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ANALYSIS OF A STAGE-STRUCTURED PREDATOR-PREY SYSTEM WITH IMPULSIVE PERTURBATIONS AND TIME DELAYS

Xinyu Song, Senlin Li, and An Li

Abstract. In this paper, a stage-structured predator-prey system with impulsive perturbations and time delays is presented to investigate the ecological problem of how a pest population and natural enemy population can coexist. Sufficient conditions are obtained using a discrete dynamical system determined by a stroboscopic map, which guarantee that a 'predator-extinction' periodic solution is globally attractive. When the impulsive period is longer than some time threshold or the impulsive harvesting rate is below a control threshold, the system is permanent. Our results provide some reasonable suggestions for pest management.

1. Introduction

Efficiency in controlling pests is a very complicated, but it is very necessary. There are many ways to control agricultural pests, such as biological, cultural, physical and chemical tools. However farmers often use pesticides to control pests because of its efficiency and convenience. So all kinds of pesticides are used to kill pests. Because of overuse of pesticides the residual pests evolve a large number of pests with resistance to the pesticides. Furthermore the chemical pesticide kills not only pests but also their natural enemies. Actually, when pests (prey) are caught or poisoned to a large extent, their natural enemies (predators) become extinct due to no food, and then the pests increase rapidly. Moreover most pesticides contribute to environmental contamination which is dangerous to humans and animals. Overuse of chemical pesticides has brought many ecological and sociologic problems.

The use of natural enemies to suppress pests is an important approach in pests control. P. Debach and H. I. Freedman et al. in [7, 8, 9, 12] brought that

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biological control is the purposeful introduction and establishment of one or more natural enemies from region of an exotic pest to kill the abundance of the pest in a new region to a level at which it no longer causes economic damage. All pests have their natural enemies. One approach to biological control is using natural enemies to kill pests. When people kill most of pests by using pesticides, their natural enemies will die out. Considering ecological balance and saving of resources, we hope that the pest population and the natural enemy population can coexist at a level where economic damage caused by pests is acceptable. The harvest for pests may cause their natural enemies to be extinct, therefore we need consider the effect of harvest for pests on their natural enemies.

According to the ecological background, we need to build a model with age structure. In [1] a model of single species growth incorporating stage-structure was proposed as a reasonable generalization of the logistic model, which takes the form

$$(1.1)\qquad \begin{cases} \dot{x}_1(t) = \alpha x_2(t) - \gamma x_1(t) - \alpha e^{-\gamma\tau} x_2(t-\tau), \\ \dot{x}_2(t) = \alpha e^{-\gamma\tau} x_2(t-\tau) - \beta y_2^2(t), \end{cases}$$

where x_1 is the immature population density, x_2 is the mature population density. α represent the birth rate, γ is the immature death rate, β is the mature death and overcrowding rate, and τ is the time to maturity. There are also several mathematical models of stage-structured population growth in the literature [1, 2, 5, 6, 11, 14, 16]. But these systems are mostly assumed continuous, and impulses have seldom been used. In this paper, we build a new model with age structure for the predator, which assumes that the prey is killed impulsively by harvesting or by pesticides. It is more appropriate to biological environments.

The remainder of this paper is organized as follows. In Section 2, we propose a stage-structured predator-prey model with impulsive perturbations and time delays. We also give necessary preliminaries for our results. In Section 3, we prove sufficient conditions for a predator-extinction periodic solution. We also obtain, when the impulsive period (T) is below some threshold or impulsive harvesting proportion (δ) is at a appropriate extent, the periodic solution of 'predator-extinction' is globally attractive. In Section 4, we find sufficient conditions for permanence. We show the system is permanent when the impulsive period (T) is longer than some threshold or the impulsive harvesting proportion (δ) is below a control threshold.

