J. Korean Math. Soc. **46** (2009), No. 1, pp. 31–39

GLOBAL BIFURCATION FOR GENERALIZED LAPLACIAN OPERATORS

In-Sook Kim

Abstract. A bifurcation problem for nonlinear partial differential equations of the form

$$\operatorname{div}(\varphi(|\nabla u|)\nabla u + \mu_0\varphi(|u|)u = q(\lambda, x, u, \nabla u)$$

subject to Dirichlet boundary conditions is discussed. Using a global bifurcation theorem of Rabinowitz type, we show that under certain conditions on φ and q, the above equation has an unbounded connected set of solutions (u, λ).

1. Introduction

This paper is concerned with global bifurcation for the following boundary value problem

$$(A_\varphi)\quad \begin{cases} \operatorname{div}(\varphi(|\nabla u(x)|)\nabla u(x)) + \mu_0\varphi(|u(x)|)u(x) = q(\lambda, x, u(x), \nabla u(x)) & \text{in } U \\ u(x) = 0 \quad \text{on } \partial U \end{cases}$$

when μ_0 is not an eigenvalue of the p-Laplacian. Here U is a bounded smooth domain in $\mathbb{R}^n$, Λ is a finite dimensional space, $q : \Lambda \times U \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function depending on a parameter λ, and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function.

When considering $\varphi(t) = t^{p-2}$, we have the following differential equation

$$(A_p)\quad \begin{cases} \Delta_p u(x) + \mu_0|u(x)|^{p-2}u(x) = q(\lambda, x, u(x), \nabla u(x)) & \text{in } U \\ u(x) = 0 \quad \text{on } \partial U, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \sum_{i=1}^n D_i(|D_i u|^{p-2}D_i u)$ is the p-Laplace operator. In [11], Väth gave some topological bifurcation results in a more general setting, by means of the notion of epiness, which is based on the work of Furi and Pera [5]. Moreover, it was shown that a global bifurcation theorem of

---

Received August 16, 2006; Revised April 14, 2008.
2000 *Mathematics Subject Classification.* 47J05, 47J10, 34B15.
Key words and phrases. bifurcation, generalized Laplacian, unbounded component.
This work is supported by the BK 21 Mathematical Modelling HR Division, Sungkyunkwan University.

Rabinowitz type [9] is applied to the above equation (A_p) when μ_0 is not an eigenvalue of the p-Laplace operator Δ_p.

In the present paper, we prove that under certain conditions on φ, problem (A_φ) has an unbounded connected set of solutions (u, λ) which contains a point of the form $(u, 0)$, where μ_0 is not an eigenvalue of Δ_p, by using some results and methods in [11]. This is an extension of a global bifurcation result for the homogeneous p-Laplacian to a large class of operators that contains nonhomogeneous operators.

Bifurcation problems for nonlinear elliptic equations have intensively been studied by many researchers. They mainly use Ljusternik-Schnirelman theory to obtain eigenvalues and to yield global bifurcation from the first eigenvalue of the p-Laplacian; see [3, 12]. At the first eigenvalue of Δ_p, global bifurcation for quasilinear elliptic equations of the form

$$-\operatorname{div}(\varphi(|\nabla u(x)|)\nabla u(x)) = f(\lambda, x, u(x))$$

with Dirichlet boundary condition $u = 0$ on ∂U has been investigated in [6, 8], in case that the asymptotic behaviors of φ near 0 and at infinity may be different; see [4] for local bifurcation results. A variational inequality approach with a homogenization method is employed in [6]; see [7]. In addition, more general situations in this direction are recently considered in [10]. For the existence of a global bifurcation of problem (A_φ), it is required that $|\varphi(t)t - t^{p-1}| / t^{p-1} \to 0$ as $t \to \infty$ and φ satisfies the uniform monotonicity condition. As we mentioned above, the proof of our main result is completely elementary in such a topological approach as in [5, 11].

