chi_0^{(\mu-1)}(t)\eta^{1,0}(F\theta(r_N)),\\ r\chi_{\mu\,I}(t) = {}&\text{the } \mu\text{-th homogeneous term of } \phi^{(\mu-1)}(t)\big((1-\rho_1)\psi^{(\mu-1)}(t)\big)\\ &-(1-\rho_1)^2\psi^{(\mu-1)}(t)\chi_I^{(\mu-1)}(t)\eta^{1,0}(F\theta(r_N)),\\ r\chi^\lambda_{\mu\,II\,i}(t) = {}&\text{the } \mu\text{-th homogeneous term of } (1-\rho_1)\psi^{(\mu-1)}(t)\phi_{0\,i}^{(\mu-1)\,\lambda}(t)\\ &-(1-\rho_1)^2\psi^{(\mu-1)}(t)\chi_{II\,i}^{(\mu-1)\,\lambda}(t)\eta^{1,0}(F\theta(r_N))\ (1\le\lambda\le n-1),\end{aligned}$$

$$
\begin{aligned}
r\chi^{n}_{\mu\, II\, i}(t) = {} & \text{the } \mu\text{-th homogeneous term of } (1-\rho_1)\psi^{(\mu-1)}(t)\phi_{0\,i}^{(\mu-1)\,n}(t) \\
& - (1-\rho_1)^2\psi^{(\mu-1)}(t)\chi_{II\,i}^{(\mu-1)\,n}(t)\eta^{1,0}(F\theta(r_N)), \\
r\chi_{\mu\, III}(t) = {} & \text{the } \mu\text{-th homogeneous term of } (1-\rho_1)\psi^{(\mu-1)}(t) \\
& \cdot\bar\partial((1-\rho_1)\psi^{(\mu-1)}(t)) - (1-\rho_1)^2\psi^{(\mu-1)}(t)\chi_{III}^{(\mu-1)}(t)\eta^{1,0}(F\theta(r_N)), \\
r\chi_{\mu\, IV}(t) = {} & \text{the } \mu\text{-th homogeneous term of } (1-\rho_1)^2\psi^{(\mu-1)}(t)^2 \\
& - (1-\rho_1)^2\psi^{(\mu-1)}(t)\chi_{IV}^{(\mu-1)}(t)\eta^{1,0}(F\theta(r_N)),
\end{aligned}
$$

where we denote $\psi^{(\mu-1)}(t) = \sum_{\nu=1}^{\mu-1}\psi_\nu(t)$, $\chi_0^{(\mu-1)}(t) = \sum_{\nu=1}^{\mu-1}\chi_{\nu\,0}(t)$ and use the same notations for $\chi_I^{(\mu-1)}(t), \dots, \chi_{IV}^{(\mu-1)}(t)$, and $\phi_0^{(\mu-1)}(t)_{|U_i} = \sum_{\lambda=1}^{n}\phi_{0\,i}^{(\mu-1)\lambda}(t)e_{i,\lambda}$ is an expression with respect to the local frame $(e_{i,1},\dots,e_{i,n-1})$ of ${}^{\circ}\widetilde{T}^{1,0}X$ over U_i and $e_{i,n} = \eta^{1,0}$.

We remark that $\chi_{\mu\,0}(t) = 0$ on $\overline{\Omega}\setminus\Omega_1$ and $\chi^{n}_{\mu\, II\, i}(t) = 0$ on $U_i \cap (\overline{\Omega}\setminus\Omega_1)$. Hence, the following proposition is proved by the same argument of Proposition 3.9 (2) in [10].

Proposition 3.2. *For $\mu \geq 2$, it holds*

$$
\begin{aligned}
\chi_\mu(t) = {} & \chi_{\mu\, I}(t)F\theta(r_N) + \sum_{i\in I_0} p_i \sum_{\lambda=1}^{n-1}\chi^{\lambda}_{\mu\, II\, i}(t)e_{i,\lambda}(F\theta(r_N)) \\
& + \chi_{\mu\, III}(t)\theta(F\theta(r_N)) + \chi_{\mu\, IV}(t)\theta_1(F\theta(r_N))
\end{aligned}
$$

over $\overline{\Omega}\setminus\Omega_1$, where $\sum_{i\in I} p_i = 1$ is a partition of unity subordinate with the covering $\{U_i\}_{i\in I}$.

We denote $\phi_0(t) := \lim_{\mu\to\infty}\phi_0^{(\mu)}(t)$ and use the same notation for $\psi(t)$, $\chi_0(t)$, $\chi_I(t)$, $\chi^{\lambda}_{IIi}(t)$, $\chi^{n}_{IIi}(t)$, $\chi_{III}(t)$ and $\chi_{IV}(t)$.

Corollary 3.3. *$\phi(t)$ has the following decomposition on $\overline{\Omega}\setminus\Omega_1$:*

$$
\begin{aligned}
\phi(t) = {} & \phi_0(t) - \bar\partial(\psi(t))\otimes\theta + \psi(t)\theta_1 + \chi_I(t)\otimes(F\theta(r_N))\eta^{1,0} \\
& + \sum_{i\in I_0}\sum_{\lambda=1}^{n-1}\chi^{\lambda}_{IIi}(t)\otimes(p_i e_{i,\lambda}(F\theta(r_N)))\eta^{1,0} \\
& + \chi_{III}(t)\otimes(\theta(F\theta(r_N)))\eta^{1,0} + \chi_{IV}(t)\otimes(\theta_1(F\theta(r_N)))\eta^{1,0}.
\end{aligned}
$$

Note that $\phi_0(t)$ and $\rho_0\eta^{1,0}$ are $\widetilde{T}^{1,0}X$-valued on $\overline{\Omega}\setminus\Omega_1$ and θ, θ_1, $F\theta(r_N)\eta^{1,0}$, $p_i e_{i,\lambda}(F\theta(r_N))\eta^{1,0}$, $\theta(F\theta(r_N))\eta^{1,0}$ and $\theta_1(F\theta(r_N))\eta^{1,0}$ are all admissible vector fields.

3.2. Proof of convergence

In this section, we prove the convergence of $\phi(t)$ and $f(t)$.

Let $||\phi||_k$ denote the Sobolev k-norm of $\phi \in A^{0,1}_{\overline{\Omega}}(T'X)$ and introduce the norms $||\psi||'_k$ for $\psi \in C^\infty(\overline{\Omega})$ and $||f||'_k$ for $f \in A^0_{\overline{\Omega}}(f_0^*T'\mathbb{C}^N)$ as follows: For $\psi \in A^{0,1}_{\overline{\Omega}}$ with Supp $\psi \subset U_i$ $(i \in I_0)$,

$$\begin{aligned} ||\psi||'_k := {} & ||\bar{\partial}\psi||_k + \sum_{\lambda=1}^{n-1} ||e_{i,\lambda}(\psi)||_k + ||\theta\psi||_k + ||\theta_1\psi||_k \\ & + ||(F(\theta(r)))\eta\psi||_k + \sum_{\lambda=1}^{n-1} ||(e_{i,\lambda}(F(\theta(r))))\eta\psi||_k \\ & + ||(\theta(F\theta(r)))\eta\psi||_k + ||(\theta_1(F\theta(r)))\eta\psi||_k + ||\psi||_k. \end{aligned} \tag{3.3}$$

Here we note that $(F\theta(r))_{|\partial\Omega} = (e_{i,\lambda}F\theta(r))_{|\partial\Omega} = (\theta F\theta(r))_{|\partial\Omega} = 0$ and $(\theta_1 F\theta(r))_{|\partial\Omega} = 0$ hold.