2. Model and preliminaries

Song et al. [15] in 2002 proposed a model consisting of two species, namely, predators and prey, with stage structure and harvesting for the predators

$$(2.1)\qquad \begin{cases} \dot{x}(t) = x(t)(r - a_1 y_2(t) - bx(t)), \\ \dot{y}_1(t) = \alpha y_2(t) - \gamma y_1(t) - \alpha e^{-\gamma\tau} y_2(t-\tau), \\ \dot{y}_2(t) = \alpha e^{-\gamma\tau} y_2(t-\tau) - \beta y_2^2(t) + a_2 x(t) y_2(t) - E y_2(t), \end{cases}$$

where $x(t), y_1(t), y_2(t)$ represent the densities of prey, immature and mature predators, respectively. Motivated by [15], we consider

(2.2)
$$\begin{cases} \left.\begin{array}{l} \dot{x}(t) = x(t)(r - a_1 y_2(t) - bx(t)), \\ \dot{y}_1(t) = \alpha y_2(t) - \gamma y_1(t) - \alpha e^{-\gamma\tau} y_2(t-\tau), \\ \dot{y}_2(t) = \alpha e^{-\gamma\tau} y_2(t-\tau) - \beta y_2^2(t) + a_2 x(t) y_2(t) - \gamma y_2(t), \end{array}\right\} & t \neq n\tau, \\ \left.\begin{array}{l} x(t^+) = (1-\delta)x(t), \\ y_1(t^+) = y_1(t), \\ y_2(t^+) = y_2(t), \end{array}\right\} & t = n\tau, \end{cases}$$

where r is the logistic intrinsic growth rate of the prey in the absence of the predators, a_1 is the predation rate of the predators, a_2 has a similar meaning as a_1, b and β are overcroweding rates, γ is the death rate of the predators, α is the birth rate, τ is a constant time to maturity. $\delta\,(0 \le \delta < 1)$ represents a partial impulsive harvest of the prey by catching or pesticides. We only consider system (2.2) in the biological meaning region: $D = \{\,(x, y_1, y_2) |\, x, y_1, y_2 \ge 0\}$.

Since y_1 does not appear in the first and third equations of system (2.2), we can modify system (2.2) to the following model:

$$(2.3) \qquad \begin{cases} \dot{x}(t) = x(t)(r - a_1 y_2(t) - bx(t)), \\ \dot{y}_2(t) = \alpha e^{-\gamma\tau} y_2(t-\tau) - \beta y_2^2(t) + a_2 x(t) y_2(t) - \gamma y_2(t), \\ x(t^+) = (1-\delta)x(t), \\ y_2(t^+) = y_2(t). \end{cases}$$

For continuity of initial conditions and consider y_1 separately, we require

$$(2.4) \qquad y_1(0) = \int_{-\tau}^{0} \alpha y_2(t) e^{\gamma t} dt.$$

In the following, we give several lemmas which are very useful in the next investigations.

Lemma 1. *Let* $(x(t), y_1(t), y_2(t)) > 0$ *for* $-\tau < t < 0$. *Then any solution of system* (2.2) *is positive.*

Proof. Since

$$\begin{cases} \dot{x}(t) = x(t)(r - a_1 y_2(t) - bx(t)), & t \neq nT,\ n \in \mathbb{N}, \\ x(t^+) = (1-\delta)x(t), & t = nT,\ n \in \mathbb{N}, \end{cases}$$

we can easily see that $x(t) > 0$ for $x(0) > 0$. Next we will show that $y_2(t) > 0$ for all $t > 0$. Otherwise, there exists a $t_0 > 0$ such that $y_2(t_0) = 0$. Assume that t_0 is the first time such that $y_2(t_0) = 0$, that is,

$$t_0 = \inf\{t > 0\ ;\ y_2(t) = 0\}.$$

Then $\dot{y}_2(t_0) = \alpha e^{-\gamma t} y_2(t_0 - \tau) > 0$. Hence for a sufficiently small $\varepsilon > 0$, $\dot{y}_2(t_0 - \varepsilon) > 0$. But by the definition of t_0, $\dot{y}_2(t_0 - \varepsilon) \le 0$. This contradiction shows that $y_2(t) > 0$ for all $t \ge 0$. Now consider the equation:

$$\dot{\varphi}(t) = -\gamma\varphi(t) - \alpha e^{-\gamma\tau} y_2(t-\tau), \quad \varphi(0) = y_1(0).$$

Obviously, $\varphi(t) = e^{-\gamma\tau}[y_1(0) - \int_0^t \alpha e^{r(s-\tau)} y_2(s-\tau)ds]$. From (2.3), we have $\varphi(\tau) = e^{-\gamma\tau}[\int_{-\tau}^0 \alpha y_2(\xi)e^{\gamma\xi}d\xi - \int_0^\tau e^{\gamma(s-\tau)}\alpha y_2(s-\tau)ds]$. Then $\varphi(\tau) = 0$ and $\varphi(t) > 0$ for $t \in [0,\tau)$. □

