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LINEAR OPERATORS THAT PRESERVE PERIMETERS OF MATRICES OVER SEMIRINGS

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Abstract. A rank one matrix can be factored as $\mathbf{u}^t\mathbf{v}$ for vectors $\mathbf{u}$ and $\mathbf{v}$ of appropriate orders. The perimeter of this rank one matrix is the number of nonzero entries in $\mathbf{u}$ plus the number of nonzero entries in $\mathbf{v}$. A matrix of rank k is the sum of k rank one matrices. The perimeter of a matrix of rank k is the minimum of the sums of perimeters of the rank one matrices. In this article we characterize the linear operators that preserve perimeters of matrices over semirings.

1. Introduction

Lately there have been many articles on linear preserver problems. For an excellent survey see [3, 4]. In [2], the linear operators that preserve the term rank and other combinatorial properties of matrices were characterized. In [5], the linear operators that preserve rank and perimeter of Boolean rank-1 matrices were characterized. In this article we investigate the linear operators that preserve perimeters of matrices over semirings.

A *semidomain* $\mathcal{D}$ is a semiring which is commutative, has a multiplicative identity different from 0 and has no zero divisors.

Let $\mathcal{M} \equiv \mathcal{M}_{m,n}(\mathcal{D})$ denote the set of all $m \times n$ matrices with entries in a semidomain $\mathcal{D}$. If $m = n$, we use the notation $\mathcal{M}_n(\mathcal{D})$ instead of $\mathcal{M}_{n,n}(\mathcal{D})$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J and O, respectively. For matrices A and B, $A \oplus B$ is the direct sum of A and B so that $A \oplus B = [\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}]$.

The *rank* or *factor rank*, $r(A)$, of a nonzero $A \in \mathcal{M}$ is defined as the least integer k for which there exist $B \in \mathcal{M}_{m,k}(\mathcal{D})$ and $C \in \mathcal{M}_{k,n}(\mathcal{D})$ such that $A = BC$. The rank of the zero matrix is zero. If $A \in \mathcal{M}$ has rank 1, there exist vectors $\mathbf{u} \in \mathcal{D}^m = \mathcal{M}_{m,1}(\mathcal{D})$ and $\mathbf{v} \in \mathcal{D}^n = \mathcal{M}_{n,1}(\mathcal{D})$ such that $A = \mathbf{u}\mathbf{v}^t$.

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The *perimeter* of this rank-1 matrix A, $p(A)$, is $\sharp(\mathbf{u}) + \sharp(\mathbf{v})$ where $\sharp(\mathbf{u})$ denotes the number of nonzero entries in $\mathbf{u}$ (see [1]).

A *rank one decomposition* of $A \in \mathcal{M}$ is a sum of rank-1 matrices which equals A. Let $ROD(A)$ denote the set of all rank one decompositions of $A \in \mathcal{M}$. For $A \in \mathcal{M}$, the *perimeter* of A, $p(A)$, is defined as

$$\min\left\{\sum_{i\in I} p(A_i) \,\middle|\, \sum_{i\in I} A_i \in ROD(A) \text{ where } I \text{ is an index set}\right\}.$$

For $k = 2, 3, \ldots, \min\{m(n+1), n(m+1)\}$, let $\mathcal{P}_k$ denote the set of matrices in $\mathcal{M}$ whose perimeter is k. An easy observation is that every matrix in $\mathcal{M}$ whose perimeter is either 2 or 3 has rank 1.

A matrix in $\mathcal{M}$ is called a *cell* if it has exactly one nonzero entry, that being a 1. We denote the cell whose nonzero entry is in the $(i,j)^{th}$ position by $E_{i,j}$. Let $\mathcal{E} = \{E_{i,j} \mid 1 \le i \le m,\ 1 \le j \le n\}$. A matrix E in $\mathcal{M}$ is a *weighted cell* if $E = \alpha E_{i,j}$ for some cell $E_{i,j} \in \mathcal{E}$ and for some nonzero $\alpha \in \mathcal{D}$.

A *line* of a matrix is defined to be a row or column of the matrix. The *term rank* of a matrix A, $t(A)$, is the minimum number of lines that contain all nonzero entries of A (see [2]). The matrix X *dominates* the matrix Y, $X \ge Y$, if $y_{i,j} \ne 0$ implies $x_{i,j} \ne 0$ for all i and j. The matrix X *dominates precisely* the matrix Y, if $X \ge Y$ and $Y \ge X$. An $m \times n$ matrix X is a *generalized diagonal* if X is a sum of $\min\{m,n\}$ weighted cells no two in any one line.

A mapping $T : \mathcal{M} \to \mathcal{M}$ is called a *linear operator* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $A, B \in \mathcal{M}$ and for all $\alpha, \beta \in \mathcal{D}$. A linear operator $T : \mathcal{M} \to \mathcal{M}$ is called a (P,Q,B)-*operator* (see [2]) if there exist permutation matrices P and Q, and a matrix B with $B \ge J$, such that $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}$, or $m = n$ and $T(X) = P(X^t \circ B)Q$ for all $X \in \mathcal{M}$, where X^t denotes the transpose of X and $X \circ B$ is the *Hadamard* or *Schur product*, i.e., the $(i,j)^{th}$ entry of $X \circ B$ is $x_{i,j}b_{i,j}$. A linear operator T is said to *preserve* a set $\mathcal{Q}$ if $A \in \mathcal{Q}$ implies $T(A) \in \mathcal{Q}$.

In this article we will restrict our arguments to operators over semidomains, however, many of the results hold over more general semirings without zero divisors with appropriate compensation for the lack of an identity or commutativity.

Throughout this article we assume that $1 \le m \le n$ and that $\mathcal{D}$ is a semidomain which has at least $mn + 1$ elements.