2. The main result

Let U be a bounded domain in $\mathbb{R}^n$ such that the boundary of U is locally Lipschitzian, Λ a finite dimensional space, $q : \Lambda \times U \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ a function, and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function. Assume that for each $\lambda \in \Lambda$ the function $q(\lambda, \cdot, \cdot)$ satisfies a Carathéodory condition, that is, $q(\lambda, \cdot, u, v)$ is measurable on U for all $(u, v) \in \mathbb{R} \times \mathbb{R}^n$ and $q(\lambda, x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost all $x \in U$. Let $1 < p < \infty$ and $p' := p/(p-1)$.

Consider the following boundary value problem

$$(A_\varphi) \quad \begin{cases} \operatorname{div}(\varphi(|\nabla u(x)|)\nabla u(x)) + \mu_0 \varphi(|u(x)|)u(x) = q(\lambda, x, u(x), \nabla u(x)) & \text{in } U \\ u(x) = 0 \quad \text{on } \partial U. \end{cases}$$

Assume that $\varphi : [0, \infty) \to [0, \infty)$ has the following behavior at infinity:

$$\lim_{t \to \infty} \frac{|\varphi(t)t - t^{p-1}|}{t^{p-1}} = 0 \tag{1}$$

and it satisfies the uniform monotonicity condition:

$$\langle \varphi(|\xi|)\xi - \varphi(|\eta|)\eta,\ \xi - \eta \rangle \geq c\,|\xi - \eta|^p \tag{2}$$

for some constant $c > 0$ and for all $\xi, \eta \in \mathbb{R}^n$, where $|\cdot|$ is the sum-norm on $\mathbb{R}^n$. The former condition will be needed in Lemma 2.2 to know the behavior of the corresponding integral operators at infinity; the latter is used in Lemma 2.1 to apply the theory of monotone operators.

Assume that q satisfies a growth condition of the form:

$$|q(\lambda, x, u, v_1, \dots, v_n)| \leq a_\lambda(x) + b_\lambda \cdot (|u| + |v_1| + \cdots + |v_n|)^{p-1}, \tag{3}$$

where $a_\lambda \in L_{p'}(U)$ and $b_\lambda \geq 0$ is finite, and that the dependence of q from λ is L_p-uniformly continuous in the sense that

$$\begin{aligned} &|q(\lambda, x, u, v_1, \dots, v_n) - q(\lambda_0, x, u, v_1, \dots, v_n)| \\ &\leq C(\lambda, \lambda_0)(a_{\lambda,\lambda_0}(x) + (|u| + |v_1| + \cdots + |v_n|)^{p-1}), \end{aligned} \tag{4}$$

where $\|a_{\lambda,\lambda_0}\|_{L_{p'}(U)} \leq 1$ and $\lim_{\lambda\to\lambda_0} C(\lambda, \lambda_0) = 0$ for each $\lambda_0 \in \Lambda$.

Let $X := W_0^{1,p}(U)$ be the Sobolev space that consists of functions from $W^{1,p}(U)$ which vanish on the boundary of U, with the norm

$$\|u\|_X = \Big(\sum_{i=1}^{n} \|D_i u\|^p_{L_p(U)}\Big)^{\frac{1}{p}}.$$

Let $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ be the usual pairing of $X^* = W^{-1,p'}(U)$ and X. See e.g. [1]. Define operators $J_\varphi : X \to X^*$, $G_\varphi : X \to X^*$, and $\psi : X \times \Lambda \to X^*$ by the relations

$$\langle J_\varphi(u), v\rangle = \int_U \langle \varphi(|\nabla u(x)|)\nabla u(x), \nabla v(x)\rangle\, dx \qquad (v \in X),$$

$$\langle G_\varphi(u), v\rangle = \int_U \varphi(|u(x)|)u(x)v(x)\, dx \qquad (v \in X),$$

$$\langle \psi(u, \lambda), v\rangle = \int_U q(\lambda, x, u(x), \nabla u(x))v(x)\, dx \qquad (v \in X).$$

A function u is called a *weak solution* of (A_φ) if

$$\langle -J_\varphi(u) + \mu_0 G_\varphi(u), v\rangle = \langle \psi(u, \lambda), v\rangle \qquad \text{for all } v \in X.$$