Using standard comparison theorem of ordinary difference equation (see [13]) we have $y_1(t) > 0$ for $t \in [0,\tau)$. By induction, we can show that

$$y_1(t) > 0 \quad \text{for all } t \geq 0.$$

Lemma 2 (see [4, 14]). *Let the function $m \in PC^1[R^+, R]$ satisfy the inequalities*

$$\begin{cases} \dot{m}(t) \leq p(t)m(t) + q(t), & t \neq t_k, \\ m(t_k^+) \leq d_k m(t_k) + b_k, & t = t_k, \end{cases}$$

$t \geq t_0$, where $p, q \in PC[R_+, R]$ and $d_k \geq 0$, b_k are constants. Then

$$m(t) \leq m(t_0) \prod_{t_0<t_k<t} d_k \exp(\int_{t_0}^t p(s)ds) + \sum_{t_0<t_k<t} (\prod_{t_k<t_j<t} d_j \exp(\int_{t_k}^t p(s)ds))b_k$$
$$+ \int_{t_0}^t \sum_{s<t_k<t} d_k \exp(\int_s^t p(\sigma)d\sigma)q(s)ds, \ t \geq t_0.$$

Lemma 3. *There exists a constant $M > 0$ such that $x(t) \leq rb^{-1}$ and $y_1(t)$, $y_2(t) \leq M$ for each solution of (2.2) with t being sufficiently large.*

Proof. Since

$$\begin{cases} \dot{x}(t) = x(t)(r - a_1 y_2(t) - bx(t)), & t \neq nT, \\ x(t^+) = (1-\delta)x(t), \quad t = nT, \end{cases}$$

we can show that there exists a $T_0 > 0$ such that $x \leq rb^{-1}$ for $t > T_0$. Defining the function

$$u(t) = x(t) + y_1(t) + y_2(t),$$

we have

$$\begin{aligned} D^+u(t) + \gamma u(t) &= \alpha y_2 - \beta y_2^2 + a_2 x y_2 - bx^2 + (r+\gamma)x - a_1 x y_2 \\ &\leq (+a_2 rb^{-1} y_2 \alpha - \beta y_2^2 + (r+\gamma)rb^1 \\ &= -\beta(y_2 - \frac{\alpha + a_2 rb^{-1}}{2\beta})^2 + \frac{(\alpha + a_2 rb^{-1})^2}{4\beta^2} + (r+\gamma)rb^{-1} \\ &\leq M \quad t > T, t > t_0, t \neq nT, \end{aligned}$$

where $M = \frac{(\alpha + a_2 rb^{-1})^2}{4\beta^2} + (r+\gamma)rb^{-1}$.

Consider the following impulsive differential inequalities

$$\frac{du(t)}{dt} \leq M - \gamma du(t).$$

According to Lemma 2, we have

$$u(t) \leq (u(0^+) - M)e^{-dt} + M,$$

so $\lim_{t\to\infty} u(t) \leq M$. Hence $u(t)$ is uniformly ultimately bounded, and therefore $y_1(t), y_2(t) \leq M$ for each solution of (2.2) for all sufficiently large t. □

Lemma 4 (see [15]). *Let* $\dot{x}(t) = ax(t-\tau) - bx(t) - cx^2(t)$, $a, b, c, \tau > 0$, $x(t) > 0$ $(-\tau \leq t \leq 0)$.

(i) *If* $a > b$, *then* $\lim_{t\to\infty} x(t) = \frac{a-b}{c}$.

(ii) *If* $a < b$, *then* $\lim_{t\to\infty} x(t) = 0$.