For $\psi \in A^{0,1}_{\overline{\Omega}}$ with Supp $\psi \subset U_i$ $(i \notin I_0)$, $||\psi||'_k := ||\psi||_{k+1}$. $||f||'_k$ is defined similarly for $f \in A^0_{\overline{\Omega}}(f_0^*T'\mathbb{C}^N)$ with Supp $f \subset U_i$.

For global ψ and f, $||\psi||'_k$ and $||f||'_k$ are defined by patching the local definitions using a partition of unity.

For the argument of the convergence, we utilize the Grauert pseudo-norm for the convergent power series (see [5]). In the followings, we use the notations $t := (t_1, \dots, t_d)$ and $t^\nu := t_1^{\nu_1} \cdots t_d^{\nu_d}$ for $\nu := (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$. Let

$$M(t) := \Pi_{i=1}^d \left(\frac{1}{2} - \frac{1}{4}\sqrt{1-t_i} \right).$$

Then, it satisfies $M(t)^2 << M(t)$.

For $\rho := (\rho_1, \dots, \rho_d) \in \mathbb{R}^d_+$, a pseudo-norm $|| \ ||_\rho$ is introduced as follows. Let A be a Banach space. For $u(t) = \sum_{\nu\in\mathbb{N}^d} u_\nu t^\nu \in A[[t_1, \dots, t_d]]$, we define

$$||u||_\rho := \inf \left\{ c \in \mathbb{R}_+ \mid \sum_{\nu\in\mathbb{N}^d} ||u_\nu|| t^\nu << cM\left(\frac{t}{\rho}\right) \right\}.$$

We collect fundamental properties of the pseudo-norm $|| \ ||_\rho$ (see [5] for details).

(i) $u(t)$ is convergent if and only if $||u(t)||_\rho$ is finite for some $\rho \in \mathbb{R}^d_+$.

(ii) (a) Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{C}\{t_1, \dots, t_d\}$. If $u(t) \in \mathfrak{m}^e$, then $||u(t)||_{\theta\rho} \le \theta^e ||u(t)||_\rho$ holds for $0 < \theta \le 1$.

(b) $||u(t)||_\rho = 0$ if and only if $u(t) = 0$.

(c) $||cu(t)||_\rho = |c| ||u(t)||_\rho$ for $c \in \mathbb{C}$.

(d) $||u(t) + v(t)||_\rho \le ||u(t)||_\rho + ||v(t)||_\rho$.

(iii) If $||ab|| \le C||a|| ||b||$ holds for $a, b \in A$, then

$$||u(t)v(t)||_\rho \le C||u(t)||_\rho ||v(t)||_\rho$$

holds for $u(t), v(t) \in A\{t_1, \dots, t_d\}$.

(iv) If $u_i(t) = \sum_{\nu \in \mathbb{N}^d} a_{i\nu} t^\nu$ $(i \in I)$ are disjoint in the sense that there exist disjoint subsets $D_i \in \mathbb{N}^d$ $(i \in I)$ such that $a_{i\nu} = 0$ for $\nu \notin D_i$, then

$$||\sum_{i\in I} u_i(t)||_\rho = \sup_{i\in I} ||u_i(t)||_\rho$$

holds.

By (i), it is enough to prove that $||\phi^{(\nu)}(t)||_{k,\rho} + ||f^{(\nu)}(t)||'_{k,\rho}$ is finite for some $\rho \in \mathbb{R}^d_+$.

By (ii) (a), for any $\sigma > 0$, there exists $\rho_0 \in \mathbb{N}^d$ such that

$$||\phi^{(1)}(t)||_{k,\rho} + ||f^{(1)}(t)||'_{k,\rho} \leq \sigma$$

holds for $0 < \rho \leq \rho_0$.

Proposition 3.4. *There exists $\rho \in \mathbb{R}^d_+$ such that*

$$(1_\mu) \qquad ||\phi^{(\mu)}(t)||_{k,\rho} + ||f^{(\mu)}(t)||'_{k,\rho} \leq \sigma,$$

$$(2_\mu) \qquad ||\phi_0^{(\mu)}(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} + ||f_0^{(\mu)}(t)||'_{k,\overline{\Omega}\backslash\Omega_1,\rho} \leq \sigma,$$

(3_μ)

$$||\chi_I^{(\mu)}(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} + \sum_{\lambda=1}^{n-1} ||\chi_{II\,i}^{(\mu)\,\lambda}(t)||_{k,U_i\cap(\overline{\Omega}\backslash\Omega_1),\rho} + ||\chi_{II\,i}^{(\mu)\,n}(t)||_{k,U_i\cap(\Omega_1\backslash\Omega_1),\rho}$$
$$+ ||\chi_{III}^{(\mu)}(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} + ||\chi_{IV}^{(\mu)}(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} \leq \sigma,$$

$$(4_\mu) \qquad ||\psi^{(\mu)}(t)||'_{k,\overline{\Omega}\backslash\Omega_1,\rho} \leq \sigma$$

hold for all $\mu \geq 1$.

Proof. We prove them by induction on μ. It is trivial for $\mu = 1$.

Let $\mu \geq 2$ and we assume that (1_ν)-(4_ν) hold for $1 \leq \nu \leq \mu - 1$.

Proposition 3.5. *There exist positive constants $C_1, \ldots, C_{12}$ independent of μ and ρ and such that*

(1) $||p_\mu(t)||_{k-1,\rho} + ||q_\mu(t)||_{k,\rho} \leq C_1\sigma^2$,
(2) $||b''_{0\mu}(t)||'_{k,\overline{\Omega}\backslash\Omega_0,\rho} + ||a''_1(t)||_{k,\overline{\Omega}\backslash\Omega_0,\rho} \leq C_2\sigma^2$,
(3) $||\bar{\partial} a''_{0\mu}(t)||_{k,\overline{\Omega}\backslash\Omega_0,\rho} \leq C_3\sigma^2$,
(4) $||\chi_{\mu I}(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} + \sum_{\lambda=1}^{n-1} ||\chi_{\mu II,\lambda}(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} + ||\chi_{\mu III}(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho}$
$+ ||\chi_{\mu IV}(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} \leq C_4\sigma^2$,
(5) $||\psi_\mu(t)||'_{k,\overline{\Omega}\backslash\Omega_1,\rho} \leq C_5\sigma^2$,
(6) $||f_{0\mu}(t)||'_{k,\overline{\Omega}\backslash\Omega_1,\rho} \leq C_6\sigma^2$,
(7) $||f'_\mu(t)||'_{k,\overline{\Omega}\backslash\Omega_1,\rho} = ||f_\mu(t)||'_{k,\overline{\Omega}\backslash\Omega_1,\rho} \leq C_7\sigma^2$,
(8) $||\phi'_\mu(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} = ||\phi_\mu(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} \leq C_8\sigma^2$,
(9) $||\phi_{0\mu}(t)||_{k,\overline{\Omega}\backslash\Omega_1,\rho} \leq C_9\sigma^2$,