2. Preservers of perimeters 2 and k

Lemma 2.1. *If $T : \mathcal{M} \to \mathcal{M}$ is a linear operator which preserves $\mathcal{P}_2$, then there exist nonzero scalars $b_{i,j}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$, and a mapping $f : \mathcal{E} \to \mathcal{E}$, such that $T(X) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j} b_{i,j} f(E_{i,j})$ for all $X = (x_{i,j}) \in \mathcal{M}$.*

Proof. Since $\mathcal{P}_2$ is the set of all weighted cells, the result easily follows. □

Let $\mathcal{P}_k^* = \{A \in \mathcal{M} \mid A$ is an element of $\mathcal{P}_k$ with fewest number of nonzero entries$\}$. For each $A \in \mathcal{M}$, we denote $\sharp(A)$ as the number of nonzero entries in A. Furthermore $\sharp(\mathcal{D})$ is the number of elements of a semidomain $\mathcal{D}$.

Proposition 2.2. *For $1 \leq r \leq m(n-1)$:*

(i) *there is a matrix X in $\mathcal{P}_{2m+r}$ such that $\sharp(X) = m + r$;*

(ii) *if A is an element in $\mathcal{P}_{2m+r}^*$, then $\sharp(A) = m + r$ and $t(A) = m$. Furthermore if B is a submatrix of A with $\sharp(B) = 6$ and size either 2×3 or 3×2, then B has rank 2.*

Proof. (i) Let $X = \sum_{i=1}^m E_{i,i} + \sum_{j=1}^r x_j E_{\alpha_j,\beta_j}$, where E_{α_j,β_j} are distinct cells with $\alpha_j \neq \beta_j$ for all j . Since $\sharp(\mathcal{D}) \geq mn+1$ it is easy to check that we can select nonzero scalars x_j for $j = 1, 2, \ldots, r$ such that $X \in \mathcal{P}_{2m+r}$.

(ii) Suppose that A is a matrix in $\mathcal{P}_{2m+r}^*$. Let s_i be the number of nonzero entries in the i^{th} row of A. If $\sharp(A) \leq m+(r-1)$, then $\sum_{i=1}^m s_i \leq m+(r-1)$, and hence, $p(A) \leq \sum_{i=1}^m (s_i + 1) \leq 2m + (r-1)$, a contradiction. So $\sharp(A) \geq m + r$, and thus, $\sharp(A) = m + r$ by (i). If $t(A) < m$, then there exist permutation matrices P and Q such that $PAQ = [\begin{smallmatrix} X & Y \\ Z & O \end{smallmatrix}]$, where X is an $l \times (m-l-1)$ matrix for some l. Let g_i be the number of nonzero entries in the i^{th} row of PAQ, and h_j be the number of nonzero entries in the j^{th} column of Z. Then we have $\sharp(A) = \sharp(PAQ) = m + r = \sum_{i=1}^l g_i + \sum_{j=1}^{m-l-1} h_j$. But then,

$$\begin{aligned} p(A) &= p(PAQ) \leq p(\begin{bmatrix} X & Y \end{bmatrix}) + p(Z) \leq \sum_{i=1}^{l} (g_i + 1) + \sum_{j=1}^{m-l-1} (h_j + 1) \\ &= \Big(\sum_{i=1}^{l} g_i + \sum_{j=1}^{m-l-1} h_j \Big) + l + (m-l-1) = 2m + (r-1), \end{aligned}$$

a contradiction. Hence $t(A) = m$.

Suppose that A has a 2×3 submatrix B with $\sharp(B) = 6$ and $r(B) = 1$. Without loss of generality, we assume $A = \left[\begin{smallmatrix} B & X' \\ Y' & Z' \end{smallmatrix}\right]$. Since $\sharp(A) = m + r$ we have $\sum_{i=1}^m s_i = m + r$ and

$$\begin{aligned} p(A) &\leq p(B) + p(X') + p(\begin{bmatrix} Y' & Z' \end{bmatrix}) \\ &\leq 5 + ((s_1 - 3 + 1) + (s_2 - 3 + 1)) + \sum_{i=3}^{m} (s_i + 1) \\ &= m - 1 + \sum_{i=1}^{m} s_i = 2m + (r-1), \end{aligned}$$

a contradiction. Hence B must have rank 2. Similarly, if B is a 3×2 submatrix of A with $\sharp(B) = 6$, then the rank of B is also 2. □

Lemma 2.3. *Suppose that $T : \mathcal{M} \to \mathcal{M}$ is a linear operator defined by $T(X) = \sum_{i=1}^m \sum_{j=1}^n x_{i,j} b_{i,j} f(E_{i,j})$ for some function $f : \mathcal{E} \to \mathcal{E}$ and for nonzero scalars $b_{i,j}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$. Then*

(i) *for* $1 \le r \le m(n-1)$ *if* T *preserves* $\mathcal{P}_{2m+r}$, *then* f *is bijective;*
(ii) *for* $1 \le r \le mn - 2n + 4$ *if* T *preserves* $\mathcal{P}_{2m+r}$, *then* T *preserves lines.*

Proof. (i) If f is not bijective, then there are two cells E and G such that $f(E) = f(G)$. Let X be an element in $\mathcal{P}^*_{2m+r}$ which dominates E and G. Then $T(X)$ cannot be an element of $\mathcal{P}_{2m+r}$ by Proposition 2.2 because $\sharp(T(X)) \le m + (r-1)$. This contradicts the fact that T preserves $\mathcal{P}_{2m+r}$. Hence, f is bijective.