Lemma 5 (see [3] Lemma 1.2). *Consider the impulsive differential inequality*

$$(2.5)\qquad \begin{cases} r'(t) & \leq & w(t, r(t)), & t \neq \tau_k, \\ \Delta r(\tau_k) & \leq & \psi_k(r(\tau_k)), & t = \tau_k \end{cases}$$

and the respective comparison equation

$$(2.6)\qquad \begin{cases} u'(t) & = & w(t, u), & t \neq \tau_k, \\ \Delta u & = & \Psi_k(u(t), & t = \tau_k, \\ u(t_0) & = & u_0. \end{cases}$$

Assume that:

1. $w \in C_\tau(R_+ \times R, R)$ *and the functions* $\Psi_k : R \to R$ *are nondecreasing in* R.
2. *The function* $r(t) \in PC^1(R_+, R)$ *satisfies* (2.5) *fort* $\in [t_0, T) \subset R_+$.
3. *The maximal solution* $u^+(t)$ *of* (2.6) *is defined for* $t \in [t_0, T)$.

Then $r(t) \leq u^+(t)$ *for* $t \in [t_0, T)$ *provided that* $r(t_0) \leq u_0$.

Moreover, if in (2.5) " $\leq$ " *is replaced by* " $\geq$ " *and* $u^-(t)$ *is the minimal solution of* (2.6), *then* $r(t) \geq u^-(t)$ *provided that* $r(t_0) \geq u_0$.

3. Periodic solution of 'predator-extinction'

Since $y_1(t) = y_2(t) = 0$, $t \geq 0$, we only consider the following impulsive system

$$(3.1)\qquad \begin{cases} \dot{x}(t) = x(t)(r - bx(t)), & t \neq nT, \\ x(t^+) = (1-\delta)x(t), & t = nT. \end{cases}$$

Solving Eq.(3.1), we have

(3.2)

$$\begin{cases} x(t) = \frac{\frac{r}{b}x((n-1)T^+)}{x((n-1)T^+) + [\frac{r}{b} - x((n-1)T^+)]\exp\{-r(t-(n-1)T)\}}, & t \in ((n-1)T, nT), \\ x((n+1)T^+) = \frac{(1-\delta)\frac{r}{b}x((n)T^+)}{x(nT^+) + [\frac{r}{b} - x(nT^+)]e^{-rT}} = F(x(nT^+)). \end{cases}$$

System (3.2) has a unique trivial fixed point $x_0^* = 0$, that is to say system (3.1) has a unique trivial periodic solution. Let $\delta_0^* = 1 - e^{-rT}$, if $\delta > \delta_0^*$, then

$$\left|\frac{dF(x(nT^+))}{dx}\right|_{x(nT^+)=0} = \frac{1-\delta}{e^{-rT}} < 1.$$

Hence the trivial periodic solution x_0^* is globally asymptotically stable. If $\delta < \delta_0^*$, then

$$\left|\frac{dF(x(nT^+))}{dx}\right|_{x(nT^+)=0} = \frac{1-\delta}{e^{-rT}} > 1.$$

Hence the trivial periodic solution x_0^* is not stable. If $\delta < \delta_0^*$, then system (3.2) has a unique positive fixed point $x_0^* = \frac{\frac{r}{b}(1-\delta-e^{-rT})}{1-e^{-rT}}$. Using the discrete dynamical system determined by the stroboscopic map, we have

$$(3.3)\qquad \begin{cases} x^*(t) = \frac{\frac{r}{b}(1-\delta-e^{-rT})}{1-\delta-e^{-rT}+\delta\exp\{-r(t-(n-1)T)\}}, & t \in ((n-1)T, nT], \\ x((n-1)T^+) = x_0^* = x^*(0) = \frac{\frac{r}{b}(1-\delta-e^{-rT})}{1-e^{-rT}}, \end{cases}$$

which is a positive periodic solution of system (3.3). In the following, we show the positive periodic solution is globally asymptotically stable.

Theorem 3.1. *If $\delta < \delta_0^*$ ($\delta_0^* = 1 - e^{-rT}$), then system* (2.3) *has a 'predator-extinction' periodic solution $(x^*(t), 0)$ for $t \in (nT, (n+1)T)$, $n \in \mathbb{N}$, where*

$$x^*(t) = \frac{\frac{r}{b}(1-\delta-e^{-rT})}{1-\delta-e^{-rT}+\delta\exp\{-r(t-(n-1)T)\}}, \quad t \in ((n-1)T, nT].$$