(10) $||p'_\mu(t)||_{k-1,\rho} + ||q'_\mu(t)||_{k,\rho} = ||p'_\mu(t)||_{k-1,\overline{\Omega}_1,\rho} + ||q'_\mu(t)||_{k,\overline{\Omega}_1,\rho} \leq C_{10}\sigma^2$,
(11) $||\phi''_\mu(t)||_{k,\rho} + ||f''_\mu(t)||_{k+1,\rho} \leq C_{11}\sigma^2$,
(12) $||\phi'''_\mu(t)||_{k,\rho} + ||f'''_\mu(t)||_{k+1,\rho} \leq C_{12}\sigma^2$.

Proof. (1) Since

$$||p_\mu(t)||_{k-1,\rho} \leq (1/2)||[\phi^{(\mu-1)}(t), \phi^{(\mu-1)}(t)]||_{k-1,\rho} \leq C||\phi^{(\mu-1)}(t)||^2_{k,\rho},$$

we have $||p_\mu(t)||_{k-1,\rho} \leq C\sigma^2$ by the inductive assumption.
Since

$$||q_\mu(t)||_{k,\rho} \leq ||\phi^{(\mu-1)}(t)f^{(\mu-1)}(t)||_{k,\rho} \leq C||\phi^{(\mu-1)}(t)||_{k,\rho}||f^{(\mu-1)}(t)||'_{k,\rho},$$

we have $||q_\mu(t)||_{k,\rho} \leq C\sigma^2$ by the inductive assumption.
(2)~(9) follow by the same argument as in [10] Proposition 3.10.
(10) follows from the definition of $p'_\mu(t)$, $q'_\mu(t)$ and (7), (8).
(11) follows since Im $\rho_{\mathcal{L}_0}$ is finite dimensional.
(12) follows from (10) and Proposition 1.8. □

Hence, if we choose σ at the beginning such that $\sigma \leq 1/C_j$ $(j = 1, \ldots, 12)$ hold then (1_μ)-(4_μ) hold. □

Thus $\phi(t)$ and $f(t)$ are convergent with respect to $|| \; ||_k$-norm and $|| \; ||'_k$-norm, respectively.

§ 4. The Kuranishi versality

By the same technique as in §10 of [8], we can prove the following Kuranishi versality.

Proposition 4.1. *Let $(\phi(t), f(t))$ be the family constructed in §2 and §3. Let $\varpi : \mathcal{Y} \to S$ be a family of complex manifolds together with a holomorphic map $\Psi : \mathcal{Y} \to \mathbb{C}^N \times S$ satisfying the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\Psi} & \mathbb{C}^N \times S \\ \varpi\downarrow & & \downarrow \\ S & = & S \end{array}$$

If $\varpi^{-1}(0)$ is a neighbourhood of $\overline{\Omega}$ in X and $\Psi_{|\varpi^{-1}(0)} = f_0$ holds, then there exist a holomorphic map $\tau : S \to T$, a holomorphic family of embeddings $G : \overline{\Omega}\times S \to \mathcal{Y}$, and a formal family of holomorphic maps $\hat{\eta} : \overline{B(c)}\times S \to \mathbb{C}^N\times S$ satisfying

(1) *$\tau(0) = 0$, $G_{|\overline{\Omega}\times 0} = \mathrm{id}_{\overline{\Omega}}$, and $\hat{\eta}_{|\overline{B}\times 0} = \mathrm{id}_{\overline{B}}$,*
(2) *the family of complex structures $\phi(\tau(s))$ is induced by the holomorphic family of embeddings G,*
(3) *$\varpi \circ G$ coincides with the projection onto the second factor and $\hat{\eta} \circ \Psi = f_0 + f(\tau(s))$.*

Outline of the proof. We fix notations.

- S is a germ of an analytic subspace of a neighbourhood of $0 \in \mathbb{C}^N$ defined by the ideal $\mathcal{I}$.
- $\{\mathcal{Y}_i, (\zeta_i^1, \dots, \zeta_i^n; s^1, \dots, s^e)\}_{i \in I}$ is a system of local chart of $\mathcal{Y}$ together with the coordinate transformation $\zeta_i^\lambda = \zeta_{ij}^\lambda(\zeta_j^1, \dots, \zeta_j^n; s^1, \dots, s^e)$ ($\lambda = 1, \dots, n$).
- $X_i := \mathcal{Y}_i \cap \varpi^{-1}(0)$ and $z_i^\lambda := \zeta^\lambda_{i|\varpi^{-1}(0)}$ ($\lambda = 1, \dots, n$) provides with a system of local chart of X.
- Holomorphic map $\Psi : \mathcal{Y} \to \mathbb{C}^N \times S$ is represented by N'ple of holomorphic functions $(\Psi^1, \dots, \Psi^N)$ and the family of differentiable mappings $f_0 + f(t) : \overline{\Omega} \times S \to \mathbb{C}^N \times S$ is represented by $(\Phi^1(t), \dots, \Phi^N(t))$ with $\Phi^\beta(t) \in A^0_{\overline{\Omega}}[[t_1, \dots, t_d]] \cap A^0_k\{t_1, \dots, t_d\}$ for all $k >> 0$.

Then τ, G, and $\hat{\eta}$ are given by

$$\tau : \ t^\alpha = \tau^\alpha(s) \in \mathbb{C}[[s^1, \dots, s^e]] \ (\alpha = 1, \dots, d),$$

$$G : \ \zeta_i^\lambda = G_i^\lambda(z_i, s) \in C^\infty(\overline{\Omega} \cap X_i)[[s^1, \dots, s^e]] \ (\lambda = 1, \dots, n),$$

$$\hat{\eta} : \ w^\beta = \eta^\beta(w, s) \in H^0(\overline{B(c)}, \mathcal{O}_{\mathbb{C}^N})[[s^1, \dots, s^e]] \ (\beta = 1, \dots, N)$$

satisfying

$$\text{(4.1)} \quad \tau^\alpha(0) = 0 \ (\alpha = 1, \dots, d), \ G_i^\lambda(z_i, 0) = z_i^\lambda \ (\lambda = 1, \dots, n), \quad \eta^\beta(0) = w^\beta \ (\beta = 1, \dots, N),$$

$$\text{(4.2)} \quad G_i^\lambda(z_{ij}(z_j), s) - \zeta_{ij}^\lambda(G_j(z_j, s), s) \in \mathcal{I}A^0_{U_i \cap U_j \cap \overline{\Omega}, k}(T^{1,0}X)\{s_1, \dots, s_e\} \quad \text{for all } k >> 0 \ (\lambda = 1, \dots, n),$$

$$\text{(4.3)} \quad (\bar{\partial} - \phi(\tau(s)))G_i^\lambda(z_i, s) \in \mathcal{I}A^0_{U_i \cap \overline{\Omega}, k}(T^{1,0}X)\{s_1, \dots, s_e\} \quad \text{for all } k >> 0 \ (\lambda = 1, \dots, n),$$

$$\text{(4.4)} \quad \eta^\beta(\Psi_i(G_i(z_i, s), s), s) - \Phi_i^\beta(z_i, \tau(s)) \in \mathcal{I}A^0_{U_i \cap \overline{\Omega}}(f^*T^{1,0}\mathbb{C}^N)[[s_1, \dots, s_e]] \quad (\beta = 1, \dots, N).$$