(ii) If T does not preserve lines, since f is bijective, there exist two cells E and F not lying on one line such that their images lie on one line. Let $X \in \mathcal{M}$ be a generalized diagonal matrix dominating E and F. Then we have $t(T(X)) \le m-1$ and hence $T(X)$ has at least one zero line.

If $T(E)$ and $T(F)$ are mapped into a row then $T(X)$ has a zero row, say the i^{th} row. Let $\mathcal{E}_1$ be the set of $mn - m$ cells that are not dominated by $f(X)$. Then we can select r cells $H_1, \dots, H_r$ in $\mathcal{E}_1$ which do not lie on the i^{th} row. Since f is bijective there exist r cells $G_1, \dots, G_r$ such that $f(G_l) = H_l$ for all $l = 1, \dots, r$. It follows from $\sharp(\mathcal{D}) \ge mn + 1$ that there exist nonzero scalars $\alpha_1, \dots, \alpha_r$ such that $Y = X + \sum_{l=1}^{r} \alpha_l G_l$ is in $\mathcal{P}^*_{2m+r}$. But $t(T(Y)) \le m-1$ because $T(Y)$ has the i^{th} zero row. By Proposition 2.2, $T(Y)$ is not in $\mathcal{P}_{2m+r}$, a contradiction.

Suppose that $T(E)$ and $T(F)$ are mapped into a column, say, without loss of generality, $f(E+F) = E_{1,1} + E_{2,1}$. Then $T(X)$ has a zero column, say the 2^{nd}. If $T(X)$ has a zero row, then we are done, as above. So we may assume that $T(X)$ has no zero rows. It follows from $f(E+F) = E_{1,1} + E_{2,1}$, that $C \not\le \sum_{j=3}^{n}(E_{1,j} + E_{2,j})$ for each cell $C \le f(X)$. Let $\mathcal{E}_2$ be the set of $mn - 2(m+n) + 4$ cells that are not dominated by $f(X) + \sum_{j=3}^{n}(E_{1,j} + E_{2,j}) + \sum_{j=1}^{m} E_{j,2}$. By an argument similar to the one above, there exist r cells $H'_1, \dots, H'_k$ in $\mathcal{E}_2$ and nonzero scalars $\beta_1, \dots, \beta_r$ such that $Y' = X + \sum_{l=1}^{r} \beta_l f^{-1}(H'_l)$ is in $\mathcal{P}^*_{2m+r}$, while $t(T(Y')) \le m-1$, and hence, $T(Y') \notin \mathcal{P}_{2m+r}$, a contradiction.

Therefore T preserves lines. □

Proposition 2.4. *Suppose that* $T : \mathcal{M} \to \mathcal{M}$ *is a linear operator defined by* $T(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{i,j} b_{i,j} f(E_{i,j})$ *for a bijective map* $f : \mathcal{E} \to \mathcal{E}$ *and for nonzero scalars* $b_{i,j}$, $i = 1, \dots, m$; $j = 1, \dots, n$. *If* T *preserves lines then* T *is a* (P, Q, B)*-operator.*

Proof. Since no combination of a rows and b columns can dominate J where $a + b = m$ unless $b = 0$ (or if $m = n$, if $a = 0$), we have that either the image of each row is a row and the image of each column is a column, or $m = n$ and the image of each row is a column and the image of each column is a row. Thus, there are two cases: (1) f maps $\mathbb{R}$ onto $\mathbb{R}$ and maps $\mathbb{C}$ onto $\mathbb{C}$, or (2) f maps $\mathbb{R}$ onto $\mathbb{C}$ and maps $\mathbb{C}$ onto $\mathbb{R}$.

Case (1): We note that $f(R_i) = R_{\sigma(i)}$ and $f(C_j) = C_{\tau(j)}$ for all $i = 1, \dots, m$; $j = 1, \dots, n$, where σ and τ are permutations of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively. Then for any $E_{i,j} \in \mathcal{E}$, we have $T(E_{i,j}) = b_{i,j} E_{\sigma(i),\tau(j)}$.

Let P and Q be the permutation matrices corresponding to σ and τ, respectively. Then we have that for any $X = (x_{i,j}) \in \mathcal{M}$,

$$T(X) = T\Big(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}E_{i,j}\Big) = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{i,j}x_{i,j}E_{\sigma(i),\tau(j)} = P(X \circ B)Q.$$

Thus, T is a (P, Q, B)-operator.

Case (2): We note that $m = n$, $f(R_i) = C_{\sigma(i)}$ and $f(C_j) = R_{\tau(j)}$ for all $i, j = 1, \ldots, m$, where σ and τ are some permutations of $\{1, \ldots, m\}$. By an argument similar to Case (1), we obtain that $T(X)$ is of the form $T(X) = P(X^t \circ B)Q$, and thus, T is a (P, Q, B)-operator. □

For our purpose we use the following theorem in [6], which characterize the linear operators that preserve perimeters 2 and k for $k \in \{4, 5, \ldots, 2m + 3\}$.