In a special case when $\varphi(t) = t^{p-2}$, define operators $J_p : X \to X^*$ and $G_p : X \to X^*$ by the relations

$$\langle J_p(u), v\rangle = \int_U \langle |\nabla u(x)|^{p-2}\nabla u(x), \nabla v(x)\rangle\, dx \qquad (v \in X),$$

$$\langle G_p(u), v\rangle = \int_U |u(x)|^{p-2}u(x)v(x)\, dx \qquad (v \in X).$$

It is known that $J_p : X \to X^*$ is a homeomorphism, $G_p : X \to X^*$ is a compact continuous map, and the operators J_p and G_p are positively homogeneous of order $p - 1$.

In the nonhomogeneous case, we show that J_φ and G_φ have the same property under suitable assumptions.

Lemma 2.1. *The operator $J_\varphi : X \to X^*$ is a homeomorphism under assumption* (2) *on φ, and $G_\varphi : X \to X^*$ is a compact continuous map. Under assumptions* (3)-(4), *the operator $\psi : X \times \Lambda \to X^*$ is continuous and compact.*

Proof. (i) Notice that

$$d_1\|u\|_X \le \|\nabla u\|_{L_p(U,\mathbb{R}^n)} \le d_2\|u\|_X$$

for all $u \in X$ and for some positive constants d_1 and d_2. First, we show that J_φ is continuous on X. Let $u_k \to u$ in X as $k \to \infty$. Then $\nabla u_k \to \nabla u$ in $L_p(U,\mathbb{R}^n)$ and so

$$\|\varphi(|\nabla u_k|)\nabla u_k - \varphi(|\nabla u|)\nabla u\|_{L_{p'}(U,\mathbb{R}^n)} \to 0 \qquad \text{as } k \to \infty$$

because $L_p(U,\mathbb{R}^n) \to L_{p'}(U,\mathbb{R}^n)$, $\nabla u \mapsto \varphi(|\nabla u|)\nabla u$ is continuous. By Schwarz's and Hölder's inequalities, we have the following estimates

$$\begin{aligned}
&\|J_\varphi(u_k) - J_\varphi(u)\|_{X^*} \\
= &\sup_{\|v\|_X\le 1} |\langle J_\varphi(u_k) - J_\varphi(u), v\rangle| \\
\le &\sup_{\|v\|_X\le 1} \int_U |\langle \varphi(|\nabla u_k(x)|)\nabla u_k(x) - \varphi(|\nabla u(x)|)\nabla u(x), \nabla v(x)\rangle|\, dx \\
\le &\sup_{\|v\|_X\le 1} \int_U (\sum_{i=1}^n |\varphi(|\nabla u_k(x)|)D_i u_k(x) - \varphi(|\nabla u(x)|)D_i u(x)|)(\sum_{i=1}^n |D_i v(x)|)\, dx \\
\le &\sup_{\|v\|_X\le 1} \|\varphi(|\nabla u_k|)\nabla u_k - \varphi(|\nabla u|)\nabla u\|_{L_{p'}(U,\mathbb{R}^n)}\|\nabla v\|_{L_p(U,\mathbb{R}^n)} \\
\le &\, d_2\|\varphi(|\nabla u_k|)\nabla u_k - \varphi(|\nabla u|)\nabla u\|_{L_{p'}(U,\mathbb{R}^n)}.
\end{aligned}$$

Since $J_\varphi(u_k) \to J_\varphi(u)$ in X^* as $k \to \infty$, J_φ is continuous on X. Next, J_φ is uniformly monotone on X. In fact, for all $u, v \in X$, we have by assumption (2) on φ,

$$\begin{aligned}
&\langle J_\varphi(u) - J_\varphi(v), u - v\rangle \\
= &\int_U \langle \varphi(|\nabla u(x)|)\nabla u(x) - \varphi(|\nabla v(x)|)\nabla v(x), \nabla u(x) - \nabla v(x)\rangle\, dx \\
\ge &\, c\int_U |\nabla u(x) - \nabla v(x)|^p\, dx \\
\ge &\, c\,(d_1)^p\,\|u - v\|_X^p.
\end{aligned}$$

Since J_φ is continuous and uniformly monotone on the reflexive Banach space X, we conclude that J_φ is a homeomorphism; see e.g. Theorem 26.A of Zeidler [13].