Then the proof of Proposition 4.1 is parallel to §10 of [8] except that we use the norms $||a_0||'_k$ and $||a_1||_k$ for $a_i \in K^{0,i}_{\overline{\Omega}}$ ($i = 0, 1$) in place of the Folland-Stein norm $||a_0||_{(k+1)}$ and $||a_1||_{(k)}$ in [8], respectively. □

§ 5. General case

The conclusions (1) and (2) of the main theorem are still valid without the condition $H^2(K^{\bullet,\bullet}_{\overline{\Omega}}) = 0$. In this section, we give the outline of the modification of the argument of §2 and §3. In this case, the parameter space of the versal family is a local analytic subspace at the origin in $H^1(K^{\bullet,\bullet}_{\overline{\Omega}})$. The main task is to construct the defining ideal $\mathfrak{J}$ as well as a power series $(\phi(t), f(t), k(t)) \in K\frac{1}{\Omega}[[t_1, \dots, t_d]]$ such that (0.1)-(0.5) hold with $\mathcal{J}_T := \mathfrak{J}$.

For this generalization, We modify the argument in §2 and §3 using the Grauert division theorem (see [5]).

5.1. The Grauert division theorem and the modification of the formal construction

First of all, we recall the fundamental properties of the Grauert division theorem. We introduce the following lexicographical ordering in $\mathbb{N}^d$. For $\nu = (\nu_1, \dots, \nu_d)$ and $\lambda = (\lambda_1, \dots, \lambda_d)$, we denote $\nu < \lambda$ if $|\nu| < |\lambda|$ holds or $|\nu| = |\lambda|$ and there exists a $1 \le k \le d$ such that $\nu_1 = \lambda_1, \dots, \nu_{k-1} = \lambda_{k-1}$ and $\nu_k < \lambda_k$ hold. For a power series $f(t) = \sum_\nu a_\nu t^\nu$ we denote

$$\text{ord}\,(f(t)) := \inf\{\nu \in \mathbb{N}^d \,|\, a_\nu \neq 0\}.$$

A subset $\Lambda \subset \mathbb{N}^d$ is called a reducing system if

$$\nu \notin \lambda + \mathbb{N}^d$$

holds for all $\lambda, \nu \in \Lambda$ satisfying $\lambda \neq \nu$.

For a reducing system Λ, we have the following partition of $\mathbb{N}^d$. Let λ_0 be the minimal element of Λ. Then

$$D_{\lambda_0} := \lambda_0 + \mathbb{N}^d.$$

For $\lambda > \lambda_0$, let

$$D_\lambda := \left(\lambda + \mathbb{N}^d\right) \setminus \cup_{\nu<\lambda} D_\nu$$
$$\Delta := \mathbb{N}^d \setminus \cup_{\lambda\in\Lambda}(\lambda + \mathbb{N}^d)$$

be called the reducing multi indices.

A power series $f(t) = \sum_\nu a_\nu t^\nu$ is called reduced with respect to Λ if $a_\nu = 0$ for all $\nu \notin \Delta$.

A family $(\omega_\lambda(t))_{\lambda\in\Lambda}$ of power series of the type

$$\omega_\lambda(t) = t^\lambda + \alpha_\lambda(t) \in \mathbb{C}[[t_1, \dots, t_d]]$$

with $\text{ord}(\alpha_\lambda(t)) > \lambda$ is called a division system with respect to Λ.

The Grauert's division theorem is as follows.

Theorem 5.1 ([5] Satz 1.14). *Let Λ be a reducing system and $(\omega_\lambda(t))_{\lambda\in\Lambda}$ its division system. Then, for a Banach space A, we have a unique decomposition of a power series $f(t) \in A\{t_1, \dots, t_d\}$,*

$$f(t) = \sum_{\lambda\in\Lambda} Q_\lambda(t)\omega_\lambda(t) + R(t)$$

where $Q_\lambda(t), R(t) \in A\{t_1, \dots, t_d\}$ have the form

$$Q_\lambda(t) = \sum_{\nu\in D_\lambda-\lambda} q_\lambda t^\lambda,\ R(t) = \sum_{\nu\in\Delta} r_\nu t^\nu,$$

and satisfy $\text{ord}(R(t)) \ge \text{ord}(f(t))$ *and* $\text{ord}(Q_\lambda(t)) + \lambda \ge \text{ord}(f(t))$.

We call $R(t)$ the canonical modulus of $f(t)$ and denote it by $\mathrm{red}_\Lambda f(t)$.

For an ideal $\mathfrak{J} \subset \mathbb{C}\{t_1, \ldots, t_d\}$, we have a mono-ideal $I := \{\mathrm{ord}(f(t)) \mid f(t) \in \mathfrak{J}\}$. Then an element $\lambda \in I$ is called an extremal point if $\lambda \notin \nu + \mathbb{N}^d$ holds for all $\nu \in I$ with $\nu \neq \lambda$. The set of all extremal points is a reducing system. We call it the reducing system associated with the ideal $\mathfrak{J}$ and denote the canonical modulus by $\mathrm{red}_\mathfrak{J}$ instead of red_Λ. We remark that a division system with respect to the associating reducing system Λ is a system of generators of $\mathfrak{J}$ (cf. [5] Satz 2.8).

By applying the Grauert's division theorem, we have a division system $(\omega_\lambda(t))_{\lambda \in \Lambda}$ such that

$$\omega_\lambda(t) = t^\lambda + \alpha_\lambda(t)$$

and $\alpha_\lambda(t)$ is reduced with respect to Λ. We call a division system of this type a Weierstraß family for $\mathfrak{J}$.

In [9], we constructed a successive sequence of extensions of ideals $\mathfrak{J}^{(\mu)} \subset \mathcal{H}_\mu := \mathbb{C}\{t_1, \ldots, t_d\}/\mathfrak{m}^{\mu+1}$. We will modify it so that $\widehat{\mathfrak{J}} := \lim_{\mu \to \infty} \mathfrak{J}^{(\mu)}$ is generated by convergent power series.