Theorem 2.1. *Let $\mathcal{D}$ be a semidomain and $T : \mathcal{M} \to \mathcal{M}$ be a linear operator. Then*

(i) *T preserves $\mathcal{P}_2$ and $\mathcal{P}_4$ if and only if T is a (P, Q, B)-operator, where every 2×2 submatrix of B has rank 1;*
(ii) *T preserves $\mathcal{P}_2$ and $\mathcal{P}_5$ if and only if T is a (P, Q, B)-operator;*
(iii) *For $6 \leq k \leq 2\min\{m, n\} + 3$, if T preserves $\mathcal{P}_2$ and $\mathcal{P}_k$, then T is a (P, Q, B)-operator, where every 2×2 submatrix of B has rank 1.*

Theorem 2.2. *Let $\mathcal{D}$ be a semidomain. For $4 \leq k \leq mn - 2(n - m) + 4$, if $T : \mathcal{M} \to \mathcal{M}$ is a linear operator which preserves $\mathcal{P}_2$ and $\mathcal{P}_k$, then T is a (P, Q, B)-operator.*

Proof. If $4 \leq k \leq 2m$, by Theorem 2.1, T is a (P, Q, B)-operator. So we assume that T preserves $\mathcal{P}_2$ and $\mathcal{P}_{2m+r}$ with $1 \leq r \leq mn - 2n + 4$. In this case, T is also a (P, Q, B)-operator by Lemmas 2.1, 2.3, and Proposition 2.4. □

A semidomain $\mathcal{D}$ is called *chain* if the set $\mathcal{D}$ is totally ordered with universal lower and upper bounds, and the operations are defined by $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$ for all $a, b \in \mathcal{D}$. Examples of chain semirings are subsets of the interval $[0, 1]$ which include 0 and 1 and with addition being max and multiplication being min. Another is the set $\{0, 1\}$ with all arithmetic as in the reals except $1 + 1 = 1$, called the *binary Boolean* semiring.

Lemma 2.5. *Let $\mathcal{D}$ be a chain semidomain. Suppose that $T : \mathcal{M} \to \mathcal{M}$ is a linear operator defined by $T(X) = X \circ B$ with $B \geq J$. If T preserves $\mathcal{P}_k$ with $6 \leq k \leq 2m + 3$, then $B = J$.*

Proof. First we claim that all entries of B are equal. If not, B has a 2×2 submatrix B' which has at least two different entries. Say $B' = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$. Without loss of generality, we assume that $b_{1,1} = \min\{b_{i,j} \,|\, i, j = 1, 2\}$ and $a = \max\{b_{i,j} \,|\, i, j = 1, 2\}$ so that $b_{1,1} \neq a$. Let $A_1 = \begin{bmatrix} a & b_{1,1} \\ b_{1,1} & a \end{bmatrix} \oplus O$ and $A_2 =$

$\begin{bmatrix} a & b_{1,1} & 1 \\ b_{1,1} & a & 0 \end{bmatrix} \oplus O$. Then we have $p(A_1) = 6$ and $p(A_2) = 7$ but $p(T(A_1)) = 4$ and $p(T(A_2)) = 6$. So if $b_{1,1} \neq a$, then T preserves neither $\mathcal{P}_6$ nor $\mathcal{P}_7$, a contradiction unless $k \geq 8$. For $k \geq 8$ we let

$$X = \begin{cases} A_1 + \sum_{i=0}^{l-4} E_{3+i,3+i} & \text{if } k = 2l,\ 4 \leq l \leq m+1, \\ A_2 + \sum_{i=0}^{l-4} E_{3+i,3+i} & \text{if } k = 2l+1,\ 4 \leq l \leq m+1. \end{cases}$$

Then we have $X \in \mathcal{P}_k$ while $p(T(X)) \leq k-1$, a contradiction. Thus all entries of B are equal and hence $B = \alpha J$ for some nonzero $\alpha \in \mathcal{D}$.

If $\alpha \neq 1$, we let $C_1 = \begin{bmatrix} \alpha & 1 \\ 1 & 1 \end{bmatrix} \oplus O$ and $C_2 = \begin{bmatrix} \alpha & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \oplus O$. Then $C_1 \in \mathcal{P}_6$ and $C_2 \in \mathcal{P}_7$ but $T(C_1) \in \mathcal{P}_4$ and $T(C_2) \in \mathcal{P}_6$, a contradiction unless $k \geq 8$. For $k \geq 8$ we get a contradiction by an argument similar to the above. Therefore $B = J$. □

Lemma 2.6. *Suppose that $T : \mathcal{M} \to \mathcal{M}$ is a linear operator defined by $T(X) = X \circ B$ with $B \geq J$, where $\mathcal{D}$ is a chain semidomain. If T preserves $\mathcal{P}_k$ with $6 \leq k \leq mn - 2(n-m) + 4$, then $B = J$.*

Proof. By Lemma 2.5, it suffices to show that if T preserves $\mathcal{P}_{2m+r}$ with $4 \leq r \leq mn - 2n + 4$, then $B = J$. Suppose that B has at least two different entries. Then B has a 2×2 submatrix B', say $B' = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$, which has at least two different entries. Without loss of generality, we assume that $b_{1,1} = \min\{b_{i,j} \mid i,j = 1,2\}$ and $a = \max\{b_{i,j} \mid i,j = 1,2\}$, so that $b_{1,1} \neq a$. Let $\alpha = \min\{b_{1,1}, b_{1,3}, b_{2,3}\}$ and $A = \begin{bmatrix} a & b_{1,1} & \alpha \\ b_{1,1} & a & \alpha \end{bmatrix} \oplus O$. Then, we have $r(A) = 2$ but the rank of $T(A)$ is 1 because

$$T(A) = \begin{bmatrix} b_{1,1} & b_{1,1} & \alpha \\ b_{1,1} & b_{2,2} & \alpha \end{bmatrix} \oplus O = \begin{bmatrix} b_{1,1} \\ b_{2,2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{2,2} & \alpha & 0 & \cdots & 0 \end{bmatrix}.$$

Let Ω be the set of cells which are not dominated by $\sum_{j=1}^{3} E_{1,j} + \sum_{j=1}^{3} E_{2,j}$. Note that if X is an element in $\mathcal{P}^*_{2m+r}$, then $\sharp(X) = m+r$. Since $\sharp(\mathcal{D}) \geq mn+1$, there exist cells E_i in Ω and nonzero scalars x_i such that $X = A + \sum_{i=1}^{m+r-6} x_i E_i$ is an element in $\mathcal{P}^*_{2m+r}$. But then, $T(X)$ has a 2×3 submatrix $X' = \begin{bmatrix} b_{1,1} & b_{1,1} & \alpha \\ b_{1,1} & b_{2,2} & \alpha \end{bmatrix}$ whose rank is 1 with $\sharp(X') = 6$. By Proposition 2.2, $T(X)$ is not a member of $\mathcal{P}_{2m+r}$, a contradiction. Thus, all entries of B are equal, and hence, $B = \beta J$ for some nonzero $\beta \in \mathcal{D}$.

