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A REMARK ON ERGODICITY OF QUANTUM MARKOVIAN SEMIGROUPS

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Abstract. The aim of this paper is to find the set of the fixed elements and the set of elements for which equality holds in Schwarz inequality for the KMS-symmetric Markovian semigroup $\{S_t\}_{t\geq 0}$ given in [10]. As an application, we study some properties such as the ergodicity and the asymptotic behavior of the semigroup.

1. Introduction

A quantum Markovian semigroup $S = \{S_t\}_{t\geq 0}$ on a von Neumann algebra $\mathcal{M}$ is a weak* continuous semigroup of normal completely positive maps on $\mathcal{M}$, and identity preserving. Quantum Markovian semigroups are the natural generalization of classical Markovian semigroups and were introduced in physics to model the decay to equilibrium of quantum open systems [2, 3, 10, 12]. The asymptotic behavior of the semigroups was studied in [5, 6, 4, 7, 9].

Assume that the semigroup S possesses a faithful normal stationary state ω, $\omega(S_t(A)) = \omega(A)$, $A \in \mathcal{M}$. Let $\mathcal{F}$ be the set of the fixed elements for S and $\mathcal{N}$ be the set of elements for which equality holds in Schwarz inequality for S, $S_t(A^*A) \geq S_t(A^*)S_t(A)$ for all $t \geq 0$, $A \in \mathcal{M}$. Then $\mathcal{F}$ is contained in $\mathcal{N}$, and $\mathcal{F}$ and $\mathcal{N}$ are von Neumann subalgebras of $\mathcal{M}$. Moreover, if $\mathcal{F} = \mathcal{N}$, then for any normal state ρ on $\mathcal{M}$, $\rho(S_t(A))$ converges as $t \to \infty$. If $\mathcal{F} = \mathbb{C}\mathbf{1}$, then ω is the unique S-invariant state and if $\mathcal{N} = \mathcal{F} = \mathbb{C}\mathbf{1}$, then for any normal state ρ on $\mathcal{M}$, $\rho(S_t(A))$ converges to $\omega(A)$ as $t \to \infty$ (mixing). See Theorem 4.3 of [4]. The subalgebras $\mathcal{F}$ and $\mathcal{N}$ have played an important role in the study of the asymptotic behavior of the semigroup.

Let $\mathcal{M}$ be a von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ and ξ_0 be a fixed cyclic and separating vector for $\mathcal{M}$. Let Δ and σ_t be the modular operator and modular group associated with the pair $(\mathcal{M}, \xi_0)$, respectively [1]. Consider the symmetric embedding map [2]:

$$i_0 : \mathcal{M} \to \mathcal{H}$$
$$i_0(A) = \Delta^{1/4} A \xi_0.$$

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In [10, 11], Park has studied the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ (noncommutative Dirichlet form in the sense of Cipriani [2]) and the symmetric Markovian semigroup $\{T_t\}_{t\geq 0}$ on the standard form of $\mathcal{M}$, and the KMS-symmetric Markovian semigroup $\{S_t\}_{t\geq 0}$ on $\mathcal{M}$ given by the symmetric map i_0 [2].

In this paper, we also consider the semigroup $\{S_t\}_{t\geq 0}$ with bounded generator $G, S_t = e^{-tG}$ on $\mathcal{M}$ given by for any $A \in \mathcal{M}$

$$
\begin{aligned}
(1.1)\qquad G(A) = \sum_{k=1}^{n} \Big(\int [&\sigma_{t+i/2}(x_k^*)\sigma_t(x_k)A + A\sigma_t(x_k)\sigma_{t-i/2}(x_k^*) \\
&- \sigma_{t+i/2}(x_k^*)A\sigma_t(x_k) - \sigma_t(x_k)A\sigma_{t-i/2}(x_k^*)] f_0(t)dt \\
+ \int [&\sigma_{t+i/2}(x_k)\sigma_t(x_k^*)A + A\sigma_t(x_k^*)\sigma_{t-i/2}(x_k) \\
&- \sigma_{t+i/2}(x_k)A\sigma_t(x_k^*) - \sigma_t(x_k^*)A\sigma_{t-i/2}(x_k)] f_0(t)dt \Big),
\end{aligned}
$$

where $\{x_k\}_{k=1}^n$ is a family of σ_t-analytic elements in $\mathcal{M}$ [1] and $f_0(t) = 2(e^{2\pi t} + e^{-2\pi t})^{-1}$ (see (2.2)). The semigroup $\{S_t\}_{t\geq 0}$ is a KMS-symmetric Markovian semigroup on $\mathcal{M}$ [10]. The generator G of the semigroup $\{S_t\}_{t\geq 0}$ is the (bounded) Lindblad type (Proposition 2.1) and satisfies $G(A^*A) \leq G(A^*)A + A^*G(A)$ for all $A \in \mathcal{M}$ (see (3.18)), which means $S_t(A^*A) \geq S_t(A^*)S_t(A)$ for all $t \geq 0,\ A \in \mathcal{M}$.