Proof. In (3.1) we carry out the change of variable $x = z^{-1}$ and obtain the linear non-homogenous impulsive equation

$$(3.4)\qquad \begin{cases} \dot{z}(t) = b - rz(t), & t \neq nT,\ n \in \mathbb{N}, \\ z(t^+) = \frac{1}{1-\delta} z(t), & t = nT,\ n \in \mathbb{N}. \end{cases}$$

Let $w(t,s) = \prod_{s \le nT < t} \frac{1}{1-\delta} e^{-r(t-s)}$ be the Cauchy matrix for the corresponding homogeneous equation. Then

$$z(t) = w(t,0)z(0) + b\int_0^t w(t,s)ds$$

is the solution of (3.4).

$$|z(t) - z^*(t)| = w(t,0)|z(0) - z^*(0)|,$$

where $z(t)$ is any solution of (3.4) with $z(0) = \frac{1}{x(0)}$, $z^*(t)$ is the periodic solution of (3.4) with $z^*(0) = \frac{1}{x^*(0)}$. Since $\delta < \delta_0^*$, that is $\frac{1}{1-\delta}e^{-rT} < 1$, then

$$w(t,0) = \prod_{0 \le nT < t} \frac{1}{1-\delta} e^{-rt} \le \left(\frac{1}{1-\delta} e^{-rT}\right)^n \to 0 (t \to \infty),$$

so

$$\lim_{t\to\infty} |x(t) - x^*(t)| = \lim_{t\to\infty} \left|\frac{1}{z(t)} - \frac{1}{z^*(t)}\right| = \lim_{t\to\infty} \frac{|z(t) - z^*(t)|}{z(t) * z^*(t)} = 0,$$

and then system (3.1) has a unique positive periodic solution $x^*(t)$, which is globally asymptotically stable. Hence system (2.3) has a 'predator-extinction' periodic solution $(x^*(t), 0)$ for $t \in (nT, (n+1)T)$, $n \in \mathbb{N}$. □

Theorem 3.2. *If* $R_1 = \frac{a_2 r(1-\delta-e^{-rT})}{b(1-\delta)(1-e^{-rT})(\gamma-\alpha e^{-\gamma t})} < 1$, *then the 'predator-extinction' periodic solution* $(x^*(t), 0)$ *of system* (2.3) *is globally attractive.*

Proof. Let $(x(t), y_2(t))$ be any solution of system (2.3) with initial condition (2.4). We consider the following impulsive differential inequalities

$$\begin{cases} \frac{dx(t)}{dt} \le x(t)(r - bx(t)), \\ x(t^+) = (1-\delta)x(t). \end{cases}$$

From (3.1) and (3.3), we get

$$\limsup_{t\to\infty} x(t) \le \frac{\frac{r}{b}(1-\delta-e^{-rT})}{(1-\delta)(1-e^{-rT})}.$$

Hence, there exist $n_0 \in \mathbb{N}$ and an arbitrarily small positive constant ε such that for all $t \ge n_0 T$,

$$x(t) \le \frac{\frac{r}{b}(1-\delta-e^{-rT})}{(1-\delta)(1-e^{-rT})} + \varepsilon = \rho. \tag{3.5}$$

From (3.5) and the second equation of (2.3), we have, for $t > n_0 T + \tau$,

$$\begin{aligned} y_2(t) &\le \alpha e^{-r\tau} y_2(t-\tau) - \beta y_2^2(t) + a_2 x(t) y_2(t) - \gamma y_2(t) \\ &\le \alpha e^{-r\tau} y_2(t-\tau) - \beta y_2^2(t) - (\gamma - a_2\rho) y_2(t). \end{aligned}$$

Consider the following comparison equation

$$\frac{dz(t)}{dt} = \alpha e^{-\gamma\tau} y_2(t-\tau) - \beta y_2^2(t) - (\gamma - a_2\rho) y_2(t).$$

Since $R_1 < 1$, we can see

$$\alpha e^{-\gamma\tau} < \gamma - a_2\rho.$$

By Lemma 4, we obtain

$$\lim_{t\to\infty} z(t) = 0.$$

By standard comparison theorem of ordinary different equation ([13]) ($y_2(t) \ge 0$), we have

$$y_2(t) \to 0 \ (t \to \infty).$$

We assume that $0 < y_2(t) < \varepsilon$ for $t \ge n_0 T$, then by the first equation of system (2.3) we get

$$\frac{dx}{dt} \ge (r - a_1\varepsilon) x(t)(1 - \frac{b}{r - a_1\varepsilon} x(t)).$$