If $\beta \neq 1$ let us consider the matrix $C = \begin{bmatrix} \beta & 1 & 1 \\ 1 & \beta & 1 \end{bmatrix} \oplus O$. Then we have $r(C) = 2$ while $r(T(C)) = 1$. Furthermore, there exist cells F_j in Ω and nonzero scalars

y_j such that $Y = C+\sum_{j=1}^{m+6-r} y_jF_j$ is an element in $\mathcal{P}^*_{2m+r}$, but $T(Y) \notin \mathcal{P}_{2m+r}$ by Proposition 2.2, a contradiction. Hence $B = J$. □

Theorem 2.3. *Let $\mathcal{D}$ be a chain semidomain and $T : \mathcal{M} \to \mathcal{M}$ be a linear operator. For $6 \leq k \leq mn - 2(n-m) + 4$, the following are equivalent:*

(i) *T preserves $\mathcal{P}_2$ and $\mathcal{P}_k$;*
(ii) *There are permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in \mathcal{M}$, or $m = n$ and $T(X) = PX^tQ$ for all $X \in \mathcal{M}$;*
(iii) *T preserves all perimeters.*

Proof. Obviously, (iii) implies (i). And we can easily check that (ii) implies (iii). So we need to show that (i) implies (ii). Assume (i). By Theorem 2.2, T is a (P,Q,B)-operator. Thus there exist permutation matrices P and Q, and a matrix B with $B \geq J$ such that $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}$, or $m = n$ and $T(X) = P(X^t \circ B)Q$ for all $X \in \mathcal{M}$. For the case of $T(X) = P(X \circ B)Q$ we define an operator $L : \mathcal{M} \to \mathcal{M}$ by $L(X) = P^tT(X)Q^t = X \circ B$. Since T preserves $\mathcal{P}_k$, so does L. By Lemma 2.6, $B = J$, and hence, $L(X) = X$, or equivalently, $T(X) = PXQ$. If $m = n$ and $T(X) = P(X^t \circ B)Q$, then by a similar method to the above we have $T(X) = PX^tQ$. Thus (ii) is satisfied. □

Let $\mathcal{U}_+$ be the set of nonnegative elements of a unique factorization domain $\mathcal{U}$ in the reals $\mathbb{R}$. Then $\mathcal{D} \equiv \mathcal{U}_+$ is a semidomain and called a *unique factorization semidomain.* A nonzero vector $\mathbf{p} = [p_1 \cdots p_n]^t$ in $\mathcal{U}^n_+$ is *irreducible* if the greatest common divisor of p_i's is 1 (that is, $\gcd(p_1, \ldots, p_n) = 1$). For vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{U}^n_+$ we use the notation $\mathbf{a} \simeq \mathbf{b}$ if there is an irreducible vector $\mathbf{p}$ such that $\mathbf{a} = \alpha\mathbf{p}$ and $\mathbf{b} = \beta\mathbf{p}$ for some $\alpha, \beta \in \mathcal{U}_+$.

Proposition 2.7. *Let $\mathbf{a} = [a_1 \ \cdots \ a_n]^t$ and $\mathbf{b} = [b_1 \ \cdots \ b_n]^t$ be nonzero vectors in $\mathcal{U}^n_+$. Then,*

(i) *if $\alpha\mathbf{a} = \beta\mathbf{b}$ for some positive $\alpha, \beta \in \mathcal{U}_+$, then $\mathbf{a} \simeq \mathbf{b}$;*
(ii) *if $a_ib_j = a_jb_i$ for all $i, j \in \{1, \ldots, n\}$, then $\mathbf{a} \simeq \mathbf{b}$.*

Proof. (i) Let $\alpha' = \gcd(a_1, \ldots, a_n)$. Then, there is an irreducible vector $\mathbf{p}$ in $\mathcal{U}^n_+$ such that $\mathbf{a} = \alpha'\mathbf{p}$. Thus, $\alpha\mathbf{a} = \beta\mathbf{b}$ becomes $\alpha\alpha'\mathbf{p} = \beta\mathbf{b}$. Let $\gamma = \gcd(\alpha\alpha', \beta)$, $\gamma_1 = \frac{\alpha\alpha'}{\gamma}$ and $\gamma_2 = \frac{\beta}{\gamma}$. Then γ_1 and γ_2 are positive, and $\alpha\alpha'\mathbf{p} = \beta\mathbf{b}$ becomes $\gamma_1\mathbf{p} = \gamma_2\mathbf{b}$. Therefore, we have γ_1 divides every γ_2b_i for all i. Since γ_1 is relatively prime to γ_2 it follows that γ_1 divides every entry in $\mathbf{b}$. Thus we have $\mathbf{b} = \gamma_1\mathbf{c}$ for some nonzero $\mathbf{c} \in \mathcal{U}^n_+$. By cancelation $\gamma_1\mathbf{p} = \gamma_2\mathbf{b}$ becomes $\mathbf{p} = \gamma_2\mathbf{c}$. Then, γ_2 is a unit in $\mathcal{U}^n_+$ because γ_2 divides every entry in the irreducible vector $\mathbf{p}$. That is, $\mathbf{c} = \gamma_2^{-1}\mathbf{p}$ and hence $\mathbf{b} = \gamma_1\gamma_2^{-1}\mathbf{p}$. Therefore $\mathbf{a} \simeq \mathbf{b}$.