Before starting the modification, we recall the relationship between the extensions of ideals and of its reducing systems.

Let μ_0 be a non-negative integer and $\mathfrak{J}^{(\mu_0)}$ an ideal of $\mathcal{H}_{\mu_0}$. Let Λ_{μ_0} be the set of extremal points of the set $\{\mathrm{ord}(f(t)) \mid f(t) \in \mathfrak{J}^{(\mu_0)}, \mathrm{ord}(f(t)) \leq \mu_0\}$. For $\mu > \mu_0$, an ideal $\mathfrak{J}^{(\mu)}$ of $\mathcal{H}_\mu$ is called an extension of $\mathfrak{J}^{(\mu_0)}$ if $\mathfrak{J}^{(\mu)} = \mathfrak{J}^{(\mu_0)}$ holds in $\mathcal{H}_{\mu_0}$. Refer to [5] for the following properties concerning extensions of $\mathfrak{J}^{(\mu_0)}$.

(α) If $\Lambda_\mu = \Lambda_{\mu_0}$ then

$$\omega_\lambda^\mu(t) \equiv \omega_\lambda^{\mu_0}(t) \bmod \mathfrak{m}^{\mu_0+1} \ (\lambda \in \Lambda)$$

holds for Weierstraß families $(\omega_\lambda^{\mu_0}(t))_{\lambda \in \Lambda}$ and $(\omega_\lambda^\mu(t))_{\lambda \in \Lambda}$ for $\mathfrak{J}^{(\mu)}$ and $\mathfrak{J}^{(\mu_0)}$, respectively ([5] Satz 3.6).

(β) The extension is minimal if and only if $\Lambda_\mu = \Lambda_{\mu_0}$ holds ([5] Satz 3.9).

(γ) Let $(\omega_\lambda^\mu(t))_{\lambda \in \Lambda}$ be the Weierstraß family for $\mathfrak{J}^{(\mu)}$ and suppose that $(\omega_\lambda^\mu(t))_{\lambda \in \Lambda_0}$ ($\Lambda_0 \subset \Lambda$) is a minimal generator of $\mathfrak{J}^{(\mu)}$. For a minimal extension $\mathfrak{J}^{(\mu+1)}$ of $\mathfrak{J}^{(\mu)}$, there exist $b_{\lambda\nu} \in \mathbb{C}$ ($\lambda \in \Lambda_0, \nu \in \Delta_{\mu+1}$) such that $\mathfrak{J}^{(\mu+1)}$ is generated by $(\omega_\lambda^{\mu\,(b)}(t))_{\lambda \in \Lambda_0}$ with denoting $(b) := (b_{\lambda\nu})_{\lambda \in \Lambda_0, \nu \in \Delta_{\mu+1}}$ and

$$\omega_\lambda^{\mu\,(b)}(t) := \omega_\lambda^\mu(t) + \sum_{\nu \in \Delta_{\mu+1}} b_{\lambda\nu} t^\nu \ (\lambda \in \Lambda_0).$$

(δ) For a successive extensions of ideals $\mathfrak{J}^{(\mu)}$ ($\mu = 1, 2, \ldots$) and the associating reducing systems Λ_μ, there exists μ_0 such that $\Lambda_\mu = \Lambda_{\mu_0}$ holds for all $\mu \geq \mu_0$.

In [9], we obtained successive sequences of families $(\phi^{(\mu)}(t), f^{(\mu)}(t), k^{(\mu)}(t)) \in K^1_{\overline{\Omega}}[t_1, \ldots, t_d]$ and of ideals $\mathfrak{J}^{(\mu)}$ of $\mathcal{H}_\mu$ such that

- $(\phi^{(\mu)}(t), f^{(\mu)}(t), k^{(\mu)}(t))$ is a polynomial of degree μ,

- $(\mathfrak{J}^{(\mu)})_{\mu\geq 1}$ is a successive sequence of extensions of ideals of $\mathcal{H}_\mu$,
- $P(\phi^{(\mu)}(t), f^{(\mu)}(t), k^{(\mu)}(t)) \equiv 0 \bmod \mathfrak{J}^{(\mu)}$,
- $(\phi^{(\mu)}(t), f^{(\mu)}(t), k^{(\mu)}(t))$ defined over $\mathrm{Spec}(\mathcal{H}_\mu/\mathfrak{J}^{(\mu)})$ is Kuranishi-versal for all μ-th order deformations where we call elements in $\mathcal{R}es_{X\to V}(T)$ with $\mathfrak{m}_T^{\mu+1} = \{0\}$ μ-th order deformations.

Let Λ_μ be the reducing system associated with $\mathfrak{J}^{(\mu)}$. Then, there exists μ_0 such that $\Lambda_\mu = \Lambda_{\mu_0}$ holds for all $\mu \geq \mu_0$ and the successive extensions $\mathfrak{J}^{(\mu+1)}$ of $\mathfrak{J}^{(\mu)}$ are all minimal extensions for $\mu \geq \mu_0$.

We denote $\Lambda := \Lambda_{\mu_0}$. Let $(\omega_\lambda^{(\mu_0)}(t))_{\lambda\in\Lambda}$ be the Weierstraß family for $\mathfrak{J}^{(\mu_0)}$. Suppose that $(\omega_\lambda^{(\mu_0)}(t))_{\lambda\in\Lambda_0}$ ($\Lambda_0 \subset \Lambda$) is a minimal generator of $\mathfrak{J}^{(\mu_0)}$ and

$$\omega_{\lambda'}^{(\mu_0)}(t) - \sum_{\lambda\in\Lambda_0} c_{\lambda'\lambda}(t)\omega_\lambda^{(\mu_0)}(t) \in \mathfrak{m}^{\mu_0+1}\ (\lambda' \in \Lambda)$$

holds for polynomials $c_{\lambda'\lambda}(t)$ of degree μ_0.

Then, each minimal extension $\mathfrak{J}^{(\mu+1)}$ of $\mathfrak{J}^{(\mu)}$ ($\mu \geq \mu_0$) is given by

$$\mathfrak{J}^{(\mu+1)} = \mathfrak{J}^{(\mu)\,(b)} := \sum_{\lambda\in\Lambda_0} \omega_\lambda^{(\mu+1)\,(b)}(t)\mathcal{H}_{\mu+1}$$

with

$$\omega_\lambda^{(\mu+1)\,(b)}(t) := \omega_\lambda^{(\mu)}(t) + \sum_{\nu\in\Delta_{\mu+1}} b_{\lambda\nu}t^\nu\ (\lambda \in \Lambda_0)$$

for some $(b) := (b_{\lambda\nu})_{\lambda\in\Lambda_0,\nu\in\Delta_{\mu+1}}$. Furthermore the Weierstraß family $(\omega_{\lambda'}^{(\mu+1)}(t))_{\lambda'\in\Lambda}$ for $\mathfrak{J}^{(\mu+1)}$ is given by

$$\omega_{\lambda'}^{(\mu+1)}(t) = t^{\lambda'} + \mathrm{red}_{\mathfrak{J}^{(\mu+1)}}\beta_{\lambda'}^{(b)}(t)\ (\lambda' \in \Lambda),$$

where $\beta_{\lambda'}^{(b)}(t) := \sum_{\lambda\in\Lambda_0} c_{\lambda'\lambda}(t)\omega_\lambda^{(\mu+1)\,(b)}(t) - t^{\lambda'}$.