Consider the following impulsive equation for $t \ge n_0 T$

$$\begin{cases} \frac{dz_1(t)}{dt} = (r - a_1\varepsilon)x(t)(1 - \frac{b}{r-a_1\varepsilon}x(t)), & t \ne nT,\ n \in \mathbb{N}, \\ z_1(t^+) = (1-\delta)z_1(t), & t = nT,\ n \in \mathbb{N}, \\ z_1(0^+) = x(0^+). \end{cases} \tag{3.6}$$

System (3.6) has a unique positive periodic solution

$$z_1^*(t) = \frac{(r-a_1\varepsilon)(1-\delta-e^{-(r-a_1\varepsilon)T})}{b(1-\delta-e^{-(r-a_1\varepsilon)T}+\delta e^{-(r-a_1\varepsilon)(t-nT)})}$$

for $nT < t < (n+1)T$, $t \geq n_0 T$.

Clearly, $z_1^*(t) \to x^*(t)$ $(t \to \infty)$. From Lemma 5, for any $\varepsilon_1 > 0$, there exists $n_1 > n_0$ such that for $t > n_1 T$

$$x(t) > z_1^*(t) - \varepsilon_1. \tag{3.7}$$

On the other hand, from the first equation of (2.3), we have

$$\frac{dx(t)}{dt} \leq x(t)(r - bx(t)).$$

From (3.3) we get the similar result, together with (3.7) we have

$$x^*(t) - \varepsilon_1 < x(t) < x^*(t) + \varepsilon_1. \tag{3.8}$$

When t is sufficiently large,

$$x(t) \to x^*(t)\ (t \to \infty).$$

□

Let

$$\begin{aligned} \delta^* &= 1 - \frac{a_2 r e^{-rT}}{a_2 r - b(1 - e^{-rT})(\gamma - \alpha e^{-\gamma T})}, \\ T^* &= \frac{1}{r} \ln \frac{a_2 r - b(1-\delta)(\gamma - \alpha e^{-\gamma T})}{(1-\delta)(a_2 r - b(\gamma - \alpha e^{-\gamma \tau}))}. \end{aligned}$$

Corollary 3.3. (i) *If $a_2 r \leq b(\gamma - \alpha e^{-\gamma\tau})$, then the 'predator-extinction' periodic solution $(x^*(t), 0)$ is globally attractive;*
(ii) *if $a_2 r > b(\gamma - \alpha e^{-\gamma\tau})$ and $\delta^* < \delta < \delta_0^*$ or $T < T^*$, then the 'predator-extinction' periodic solution $(x^*(t), 0)$ is globally attractive.*

In this section, we have proved that, when $\delta^* < \delta < \delta_0^*$ or $T < T^*$, the 'predator-extinction' periodic solution $(x^*(t), 0)$ is globally attractive. This showed that the natural enemy is more affected compared to the pest. In practice, we hope to control the pest population under the economic threshold Level (ETL) so that the pests do not bring about immense economic losses such that the pest population and the natural enemy population can coexist. Hence we need to discuss the permanence of system (2.3).

4. Permanence

In the following, we define permanence.

Definition 1 (see [9]). System (2.3) is said to be permanent if there exists a compact domain $D \in \operatorname{int} \Omega$ such that every solution of system (2.3) with initial conditions (2.4) will eventually enter and remain in D.

Theorem 4.1. *If $R_2 = \frac{a_2(r - \frac{\delta}{T})}{b(\beta M + \gamma - \alpha e^{-\gamma\tau})} > 1$, then system (2.3) is permanent.*

Proof. Let $X(t) = (x(t), y_2(t))$ be any positive solution of system (2.3) with initial condition (2.4). Note the second equation of system (2.3) can be rewritten as:

$$\dot{y}(t) = [\alpha e^{-\gamma\tau} - \gamma - \beta y_2(t) + a_2 x(t)]y_2(t) - \alpha e^{-\gamma\tau}\frac{d}{dt}\int_{t-\tau}^{t} y_2(u)du. \tag{4.1}$$