(ii) As in the above proof, it can be verified that $G_\varphi : X \to X^*$ is continuous. Now it remains to prove that G_φ is compact. Let $\{u_k\}$ be any bounded sequence in X. By the compactness of the embedding $X \hookrightarrow L_p(U)$, we may without loss of generality suppose that $\{u_k\}$ converges to some point u_0 in $L_p(U)$. Since the map $L_p(U) \to L_{p'}(U)$, $u \mapsto \varphi(|u|)u$ is continuous, it follows that

$$\varphi(|u_k|)u_k \to \varphi(|u_0|)u_0 \quad \text{in } L_{p'}(U) \quad \text{as } k \to \infty.$$

From the Hölder inequality we obtain

$$\begin{aligned} \|G_\varphi(u_k) - G_\varphi(u_0)\|_{X^*} &= \sup_{\|v\|_X \le 1} |\langle G_\varphi(u_k) - G_\varphi(u_0), v\rangle| \\ &\le \sup_{\|v\|_X \le 1} \Big| \int_U [\varphi(|u_k(x)|)u_k(x) - \varphi(|u_0(x)|)u_0(x)]v(x)\,dx \Big| \\ &\le \sup_{\|v\|_X \le 1} \|\varphi(|u_k|)u_k - \varphi(|u_0|)u_0\|_{L_{p'}(U)} \|v\|_{L_p(U)} \\ &\le d_3 \|\varphi(|u_k|)u_k - \varphi(|u_0|)u_0\|_{L_{p'}(U)} \end{aligned}$$

for some positive constant d_3, which implies $G_\varphi(u_k) \to G_\varphi(u_0)$ in X^* as $k \to \infty$. Thus, G_φ is compact on X.

(iii) The proof of continuity and compactness of the map ψ can be found in [11, Corollary 24]. □

For our aim, we need the following result whose proof is based on the nonlinear spectral theory for homogeneous operators; see [2, Theorem 2] or [11, Lemma 27]. Recall that μ_0 is an *eigenvalue* of Δ_p on U if the equation $J_p(u) = \mu_0 G_p(u)$ has a nontrivial solution.

Lemma 2.2. *Under assumption* (1), *if μ_0 is not an eigenvalue of Δ_p on U, then we have*

$$\liminf_{\|u\|_X \to \infty} \frac{\|J_\varphi(u) - \mu_0 G_\varphi(u)\|_{X^*}}{\|u\|_X^{p-1}} > 0.$$

Proof. We first claim that $\lim_{\|u\|_X \to \infty} \|J_\varphi(u) - J_p(u)\|_{X^*} / \|u\|_X^{p-1} = 0$. Let $\delta > 0$. By assumption (1) on φ, we can choose a real number $R > 0$ such that

$$|\varphi(t)t - t^{p-1}| \le \delta t^{p-1} \quad \text{for all } t \ge R.$$

Since the map $t \mapsto \varphi(t)t - t^{p-1}$ is continuous on $[0, R]$, there exists a positive real number C_R such that

$$|\varphi(t)t - t^{p-1}| \le C_R \quad \text{for all } t \in [0, R].$$

Given $v \in L_p(U, \mathbb{R}^n)$, put $U_R := \{x \in U : |v(x)| \le R\}$, where $|\cdot|$ is the sum-norm on $\mathbb{R}^n$. Note that $\|\nabla u\|_{L_p(U,\mathbb{R}^n)} \le d\|u\|_X$ for all $u \in X$ and for some positive constant d. Then we have the following estimates