Let $(\bar{\phi}^{(\mu_0)}(t), \bar{f}^{(\mu_0)}(t), \bar{k}^{(\mu_0)}(t))$ be a polynomial of degree μ_0 which agrees with $\mathrm{red}_{\mathfrak{J}^{(\mu_0)}}(\phi^{(\mu_0)}(t), f^{(\mu_0)}(t), k^{(\mu_0)}(t))$ up to the μ_0-th term. Then $(\bar{\phi}^{(\mu_0)}(t), \bar{f}^{(\mu_0)}(t), \bar{k}^{(\mu_0)}(t))$ is reduced with respect to Λ.

The Grauert division theorem asserts the decomposition

$$P(\bar{\phi}^{(\mu_0)}(t), \bar{f}^{(\mu_0)}(t), \bar{k}^{(\mu_0)}(t)) = \sum_{\lambda'\in\Lambda} Q_{\mu_0}^{\lambda'}(t)\omega_{\lambda'}^{(\mu_0)}(t) + R_{\mu_0+1}(t), \tag{5.1}$$

where $R_{\mu_0+1}(t) \equiv 0 \mod \mathfrak{m}^{\mu_0+1}$ and it is reduced with respect to Λ (cf. [5] Satz 1.14).

Let $x_{\mu_0,\lambda} \in K_\Omega^2$ ($\lambda \in \Lambda_0$) be defined by

$$x_{\mu_0,\lambda} := \sum_{\lambda'\in\Lambda} Q_{\mu_0}^{\lambda'}(0)c_{\lambda'\lambda}(0). \tag{5.2}$$

Then we have

$$dx_{\mu_0,\lambda} = 0 \tag{5.3}$$

since

$$dP(\bar{\phi}^{(\mu_0)}(t),\, \bar{f}^{(\mu_0)}(t),\, \bar{k}^{(\mu_0)}(t)) \equiv 0 \bmod \mathfrak{m}(\widetilde{\mathfrak{I}}^{(\mu_0)} + \mathfrak{m}^{\mu_0+1})$$

holds by the same reason as Proposition 2.2, where $\widetilde{\mathfrak{I}}^{(\mu_0)}$ is the ideal of $\mathbb{C}\{t_1, \ldots, t_d\}$ generated by $\omega_{\lambda'}^{(\mu_0)}(t)\,(\lambda' \in \Lambda)$. We denote $\mathrm{Ob} \subset H^2(K_{\overline{\Omega}}^{\bullet,\bullet})$ the subspace spanned by $x_{\mu_0,\lambda}\,(\lambda \in \Lambda_0)$.

Suppose that we obtained a polynomial $(\bar{\phi}^{(\mu-1)}(t),\, \bar{f}^{(\mu-1)}(t),\, \bar{k}^{(\mu-1)}(t))$ of degree $\mu - 1 \geq \mu_0$ which is reduced with respect to Λ and such that

$$P(\bar{\phi}^{(\mu-1)}(t),\, \bar{f}^{(\mu-1)}(t),\, \bar{k}^{(\mu-1)}(t)) \equiv 0 \bmod \mathfrak{I}^{(\mu-1)}$$

holds.

Let $y_\mu(t)$ be a homogeneous polynomial of degree μ such that

$$y_\mu(t) - \mathrm{red}_{\mathfrak{I}^{(\mu-1)\,(0)}} P(\bar{\phi}^{(\mu-1)}(t),\, \bar{f}^{(\mu-1)}(t),\, \bar{k}^{(\mu-1)}(t)) \equiv 0 \mod \mathfrak{m}^{\mu+1}$$

holds. Then, $y_\mu(t)$ is reduced with respect to Λ and, by the same way as (5.3), we have

$$dy_\mu(t) = 0.$$

Proposition 5.2. *The cohomology classes of the coefficients of $y_\mu(t)$ belong to* Ob.

Proof. Let $(\widetilde{\mathcal{V}}_{\mu-1}, \Phi_{\mu-1}, T_{\mu-1})$ be the family associated with $(\bar{\phi}^{(\mu-1)}(t), \bar{f}^{(\mu-1)}(t), \bar{k}^{(\mu-1)}(t))$ and $\mathfrak{I}^{(\mu-1)}$. Since both of the original $(\phi^{(\mu-1)}(t),\, f^{(\mu-1)}(t),\, k^{(\mu-1)}(t))$ and $(\bar{\phi}^{(\mu-1)}(t),\, \bar{f}^{(\mu-1)}(t),\, \bar{k}^{(\mu-1)}(t))$ together with $\mathfrak{I}^{(\mu-1)}$ are Kuranishi versal for $(\mu-1)$-th order deformations, we have an isomorphism

$$(\widehat{\mathcal{X}}_{|T_{\mu-1}}, \widehat{\Phi}_{|T_{\mu-1}}, T_{\mu-1}) \simeq (\widetilde{\mathcal{V}}_{\mu-1}, \Phi_{\mu-1}, T_{\mu-1}).$$

Hence, the obstruction class for the extension of $(\bar{\phi}^{(\mu-1)}(t), \bar{f}^{(\mu-1)}(t), \bar{k}^{(\mu-1)}(t))$ to $(\widehat{\phi}(t)_{|\mathcal{H}_\mu/\mathfrak{I}^{(\mu)}},\, \widehat{f}(t)_{|\mathcal{H}_\mu/\mathfrak{I}^{(\mu)}},\, \widehat{k}(t)_{|\mathcal{H}_\mu/\mathfrak{I}^{(\mu)}})$ vanishes.

Now we may assume that $\mathfrak{I}^{(\mu)}$ is generated by

$$\omega_\lambda^{(\mu)\,(b^*)}(t) := \omega_\lambda^{(\mu-1)}(t) + \sum_{\nu\in\Delta_\mu} b^*_{\lambda\nu} t^\nu \;(\lambda \in \Lambda_0) \tag{5.4}$$

for some $b^* := (b^*_{\lambda\nu})_{\lambda\in\Lambda_0,\nu\in\Delta_\mu}$.