Define $v(t) = y_2(t) + \alpha e^{-\gamma\tau}\int_{t-\tau}^{t} y_2(u)du$, and calculating the derivative of $v(t)$ along the solution of (2.3), it follows from (4.1) that

$$\frac{dv(t)}{dt} = [\alpha e^{-\gamma\tau} - \beta y_2(t) + a_2 x(t) - \gamma]y_2(t). \tag{4.2}$$

Let $y_2^* = -\frac{b}{a_1 a_2}(\gamma + \beta M - \alpha e^{-\gamma\tau}) + \frac{1}{a_1}(r - \frac{\delta}{T})$, since $R_2 > 1$. Then $y_2^* > 0$. Hence there exists $\varepsilon_2 > 0$ such that

$$\alpha e^{-\gamma\tau} - \gamma - \beta M + a_2\varrho > 0,$$

where $\varrho = \frac{r - a_1 y_2^*}{b}(1 - \frac{\delta}{1 - e - (r - a_1 y_2^*)T}) - \varepsilon_2$. We claim that for any $t_0 > 0$, it is impossible that $y_2(t) < y_2^*$ for all $t > t_0$. Otherwise there exists a $t_0 > 0$ such that $y_2(t) < y_2^*$ for all $t \geq t_0$. From system (2.3), we have

$$\begin{cases} \frac{dx(t)}{dt} & \geq & (r - a_1 y_2^*)x(t)(1 - \frac{b}{r - a_1 y_2^*}x(t)), & t \neq nT,\ n \in \mathbb{N}, \\ x(t^+) & = & (1-\delta)x(t), & t = nT,\ n \in \mathbb{N}. \end{cases}$$

There exists a $T_1 > t_0 + \tau$ such that for $t \geq T_1$

$$x(t) > \frac{r - a_1 y_2^*}{b}(1 - \frac{\delta}{1 - e - (r - a_1 y_2^*)T}) - \varepsilon_2 = \varrho. \tag{4.3}$$

From (4.2) and (4.3), we have

$$\frac{dv(t)}{dt} > [\alpha e^{-\gamma\tau} - \gamma - \beta M + a_2\varrho]y_2(t), \quad t > t_1.$$

Let $y_2^m = \min_{t \in [t_1, t_1 + \tau]} y_2(t)$, we claim that $y_2(t) \geq y_2^m$ for all $t > t_1$. Otherwise there exists a $T_1 > 0$ such that $y_2(t) \geq y_2^m$ for $t \in [t_1, t_1 + \tau + T_1]$ satisfying $y_2(t_1 + \tau + T_1) = y_2^m$ and $\dot{y}_2(t_1 + \tau + T_1) \leq 0$. Hence

$$\begin{aligned} \dot{y}_2(t_1 + \tau + T_1) &\geq \alpha e^{-\gamma\tau}y_2(t_1 + T_1) - \gamma y_2(t_1 + \tau + T_1) - \beta y_2^2(t_1 + \tau + T_1) \\ &\quad + a_2 x(t_1 + \tau + T_1)y_2(t_1 + \tau + T_1) \\ &\geq (\alpha e^{-\gamma\tau} - \gamma - \beta M + a_2\varrho)y_2^m > 0. \end{aligned}$$

This is a contradiction, and thus $y_2(t) \geq y_2^m$ for all $t > t_1$. As a consequence,

$$\frac{dv(t)}{dt} > [\alpha e^{-\gamma\tau} - \gamma - \beta M + a_2\varrho]y_2^m > 0$$

for all $t > t_1$, $v(t) \to \infty$ $(t \to \infty)$. This contradicts with $v(t) \leq M(1 + \alpha e^{-\gamma\tau}\tau)$. Hence the inequality $y_2(t) \geq y_2^*$ cannot hold for all $t > t_0$.