$$\begin{aligned}
&\|\varphi(|\nabla u|)\nabla u - |\nabla u|^{p-2}\nabla u\|^{p'}_{L_{p'}(U,\mathbb{R}^n)} \\
&= \int_U |\varphi(|\nabla u(x)|)|\nabla u(x)| - |\nabla u(x)|^{p-1}|^{p'}\, dx \\
&\le \int_{U_R} (C_R)^{p'}\, dx + \int_{U\setminus U_R} (\delta|\nabla u(x)|^{p-1})^{p'}\, dx \\
&\le (C_R)^{p'} \text{mes}\, U + (\delta)^{p'} \|\nabla u\|^p_{L_p(U,\mathbb{R}^n)} \\
&\le (C_R)^{p'} \text{mes}\, U + (\delta)^{p'} d^p \|u\|^p_X
\end{aligned}$$

for all $u \in X$, which implies

$$\lim_{\|u\|_X\to\infty} \frac{\|\varphi(|\nabla u|)\nabla u - |\nabla u|^{p-2}\nabla u\|_{L_{p'}(U,\mathbb{R}^n)}}{\|u\|^{p-1}_X} = 0.$$

Therefore, it follows from the Hölder inequality that

$$\begin{aligned}
&\|J_\varphi(u) - J_p(u)\|_{X^*} \\
&= \sup_{\|v\|_X\le 1} \Big| \int_U \langle \varphi(|\nabla u(x)|)\nabla u(x) - |\nabla u(x)|^{p-2}\nabla u(x), \nabla v(x)\rangle\, dx \Big| \\
&\le \sup_{\|v\|_X\le 1} \int_U \Big(\sum_{i=1}^n |\varphi(|\nabla u(x)|)D_i u(x) - |\nabla u(x)|^{p-2} D_i u(x)|\Big)\Big(\sum_{i=1}^n |D_i v(x)|\Big)\, dx \\
&\le \sup_{\|v\|_X\le 1} \|\varphi(|\nabla u|)\nabla u - |\nabla u|^{p-2}\nabla u\|_{L_{p'}(U,\mathbb{R}^n)} \|\nabla v\|_{L_p(U,\mathbb{R}^n)} \\
&\le d\, \|\varphi(|\nabla u|)\nabla u - |\nabla u|^{p-2}\nabla u\|_{L_{p'}(U,\mathbb{R}^n)}
\end{aligned}$$

for all $u \in X$. Dividing this estimate by $\|u\|^{p-1}_X$, we obtain

$$\lim_{\|u\|_X\to\infty} \frac{\|J_\varphi(u) - J_p(u)\|_{X^*}}{\|u\|^{p-1}_X} = 0.$$

A similar argument shows that

$$\lim_{\|u\|_X\to\infty} \frac{\|G_\varphi(u) - G_p(u)\|_{X^*}}{\|u\|^{p-1}_X} = 0.$$

Now we use the following result on homogeneous operators proved in [11, Lemma 27] that

$$\alpha := \liminf_{\|u\|_X\to\infty} \frac{\|J_p(u) - \mu_0 G_p(u)\|_{X^*}}{\|u\|^{p-1}_X} > 0.$$

Let $\varepsilon > 0$ be given arbitrary. Choose a positive real number R_1 such that

$$\|u\|_X \ge R_1 \text{ implies } \|J_p(u) - \mu_0 G_p(u)\|_{X^*} > (\alpha - \varepsilon)\|u\|^{p-1}_X.$$

Since $\lim_{\|u\|_X\to\infty} \|J_\varphi(u) - J_p(u)\|_{X^*}/\|u\|_X^{p-1} = 0$, there exists a real number $R_2 > 0$ such that

$$\|u\|_X \geq R_2 \text{ implies } \|J_\varphi(u) - J_p(u)\|_{X^*} < \frac{\alpha}{4}\|u\|_X^{p-1}.$$

Analogously, since $\lim_{\|u\|_X\to\infty} \|G_\varphi(u) - G_p(u)\|_{X^*}/\|u\|_X^{p-1} = 0$, there is a real number $R_3 > 0$ such that

$$\|u\|_X \geq R_3 \text{ implies } \|\mu_0 G_\varphi(u) - \mu_0 G_p(u)\|_{X^*} < \frac{\alpha}{4}\|u\|_X^{p-1}.$$