For each $b := (b_{\lambda\nu})_{\lambda\in\Lambda_0,\nu\in\Delta_\mu}$, we set

$$R_\mu^{(b)}(t) := \mathrm{red}_{\mathfrak{I}^{(\mu-1)(b)}+\mathfrak{m}^{\mu+1}} P(\bar{\phi}^{(\mu-1)}(t),\, \bar{f}^{(\mu-1)}(t),\, \bar{k}^{(\mu-1)}(t)).$$

Then, $R_\mu^{(0)}(t) = y_\mu(t)$ and $\left[R_\mu^{(b^*)}(t)\right] = 0$. Hence, by [5] Satz 3.12, we have

$$[y_\mu(t)] = \left[R_\mu^{(0)}(t) - R_\mu^{(b^*)}(t)\right] = \left[\sum_{\lambda\in\Lambda_0}\sum_{\nu\in\Delta_\mu} x_{\mu_0\lambda} b^*_{\lambda\nu} t^\nu\right].$$

□

If we set

$$\omega_\lambda^{(\mu)}(t) := \omega_\lambda^{(\mu-1)}(t) + \sum_{\nu \in \Delta_\mu} b^*_{\lambda\nu} t^\nu \, (\lambda \in \Lambda_0) \tag{5.5}$$

as in the proof of Proposition 5.2, then

$$\widetilde{y}_\mu(t) := y_\mu(t) - \sum_{\lambda \in \Lambda_0} \sum_{\nu \in \Delta_\mu} x_{\mu_0 \lambda} b^*_{\lambda\nu} t^\nu \tag{5.6}$$

is d-exact.

Hence we can apply the construction of $(\phi_\mu(t), f_\mu(t), k_\mu(t))$ in Section 2 with replacing $(p_\mu(t), q_\mu(t), r_\mu(t))$ by $\widetilde{y}_\mu(t)$.

Let

$$(\phi^{(\mu)}(t), f^{(\mu)}(t), k^{(\mu)}(t)) := (\bar{\phi}^{(\mu-1)}(t), \bar{f}^{(\mu-1)}(t), \bar{k}^{(\mu-1)}(t)) + (\phi_\mu(t), f_\mu(t), k_\mu(t)).$$

Then $(\phi^{(\mu)}(t),\, f^{(\mu)}(t),\, k^{(\mu)}(t))$ is reduced with respect to Λ and we complete the inductive construction.

5.2. Proof of convergence

In order to prove the convergence, we use the Grauert pseudo-norm as in subsection 3.2.

In addition to (i)-(iv) in subsection 3.2, it has the following properties.

(v) Let I be a finite set and suppose $u_i(t) \in A\{t_1, \ldots, t_d\}$ satisfies $\operatorname{ord}(u_i(t)) > \nu_i$ $(i \in I)$. Then, for each $\epsilon > 0$ there exists $\rho \in \mathbb{R}^d_+$ such that

$$||u_i(t)||_\rho \le \epsilon \rho^{\nu_i} \; (i \in I)$$

holds. (Cf. [5] Satz 1.8.)

(vi) Let $\Lambda \subset \mathbb{N}^d$ be a reducing system and $\omega_\lambda(t) = t^\lambda + \alpha_\lambda(t)$ $(\lambda \in \Lambda)$ its division system. For $f(t) \in A\{t_1, \ldots, t_d\}$, we have the decomposition by the Grauert division theorem

$$f(t) = \sum_{\lambda \in \Lambda} Q_\lambda(t) \omega_\lambda(t) + R(t).$$

Suppose

$$||\alpha_\lambda(t)||_\rho \le \frac{\epsilon}{|\Lambda|} \rho^\lambda \; (\lambda \in \Lambda)$$

holds for some $\rho \in \mathbb{R}^d_+$ and $0 < \epsilon < 1$. Then

$$||R(t)||_\rho \le \frac{1}{1-\epsilon} ||f(t)||_\rho$$

holds. (Cf. [5] Satz 1.14.)

The convergence of

$$\mathfrak{I} := \lim_{\mu \to \infty} \mathfrak{I}^{(\mu)} \text{ and } (\phi(t), f(t)) := \lim_{\mu \to \infty} (\phi^{(\mu)}(t), f^{(\mu)}(t))$$

are proved as follows.

By (v), there exists $\rho_0 \in \mathbb{R}^d_+$ such that

$$(5.7) \qquad ||\alpha_\lambda^{(\mu_0)}(t)||_\rho \leq \frac{1}{2|\Lambda|}\rho^\lambda$$

holds for all $0 < \rho < \rho_0$ where $\alpha_\lambda^{(\mu_0)}(t) := \omega_\lambda^{(\mu_0)}(t) - t^\lambda$ ($\lambda \in \Lambda$).

We denote $\gamma(\rho) := \min(\rho_1, \dots, \rho_d)$.

We shall show that there exist positive constants C_1 and $\rho \in \mathbb{R}^d_+$ such that

$$(\mathrm{A}_\mu) \qquad ||\bar{\phi}^{(\mu)}(t) - \bar{\phi}^{(\mu_0)}(t)||_{k,\rho} + ||\bar{f}^{(\mu)}(t) - \bar{f}^{(\mu_0)}(t)||'_{k,\rho} \leq C_1\gamma(\rho)^{\mu_0},$$

$$(\mathrm{B}_\mu) \qquad ||\omega_\lambda^{(\mu)}(t) - \omega_\lambda^{(\mu_0)}(t)||_\rho \leq \frac{1}{8K_*|\Lambda|}\gamma(\rho)^{\mu_0} \ (\lambda \in \Lambda_0),$$

hold for $\mu \geq \mu_0$.

For the simplicity of the notation, we denote $u^{(\mu)}(t) := (\phi^{(\mu)}(t), f^{(\mu)}(t), k^{(\mu)}(t))$ and

$||u(t)||_\rho := ||a_1(t)||_{k,\rho} + ||b_0(t)||'_{k,\rho}$ for $u(t) = (a_1(t), b_0(t), c_{-1}(t)) \in K^1_{\bar{\Omega}}[[t_1, \dots, t_d]]$,

$||v(t)||_\rho := ||a_2(t)||_{k-1,\rho} + ||b_1(t)||_{k,\rho}$ for $v(t) = (a_2(t), b_1(t), c_0(t)) \in K^2_{\bar{\Omega}}[[t_1, \dots, t_d]]$.

Let $K_* := \sup_{\lambda' \in \Lambda} \sum_{\lambda \in \Lambda_0} ||c_{\lambda'\lambda}(t)||_1$ with denoting $1 := (1, \dots, 1)$ and $\sigma(\rho) := ||\bar{\phi}^{(\mu_0)}(t)||_{k,\rho} + ||\bar{f}^{(\mu_0)}(t)||'_{k,\rho}$. We denote $\rho_{\mathcal{H}^2} := 1 - dQ_2$ the projection $\mathrm{Ker}\ d \cap K^2_{\bar{\Omega}} \to \mathrm{Im}\ \rho_{\mathcal{H}^2} \simeq H^2(K^{\bullet,\bullet}_{\bar{\Omega}})$ (see [9] for the definition of Q_2) and ρ_{Ob} the projection $H^2(K^{\bullet,\bullet}_{\bar{\Omega}}) \to \mathrm{Ob}$. Let C be a positive constant such that

$$|\rho_{\mathrm{Obs}} \circ \rho_{\mathcal{H}^2} u| \leq C\,(||a_2||_{k-1} + ||b_1||_k)$$

holds for all $u = (a_2, b_1, c_0) \in \mathrm{Ker}\ d \cap K^2_{\bar{\Omega}}$ where $|\ |$ denote the Euclidean norm in the finite dimensional vector space Ob.