If $y_2(t) \geq y_2^*$ for all sufficiently large t, the conclusion is evident; if $y_2(t)$ oscillates about y_2^*, let $y_2 = \min\{\frac{y_2^*}{2}, y_2^* e^{-\gamma\tau}\}$. We claim that $y_2(t) \geq y_2$ for all sufficiently large t. There exist $\tilde{t}, \eta > 0$ such that $y_2(\tilde{t}) = y_2(\tilde{t} + \eta) = y_2^*, y_2(t) <$

y_2^* for $\tilde{t} < t < \tilde{t} + \eta$. Since $y_2(t)$ is continuous, ultimately bounded and not affected by impulses, there exists a constant t_2 independent of $\tilde{t}$ $(0 < t_2 < \tau)$ such that $y_2(t) > \frac{y_2^*}{2}$ for all $\tilde{t} \le t \le \tilde{t} + t_2$. If $\eta \le t_2$, the conclusion is obvious. If $t_2 < \eta \le \tau$, from the second equation of (2.3) we have that $\dot{y}_2(t) \ge -\gamma y_2(t)$ for $\tilde{t} < t \le \tilde{t} + \eta$. Since $y_2(\tilde{t}) = y_2^*$, it is obvious that $y_2(t) \ge y_2$ for $\tilde{t} \le t \le \tilde{t} + \eta \le \tilde{t} + \tau$. We have shown that $y_2(t) \ge y_2$ for $\tilde{t} \le t \le \tilde{t} + \eta$. Then by the same argument leading to the claim above, we can obtain $y_2(t) \ge y_2$ for $\tilde{t} + \tau \le t \le \tilde{t} + \eta$. Since $[\tilde{t}, \tilde{t} + \eta]$ is arbitrarily chosen, then $y_2(t) \ge y_2$ for all sufficiently large t.

By Lemma 3, we have $y_2(t) \le M$ for all sufficiently large t. Then, when t is sufficiently large, we have

$$\frac{dx}{dt} \ge (r - a_1 M)x(t)(1 - \frac{b}{r - a_1 M}x(t)).$$

Then there exists a sufficiently small $\varepsilon > 0$ for all sufficiently large t such that

$$x(t) > \tilde{z}(t) - \varepsilon \ge \frac{r - a_1 M}{b}(1 - \frac{\delta}{1 - e^{-(r - a_1 M)T}}) - \varepsilon = x_1.$$

Let $D = \{(x, y_2) \in R_+^2 \,|\, x_1 \le x(t) < rb^{-1}, y_2 \le y_2(t) < M\}$. Then D is bounded compact domain which has positive distance from the coordinate axes. We can see that any solution of system (2.3) with the initial condition (2.4) ultimately enters and remains in the domain D, so that system (2.3) is permanent (if $R_2 > 1$). □

Let

$$\delta_* = \frac{a_2 r - b(\beta M + \gamma - \alpha e^{-\gamma\tau}))T}{a_2},$$

$$T_* = \frac{a_2 \delta}{a_2 r - b(\beta M + \gamma - \alpha e^{-\gamma\tau})}.$$

Corollary 4.2. *If $\delta < \delta_*$ or $T > T_*$, then system* (2.3) *is permanent.*

From Theorem 4.1, we can see that a small impulsive catching or poisoning rate (δ) or a long period between impulses (T) is a sufficient condition for the permanence of system (2.3).

5. Conclusion

In this paper, we have investigated a stage structured predator-prey model with time delay and impulsive harvest on the prey. We have shown that if $a_2 r \le b(\gamma - \alpha e^{-\gamma\tau})$, or if $a_2 r > b(\gamma - \alpha e^{-\gamma\tau})$ and $\delta^* < \delta < \delta_0^*$ or $T < T^*$, then the predator (natural enemy) will become extinct, while, if $\delta < \delta_*$ or $T > T_*$, the system is permanent. By these results, harvesting pests reasonably may avoid extinction of the predator. From Theorem 4.1 and Corollary 4.2, we can choose an appropriate impulsive harvest level (δ) or impulsive period (T) such that the pest population is under an economic threshold level (ETL) as well as the pest population and the natural enemy population can coexist. By the above investigation, we can integrate pesticides and natural enemies to control the pest. But how to optimize this control at lower costs much as not to damage the environment is an important question. The variation of the environment influences greatly effects on the pest population, so we have to consider periodic changes, and this is left to future research.

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Xinyu Song
College of Mathematics and Information Science
Xinyang Normal University
Xinyang 464000, Henan, P. R. China
E-mail address: xysong88@163.com

Senlin Li
College of Mathematics and Information Science
Xinyang Normal University
Xinyang 464000, Henan, P. R. China
E-mail address: senlinli2005@163.com

An Li
College of Mathematical Sciences
Xiamen University
Xiamen 361005, Fujian, P. R. China
E-mail address: leean1980@163.com