Set $\tilde{R} := \max\{R_1, R_2, R_3\}$. For all $u \in X$ with $\|u\|_X \geq \tilde{R}$, we have

$$\begin{aligned}
&\|J_\varphi(u) - \mu_0 G_\varphi(u)\|_{X^*} \\
&\geq \|J_p(u) - \mu_0 G_p(u)\|_{X^*} - \|J_p(u) - J_\varphi(u)\|_{X^*} - \|\mu_0 G_\varphi(u) - \mu_0 G_p(u)\|_{X^*} \\
&> [(\alpha - \varepsilon) - \frac{\alpha}{4} - \frac{\alpha}{4}]\,\|u\|_X^{p-1} \\
&= (\frac{\alpha}{2} - \varepsilon)\|u\|_X^{p-1}.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that

$$\liminf_{\|u\|_X\to\infty} \frac{\|J_\varphi(u) - \mu_0 G_\varphi(u)\|_{X^*}}{\|u\|_X^{p-1}} \geq \frac{\alpha}{2} > 0.$$

□

In order to obtain a useful property of $\psi(\cdot, 0)$ at infinity given in [11, Corollary 26], the following restrictive growth condition is required:

$$|q(0, x, u, v_1, \ldots, v_n)| \leq a(x) + b(|u| + |v_1| + \cdots + |v_n|), \tag{5}$$

where $a \in L_{p'}(U)$ and $b : [0, \infty) \to \mathbb{R}$ is locally bounded such that

$$\lim_{r\to\infty} \frac{|b(r)|}{r^{p-1}} = 0.$$

Lemma 2.3. *Under assumption* (5), *the operator* $\psi(\cdot, 0) : X \to X^*$ *has the following property:*

$$\lim_{\|u\|_X\to\infty} \frac{\|\psi(u, 0)\|_{X^*}}{\|u\|_X^{p-1}} = 0.$$

The following global bifurcation result which is a key tool of our main theorem is a particular case of a topological bifurcation result of Rabinowitz type stated in [11]; see [11, Corollary 22].

Lemma 2.4. *Let* X *be a Banach space,* Y *a normed space, and* Λ *a finite-dimensional space. Let* $J : X \to Y$ *be a homeomorphism onto* Y *and* $G : X \to Y$ *a compact continuous map such that the composition* $J^{-1} \circ (-G)$ *is odd. Let* $\psi : X \times \Lambda \to Y$ *be a compact continuous map. Suppose that the set*

$$S := \bigcup_{t\in[0,1]} \{x \in X : J(x) + G(x) = t\psi(x, 0)\}$$

is bounded. Then there is an unbounded component of

$$X_1 := \{(x, \lambda) \in X \times \Lambda : J(x) + G(x) = \psi(x, \lambda)\}$$

which intersects $X \times \{0\}$.

Applying the previous results, we can now prove a global bifurcation theorem for not necessarily homogeneous operators.

Theorem 2.5. *Suppose that* μ_0 *is not an eigenvalue of* Δ_p *on* U *and the conditions* (1)-(5) *are satisfied. Then there is an unbounded component* $C \subseteq X \times \Lambda$ *intersecting* $X \times \{0\}$ *such that every point* (u, λ) *in* C *is a weak solution of problem* (A_φ).

Proof. Apply Lemma 2.4 with $Y = X^*, J = -J_\varphi$, and $G = \mu_0 G_\varphi$. By Lemma 2.1, J_φ is a homeomorphism and the maps G_φ and ψ are continuous and compact. Since J_φ and G_φ are odd maps, the composition $(-J_\varphi)^{-1} \circ (-\mu_0 G_\varphi)$ is also odd. Since μ_0 is not an eigenvalue of Δ_p on U, Lemmas 2.2 and 2.3 imply that given $\beta > 0$, there exists a positive real number R such that

$$\|J_\varphi(u) - \mu_0 G_\varphi(u)\|_{X^*} > \beta \|u\|_X^{p-1} > \|\psi(u, 0)\|_{X^*}$$