Let

$$P(\bar{u}^{(\mu_0)}(t)) = \sum_{\lambda \in \Lambda_0} q_\lambda(t)\omega_\lambda^{(\mu_0)}(t) + r_{\mu_0+1}(t)$$

be the decomposition by the Grauert division theorem. Then $r_{\mu_0+1}(t) \equiv 0 \mod \mathfrak{m}^{\mu_0+1}$ and it is reduced with respect to Λ.

Let $\tau(\rho) := \max_{\lambda \in \Lambda_0} ||q_\lambda(t) - q_\lambda(0)||_\rho$ and $\kappa(\rho) := ||r_{\mu_0+1}(t)||_\rho$. Then we may assume that $\sigma(\rho)$, $\tau(\rho)$ and $\kappa(\rho)$ are small enough if we replace ρ by a smaller one.

We show that (A_μ), (B_μ) and

$$(\mathrm{C}_\mu) \qquad ||\alpha_\lambda^{(\mu)}(t)||_\rho \leq \frac{1}{2|\Lambda|}\rho^\lambda \ (\lambda \in \Lambda_0)$$

hold for all $\mu \geq \mu_0$ if $\rho \in \mathbb{R}^d_+$ is sufficiently small where $\alpha_\lambda^{(\mu)}(t) := \omega_\lambda^{(\mu)}(t) - t^\lambda$ for the Weierstraß family $(\omega_\lambda^{(\mu)}(t))_{\lambda \in \Lambda}$ for $\mathfrak{I}^{(\mu)}$.

Proposition 5.3. *Suppose that* $(\mathrm{A}_{\mu-1})$, $(\mathrm{B}_{\mu-1})$ *and* $(\mathrm{C}_{\mu-1})$ *hold. Then we have*

$$(\mathrm{D}_\mu) \qquad ||y_\mu(t)||_\rho \leq \frac{1}{8CK_*|\Lambda|}\gamma(\rho)^{\mu_0},$$

$$(\mathrm{E}_\mu) \qquad ||b^*_{\mu|\lambda\nu}t^\nu||_\rho \leq \frac{1}{8K_*|\Lambda|}\gamma(\rho)^{\mu_0}$$

for sufficiently small $\rho \in \mathbb{R}^d_+$.

Proof. Note that

$$\begin{aligned}
&P(\bar{u}^{(\mu-1)}(t)) \\
&= P(\bar{u}^{(\mu_0)}(t)) + \left(P(\bar{u}^{(\mu-1)}(t)) - P(\bar{u}^{(\mu_0)}(t))\right) \\
&= \sum_{\lambda\in\Lambda_0} q_\lambda(t)\omega_\lambda^{(\mu-1)}(t) + \sum_{\lambda\in\Lambda_0}(q_\lambda(t) - q_\lambda(0))\left(\omega_\lambda^{(\mu-1)}(t) - \omega_\lambda^{(\mu_0)}(t)\right) \\
&\quad + \sum_{\lambda\in\Lambda_0} q_\lambda(0)\left(\omega_\lambda^{(\mu-1)}(t) - \omega_\lambda^{(\mu_0)}(t)\right) + r_{\mu_0+1}(t) \\
&\quad + \left(P(\bar{u}^{(\mu-1)}(t)) - P(\bar{u}^{(\mu_0)}(t))\right).
\end{aligned}$$

Since $\mathrm{red}_{\mathfrak{J}^{(\mu-1)}(0)}P(\bar{u}^{(\mu-1)}(t)) \equiv 0 \mod \mathfrak{m}^{\mu+1}$ and $\omega_\lambda^{(\mu-1)}(t) - \omega_\lambda^{(\mu_0)}(t)$ is reduced with respect to Λ, we have

$$\begin{aligned}
||y_\mu(t)||_\rho \leq 2\Bigg(&\sum_{\lambda\in\Lambda_0}||q_\lambda(t) - q_\lambda(0)||_\rho||\omega_\lambda^{(\mu-1)}(t) - \omega_\lambda^{(\mu_0)}(t)||_\rho \\
&+ ||r_{\mu_0+1}(t)||_\rho + ||P(\bar{u}^{(\mu-1)}(t)) - P(\bar{u}^{(\mu_0)}(t))||_\rho\Bigg).
\end{aligned}$$

By the same calculation in Proposition 3.5, there exists constants such that

$$||P(\bar{u}^{(\mu-1)}(t)) - P(\bar{u}^{(\mu_0)}(t))||_\rho \leq (C_2\sigma(\rho) + C_3\gamma(\rho)^{\mu_0})\,\gamma(\rho)^{\mu_0}$$

holds.

Therefore we have

$$||y_\mu(t)||_\rho \leq 2\left(\frac{|\Lambda_0|}{8K_*|\Lambda|}\tau(\rho) + \kappa(\rho) + C_2\sigma(\rho) + C_3\gamma(\rho)^{\mu_0}\right)\gamma(\rho)^{\mu_0}$$

holds for sufficiently small ρ.

Since we can choose ρ so small that $\frac{|\Lambda_0|}{8K_*|\Lambda|}\tau(\rho)+\kappa(\rho)+C_2\sigma(\rho)+C_3\gamma(\rho)^{\mu_0} \leq \frac{1}{16CK_*|\Lambda|}$ holds, we have (D_μ).

(E_μ) follows from (D_μ). □

Proposition 5.4. (A_μ), (B_μ) *and* (C_μ) *follow from* (D_μ) *and* (E_μ).

Proof. It is proved in [5] 3.11 that (E_μ) implies (B_μ) and (C_μ). Since $(\phi_\mu(t)$, $f_\mu(t)$, $k_\mu(t))$ is obtained by the same way as in §3 with replacing $(p_\mu(t)$, $q_\mu(t)$, $r_\mu(t))$ by $\widetilde{y}_\mu(t)$, (A_μ) follows from (E_μ) and (D_μ) by the same calculation as Proposition 3.5. □

Since (A_{μ_0}), (B_{μ_0}) and (C_{μ_0}) hold, we complete the proof of convergence of $\mathfrak{J} := \lim_{\mu\to\infty}\mathfrak{J}^{(\mu)}$ and $(\phi(t), f(t)) := \lim_{\mu\to\infty}(\phi^{(\mu)}(t), f^{(\mu)}(t))$.

§ 6. Comparison with the versal family for $\mathcal{R}es_{\tilde{V}\to V}$

By the same comparison method as in §11 of [8], we have the following

Proposition 6.1. *Let $(\mathcal{Y}, \Psi)$ be the versal family for $\mathcal{R}es_{\tilde{V}\to V}$ and S its parameter space. Then we have an isomorphism $\tau : S \simeq T$ and a holomorphic family of embeddings $G : \overline{\Omega} \times S \to \mathcal{Y}$ such that the family of complex structures $\phi(\tau(s))$ is induced by the holomorphic family of embeddings G.*

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