(ii) If $n = 1$ or 2, the result is obvious. So we may assume that $n \geq 3$. Since $\mathbf{a}$ is nonzero, $a_i \neq 0$ for some i. Now we claim $b_i \neq 0$. Since $\mathbf{b}$ is also nonzero, $b_j \neq 0$ for some j. If $i \neq j$, it follows from $(0 \neq)\, a_ib_j = a_jb_i$ that $b_i \neq 0$. Since $\mathcal{U}_+$ is a unique factorization semidomain, there are positive $\alpha, \beta \in \mathcal{U}_+$ such that $\alpha a_i = \beta b_i$. Let j be arbitrary in $\{1, \ldots, n\}$. Since $a_ib_i \neq 0$ and $a_ib_j = a_jb_i$, we have $a_j = 0$ if and only if $b_j = 0$. Let l be an arbitrary index such that $a_l b_l \neq 0$. From $a_i b_l = a_l b_i$, we have $\alpha\beta a_i b_l = \alpha\beta a_l b_i$, and hence, $\alpha a_l = \beta b_l$ because $\alpha a_i = \beta b_i$. Since l is arbitrary, it follows that $\alpha \mathbf{a} = \beta \mathbf{b}$. Thus, by (i), we have $\mathbf{a} \simeq \mathbf{b}$. □

Let $X = \left[\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right]$ and $Y = \left[\begin{smallmatrix} y_1 & y_2 \\ y_3 & y_4 \end{smallmatrix}\right]$ be 2×2 matrices over $\mathcal{U}_+$, where $x_i \neq 0$ and $y_i = \prod_{l=1, l\neq i}^{4} x_l$ for all $i = 1, 2, 3, 4$. Then using Proposition 2.7(ii), we can easily show that

$$(2.1) \qquad r(X) = 1 \quad \text{if and only if} \quad x_1 x_4 = x_2 x_3 \quad \text{if and only if} \quad r(Y) = 1.$$

Lemma 2.8. *Let $B = (b_{i,j})$ be a matrix in $\mathcal{M}_{m,n}(\mathcal{U}_+)$ with $B \geq J$. Then, $r(B) = 1$ if and only if every 2×2 submatrix of B has rank 1.*

Proof. If $r(B) = 1$, then there are two vectors $\mathbf{u} = [u_1 \ \cdots \ u_m]^t \in \mathcal{U}_+^m$ and $\mathbf{v} = [v_1 \ \cdots \ v_n]^t \in \mathcal{U}_+^n$ such that $B = \mathbf{u}\mathbf{v}^t$. Let B' be an arbitrary 2×2 submatrix of B. Then B' is of the form $\left[\begin{smallmatrix} b_{i,j} & b_{i,k} \\ b_{l,j} & b_{l,k} \end{smallmatrix}\right]$. It follows from $B = \mathbf{u}\mathbf{v}^t$ that $b_{i,j} b_{l,k} = (u_i v_j)(u_l v_k) = (u_i v_k)(u_l v_j) = b_{i,k} b_{l,j}$, and hence, $r(B') = 1$ by (2.1).

Conversely if every 2×2 submatrix of B has rank 1, then by (2.1) and Proposition 2.7, we have $\mathbf{b}_1 \simeq \mathbf{b}_j$ for all j, where $\mathbf{b}_j$ is the j^{th} column of B. Thus, there are an irreducible vector $\mathbf{p} \in \mathcal{U}_+^m$ and positive scalars q_j such that $\mathbf{b}_j = q_j \mathbf{p}$ for all j. If we let $\mathbf{q} = [q_1 \ \cdots \ q_n]^t$, then $B = \mathbf{p}\mathbf{q}^t$, and hence, $r(B) = 1$. □

Lemma 2.9. *Suppose that $T : \mathcal{M}_n(\mathcal{U}_+) \to \mathcal{M}_n(\mathcal{U}_+)$ is a linear operator defined by $T(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} b_{i,j} f(E_{i,j})$ for some function $f : \mathcal{E} \to \mathcal{E}$ and for positive scalars $b_{i,j}$, $i, j = 1, \ldots, n$. For $2n + 1 \leq k \leq n(n+1)$, if T preserves $\mathcal{P}_k$, then T preserves lines.*

Proof. By Lemma 2.3(i) f is bijective, and hence, by Lemma 2.3(ii), it is sufficient to show that for $n^2 - 2n + 5 \leq r \leq n^2 - n$, if T preserves $\mathcal{P}_{2n+r}$, then T preserves lines.

Suppose that T does not preserve lines. Since f is bijective there exist two cells E and F not lying on one line such that their images lie on one line. Without loss of generality, we assume $f(E_{1,1}) = E_{1,1}$ and $f(E_{2,2}) = E_{1,2}$. Let $X = (x_{i,j})$ be a matrix in $\mathcal{P}_{2n+r}$ dominating $E_{1,1}$ and $E_{2,2}$, and let $T(X) = Y = (y_{i,j})$. It follows from $f(E_{1,1}) = E_{1,1}$ and $f(E_{2,2}) = E_{1,2}$ that $y_{1,1} y_{1,2} \neq 0$. Since $\sharp(X) = n + r \geq n^2 - n + 5$, $T(X) = Y$ has a 2×3 or 3×2 submatrix Y_1 with $\sharp(Y_1) = 6$ containing elements $y_{1,1}$ and $y_{1,2}$. Without loss of generality, we assume $Y = \left[\begin{smallmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{smallmatrix}\right]$, where $Y_1 = \left[\begin{smallmatrix} y_{1,1} & y_{1,2} & y_{1,3} \\ y_{2,1} & y_{2,2} & y_{2,3} \end{smallmatrix}\right]$ is a 2×3 submatrix of Y with $\sharp(Y_1) = 6$. Consider the matrix