for all $u \in X$ with $\|u\| \geq R$ and hence

$$\|J_\varphi(u) - \mu_0 G_\varphi(u)\|_{X^*} > \| - t\psi(u, 0)\|_{X^*}$$

for all $t \in [0, 1]$. Thus the set

$$S = \bigcup_{t \in [0,1]} \{u \in X : -J_\varphi(u) + \mu_0 G_\varphi(u) = t\psi(u, 0)\}$$

is bounded. By Lemma 2.4, problem (A_φ) has an unbounded component C of weak solutions in $X \times \Lambda$ which intersects $X \times \{0\}$. □

As a consequence of Theorem 2.5, we get Theorem 28 of [11].

Corollary 2.6. *Suppose that* μ_0 *is not an eigenvalue of* Δ_p *on* U *and conditions* (3)-(5) *are satisfied. Then there is an unbounded component* $C \subseteq X \times \Lambda$ *intersecting* $X \times \{0\}$ *such that every point* (u, λ) *in* C *is a weak solution of equation* (A_p). *In case where* $q(0, \cdot, \cdot) = 0$, *we have* $(0, 0) \in C$.

Proof. Applying Theorem 2.5 with $\varphi(t) = t^{p-2}$ for $t \geq 0$, equation (A_p) has an unbounded component C of weak solutions in $X \times \Lambda$ such that $C \cap (S \times \{0\}) \neq \emptyset$, where $S = \{u \in X : -J_p(u) + \mu_0 G_p(u) = t\psi(u, 0)$ for some $t \in [0, 1]\}$. When $q(0, \cdot, \cdot) = 0$, we have $S = \{0\}$ and so $(0, 0) \in C$ because μ_0 is not an eigenvalue of Δ_p. □

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] J. Appell, E. Giorgieri, and M. Väth, *Nonlinear spectral theory for homogeneous operators*, Nonlinear Funct. Anal. Appl. **7** (2002), no. 4, 589–618.

[3] M. A. Del Pino and R. F. Manásevich, *Global bifurcation from the eigenvalues of the p-Laplacian*, J. Differential Equations **92** (1991), no. 2, 226–251.

[4] N. Fukagai, M. Ito, and K. Narukawa, *A bifurcation problem of some nonlinear degenerate elliptic equations*, Adv. Differential Equations **2** (1997), no. 6, 895–926.

[5] M. Furi and M. P. Pera, *On the existence of an unbounded connected set of solutions for nonlinear equations in Banach spaces*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **67** (1979), no. 1-2, 31–38.

[6] V. K. Le, *Global bifurcation in some degenerate quasilinear elliptic equations by a variational inequality approach*, Nonlinear Anal. **46** (2001), no. 4, 567–589.

[7] V. K. Le and K. Schmitt, *Global Bifurcation in Variational Inequalities*, Applications to obstacle and unilateral problems. Applied Mathematical Sciences, 123. Springer-Verlag, New York, 1997.

[8] K. Narukawa, *Global bifurcation for quasilinear elliptic equations*, Nonlinear Anal. **30** (1997), no. 8, 5241–5249.

[9] P. H. Rabinowitz, *A note on a nonlinear elliptic equation*, Indiana Univ. Math. J. **22** (1972/73), 43–49.

[10] K. Schmitt and I. Sim, *Bifurcation problems associated with generalized Laplacians*, Adv. Differential Equations **9** (2004), no. 7-8, 797–828.

[11] M. Väth, *Global bifurcation of the p-Laplacian and related operators*, J. Differential Equations **213** (2005), no. 2, 389–409.

[12] E. Zeidler, *Nonlinear Functional Analysis and its Applications. III*, Variational methods and optimization. Translated from the German by Leo F. Boron. Springer-Verlag, New York, 1985.

[13] ———, *Nonlinear Functional Analysis and its Applications. II/B*, Nonlinear monotone operators. Translated from the German by the author and Leo F. Boron. Springer-Verlag, New York, 1990.

Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
E-mail address: iskim@math.skku.ac.kr