$$Y_1' = \begin{bmatrix} (a^2 y_{1,1} y_{1,2} y_{2,1} y_{2,3}) y_{1,1} & (a^2 {y_{1,1}}^2 y_{2,2} y_{2,3}) y_{1,2} & (ab y_{1,1} y_{1,2} y_{2,1} y_{2,3}) y_{1,3} \\ \\ (a y_{1,1} y_{1,2} y_{2,1} y_{2,3}) y_{2,1} & (a y_{1,1} y_{1,2} y_{2,1} y_{2,3}) y_{2,2} & (b y_{1,2} {y_{2,1}}^2 y_{1,3}) y_{2,3} \end{bmatrix},$$

where $ab \neq 0$. Then $r(Y_1') = 1$ by Lemma 2.8. Notice that $E_{i,j} \leq Y = T(X)$ for all $i = 1, 2$ and $j = 1, 2, 3$. Since f is bijective there exist cells E_0, E_1, E_2 and E_3 such that $f(E_0) = E_{1,3}$ and $f(E_l) = E_{2,l}$ for all $l = 1, 2, 3$. It follows that there are positive entries x_{s_l,t_l} of X such that $T(x_{s_0,t_0}E_0) = y_{1,3}E_{1,3}$ and $T(x_{s_l,t_l}E_l) = y_{2,l}E_{2,l}$ for all $l = 1, 2, 3$. Let $\Delta = \{(i,j) | x_{i,j} \neq 0;\ i,j = 1, \ldots, n\}$ and $\Delta' = \{(1,1), (2,2), (s_0,t_0), (s_1,t_1), (s_2,t_2), (s_3,t_3)\}$. Define a matrix $X' = (x'_{i,j})$ by

$$x'_{i,j} = \begin{cases} (a^2 y_{1,1} y_{1,2} y_{2,1} y_{2,3}) x_{1,1} & \text{if } (i,j) = (1,1), \\ (a^2 {y_{1,1}}^2 y_{2,2} y_{2,3}) x_{2,2} & \text{if } (i,j) = (2,2), \\ (ab y_{1,1} y_{1,2} y_{2,1} y_{2,3}) x_{s_0,t_0} & \text{if } (i,j) = (s_0,t_0), \\ (a y_{1,1} y_{1,2} y_{2,1} y_{2,3}) x_{s_l,t_l} & \text{if } (i,j) = (s_l,t_l)\ l = 1,2, \\ (b y_{1,2} {y_{2,1}}^2 y_{1,3}) x_{s_3,t_3} & \text{if } (i,j) = (s_3,t_3), \\ x_{i,j} & \text{if } (i,j) \in \Delta \setminus \Delta'. \end{cases}$$

Then we have $T(X') = \begin{bmatrix} Y_1' & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$ and we can select positive scalars a and b such that X' is in $\mathcal{P}_{2n+r}$. But $T(X')$ has a 2×3 submatrix Y_1' with $\sharp(Y_1') = 6$ and $r(Y_1') = 1$. By Proposition 2.2 $T(X') \notin \mathcal{P}_{2n+r}$, a contradiction. Thus T preserves lines. □

Lemma 2.10. *Suppose that* $T : \mathcal{M}_{m,n}(\mathcal{U}_+) \to \mathcal{M}_{m,n}(\mathcal{U}_+)$ *is a linear operator defined by* $T(X) = X \circ B$ *with* $B \geq J$. *For* $k = 4$ *or* $6 \leq k \leq m(n+1)$, *if* T *preserves* $\mathcal{P}_k$, *then there are diagonal matrices* C *and* D *such that* $T(X) = CXD$ *for all* $X \in \mathcal{M}_{m,n}(\mathcal{U}_+)$.

Proof. First we show $r(B) = 1$, or equivalently, by Lemma 2.8, every 2×2 submatrix of B has rank 1. Clearly $T(X) = X \circ B$ preserves $\mathcal{P}_2$. So if T preserves $\mathcal{P}_k$ with $k = 4$ or $6 \leq k \leq 2m+3$, then the result is obvious by Theorem 2.1. Thus, it suffices to show that if T preserves $\mathcal{P}_{2m+r}$ with $4 \leq r \leq mn - m$, then every 2×2 submatrix of B has rank 1. If not, B has a 2×2 submatrix B' of rank 2, say $B' = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \equiv \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. Consider the matrix

$$Y = \begin{bmatrix} y_1 & y_2 & b_{2,3} \\ y_3 & y_4 & b_{1,3} \end{bmatrix} \oplus O \in \mathcal{M}_{m,n}(\mathcal{U}_+),$$

where $y_i = \prod_{l=1, l\neq i}^{4} x_l$ for all $i = 1, 2, 3, 4$. By (2.1) $r(Y) = 2$, but $r(T(Y)) = 1$ because $T(Y) = \begin{bmatrix} x & x & b_{1,3}b_{2,3} \\ x & x & b_{2,3}b_{1,3} \end{bmatrix} \oplus O$, where $x = \prod_{i=1}^{4} x_i$. Let Ω be the set of $mn - 6$ cells not dominated by $T(Y)$. Now, we can select positive scalars $\alpha_i, \ldots, \alpha_{m+r-6}$ and cells $E_i, \ldots, E_{m+r-6}$ in Ω so that $X = Y + \sum_{i=1}^{m+r-6} \alpha_i E_i$ is in $\mathcal{P}^*_{2m+r}$. But then, $T(X)$ has a 2×3 submatrix $X' = \begin{bmatrix} x & x & b_{1,3}b_{2,3} \\ x & x & b_{1,3}b_{2,3} \end{bmatrix}$ of rank 1 with $\sharp(X') = 6$. By Proposition 2.2, $T(X)$ is not a member of $\mathcal{P}_{2m+r}$, a contradiction. Hence, $r(B) = 1$.

If $r(B) = 1$, then there are positive scalars c_i and d_j such that

$$B = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \begin{bmatrix} d_1 & \cdots & d_n \end{bmatrix}.$$

Let $C = \text{diag}(c_1, \ldots, c_m)$ and $D = \text{diag}(d_1, \ldots, d_n)$ be diagonal matrices. Then, the $(i,j)^{th}$ entry of $X \circ B$ is $x_{i,j}b_{i,j}$ and the $(i,j)^{th}$ entry of CXD is $c_i x_{i,j} d_j = x_{i,j}b_{i,j}$. Thus, $X \circ B = CXD$ and hence the result follows. □

Theorem 2.4. *Let $\mathcal{D}$ be a unique factorization semidomain $\mathcal{U}_+$, and $T : \mathcal{M} \to \mathcal{M}$ be a linear operator. For $k = 4$ or $6 \le k \le mn - 2(n-m) + 4$, the following are equivalent:*

- (i) *T preserves $\mathcal{P}_2$ and $\mathcal{P}_k$;*
- (ii) *There are generalized diagonal matrices $U \in \mathcal{M}_m(\mathcal{D})$ and $V \in \mathcal{M}_n(\mathcal{D})$ such that $T(X) = UXV$ for all $X \in \mathcal{M}$, or $m = n$ and $T(X) = UX^tV$ for all $X \in \mathcal{M}$;*
- (iii) *T preserves all perimeters.*

In particular, if $m = n$, the above statement is true for $k = 4$ or $6 \le k \le n(n+1)$.

Proof. (i)⇒(ii): Assume (i). By Theorem 2.2, T is a (P, Q, B)-operator. Thus, there are permutation matrices P and Q, and a matrix B with $B \ge J$ such that $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}$, or $m = n$ and $T(X) = P(X^t \circ B)Q$ for all $X \in \mathcal{M}$. For the case of $T(X) = P(X \circ B)Q$, we define an operator $L : \mathcal{M} \to \mathcal{M}$ by $L(X) = P^tT(X)Q^t = X \circ B$. Since T preserves $\mathcal{P}_k$ so does L. By Lemma 2.10, there are diagonal matrices C and D such that $L(X) = CXD$, and hence, $T(X) = PCXDQ$ for all $X \in \mathcal{M}$. If we let $U = PC \in \mathcal{M}_m(\mathcal{D})$ and $V = DQ \in \mathcal{M}_n(\mathcal{D})$, then, clearly, U and V are generalized diagonal matrices and $T(X) = UXV$. If $m = n$ and $T(X) = P(X^t \circ B)Q$, then, by a similar method to the above, we have $T(X) = UX^tV$ for some generalized diagonal matrices U and V. Thus (ii) is satisfied.

(ii)⇒(iii): Assume (ii). Without loss of generality we assume that $U = \text{diag}(u_1, \ldots, u_m)$ and $V = \text{diag}(v_1, \ldots, v_n)$ are diagonal matrices with $u_iv_j \ne 0$ for all i and j. Then we can easily show that

$$p(X) = p(X^t), \quad p(X) = p(cX) \quad \text{and} \quad p(UXV) \le p(X) \tag{2.2}$$

for all $X \in \mathcal{M}$ and for all positive $c \in \mathcal{U}_+$. Let $u = \prod_{i=1}^m u_i$, $u_i' = \prod_{l=1, l\ne i}^m u_l$, $v = \prod_{j=1}^n v_j$ and $v_j' = \prod_{k=1,k\ne j}^n v_k$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let $U' = \text{diag}(u_1', \ldots, u_m')$ and $V' = \text{diag}(v_1', \ldots, v_n')$ so that $U'U = uI_m$ and $VV' = vI_n$. Then, by (2.2), we have

$$p(X) = p(uvX) = p(U'UXVV') \le p(UXV) \le p(X).$$

This fact shows that (ii) implies (iii).

(iii)⇒(i): Obvious.

In particular, suppose that $m = n$ and $k = 4$ or $6 \le k \le n(n+1)$. Notice that if T preserves $\mathcal{P}_2$ and $\mathcal{P}_k$, then T is a (P, Q, B)-operator by Lemmas 2.1, 2.3, 2.9, and Proposition 2.4. The other implications follow arguments similar to the above. □

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References

[1] L. B. Beasley and N. J. Pullman, *Boolean-rank-preserving operators and Boolean-rank-1 spaces*, Linear Algebra Appl. **59** (1984), 55–77.

[2] ———, *Term-rank, permanent, and rook-polynomial preservers*, Linear Algebra Appl. **90** (1987), 33–46.

[3] C.-K. Li and S. Pierce, *Linear preserver problems*, Amer. Math. Monthly **108** (2001), no. 7, 591–605.

[4] S. Pierce et.al., *A survey of linear preserver problems*, Linear and Multilinear Algebra 33 (1992), no. 1-2. Gordon and Breach Science Publishers, Yverdon, 1992. pp. 1–129.

[5] S.-Z. Song, L. B. Beasley, G.-S. Cheon, and Y.-B. Jun *Rank and perimeter preservers of Boolean rank-1 matrices*, J. Korean Math. Soc. **41** (2004), no. 2, 397–406.

[6] S.-Z. Song, K.-T. Kang, and L. B. Beasley, *Perimeter preserving linear operators*, to appear.

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