In [10], Park gave the sufficient condition so that the Lindblad type generator [12] can be expressed as the type of the Dirichlet operator G in (1.1). The aim of this paper is to find two subalgebras $\mathcal{F}$ and $\mathcal{N}$ for the KMS-symmetric Markovian semigroup $\{S_t\}_{t\geq 0}$. First, using the converse course of Proposition 2.1 and Theorem 2.2 in [10], we show that the generator G in (1.1) is written as the Lindblad type, and we classify two subalgebras $\mathcal{F}$ and $\mathcal{N}$. As an application, we study some properties such as the ergodicity and the asymptotic behavior of the semigroup.

This paper is organized as follows. In Section 2, we introduce basic terminologies and the generator of KMS-symmetric Markovian semigroup on $\mathcal{M}$, and state main results. In Section 3, we give proofs of main results.

2. The quantum Markovian semigroup and main results

In this section, we introduce some terminologies and the generator of KMS-symmetric Markovian semigroup on a von Neumann algebra given in [10], and state main results.

Let $\mathcal{M}$ be a σ-finite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle\cdot,\cdot\rangle$. Let $\xi_0 \in \mathcal{H}$ be a cyclic and separating vector for $\mathcal{M}$. We use Δ and J to denote, respectively, the modular operator and the modular conjugation associated with the pair $(\mathcal{M}, \xi_0)$. The associated modular automorphism group is denoted by $\sigma_t : \sigma_t(A) = \Delta^{it}A\Delta^{-it}, A \in \mathcal{M},\ t \in \mathbb{R}$. The map $j : \mathcal{M} \to \mathcal{M}'$ is the antilinear $*$-isomorphism defined by $j(A) = JAJ,\ A \in \mathcal{M}$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$.

The positive cone $\mathcal{P}$ associated with the pair $(\mathcal{M}, \xi_0)$ is the closure of the set $\{Aj(A)\xi_0 : A \in \mathcal{M}\}$. $\mathcal{P}$ can be obtained by the closure of the set $\{\Delta^{1/4}A^*A\xi_0 : A \in \mathcal{M}\}$ and is self-dual in the sense that

$$\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \geq 0, \ \forall \eta \in \mathcal{P}\} = \mathcal{P}.$$

For the details we refer to [2] and Section 2.5 of [1].

The form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the standard form associated with the pair $(\mathcal{M}, \xi_0)$. The Hilbert space $\mathcal{H}$ is the complexification of the real subspace $\mathcal{H}^J := \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \in \mathbb{R}, \forall \eta \in \mathcal{P}\}$, whose elements are called *J-real*: $\mathcal{H} = \mathcal{H}^J \oplus i\mathcal{H}^J$. Such a positive cone $\mathcal{P}$ gives rise to a structure of ordered Hilbert space on $\mathcal{H}^J$(order relation denoted by $\leq$) and an anti-unitary involution J on $\mathcal{H}$ by $J(\xi + i\eta) := \xi - i\eta$, $\forall \xi, \eta \in \mathcal{H}^J$. For ξ, $\eta \in \mathcal{H}^J$, $\xi \leq \eta$ means $\eta - \xi \in \mathcal{P}$. Any J-real element $\xi \in \mathcal{H}^J$ can be decomposed uniquely as a difference of two orthogonal, positive elements, called the positive and the negative part of ξ: $\xi_+, \xi_- \in \mathcal{P}$, $\xi = \xi_+ - \xi_-$, $\langle \xi_+, \xi_- \rangle = 0$. The order interval $\{\eta \in \mathcal{H} : 0 \leq \eta \leq \xi_0\}$, denoted by $[0, \xi_0]$, is a closed convex subset of $\mathcal{H}$, and we denote the nearest point projection onto $[0, \xi_0]$ by $\eta \to \eta_I$.

For any $\lambda > 0$, denote by $\mathcal{M}_\lambda$ the dense subset of $\mathcal{M}$ consisting of every σ_t-analytic element with a domain containing $I_\lambda = \{z \in \mathbb{C} : |\mathrm{Im}\, z\,| \leq \lambda\}$. Let $\mathcal{M}_0 \subset \mathcal{M}$ be the $*$-subalgebra of the σ_t-entire analytic elements [1] and $\mathcal{M}_+$ the subset of positive elements of $\mathcal{M}$, i.e., $\mathcal{M}_0 = \cap \mathcal{M}_\lambda$. Let ω be a vector state on $\mathcal{M}$ such that $\omega(A) := \langle \xi_0, A\xi_0 \rangle$, $A \in \mathcal{M}$.

Consider the semigroup $\{S_t\}_{t \geq 0}$ of everywhere defined linear maps on $\mathcal{M}$. A semigroup $\{S_t\}_{t \geq 0}$ is said to be *KMS-symmetric* if for all $t \in \mathbb{R}$ and for all $A, B \in \mathcal{M}_0$, one has

$$\omega(S_t(A)\sigma_{-i/2}(B)) = \omega(\sigma_{i/2}(A)S_t(B)). \tag{2.1}$$

A semigroup $\{S_t\}_{t \geq 0}$ is said to be *real* if $S_t(A^*) = (S_t(A))^*$ for all $A \in \mathcal{M}$ and for all $t \geq 0$, *positive preserving* if $S_t(A) \in \mathcal{M}_+$ for all $A \in \mathcal{M}_+$ and for all $t \geq 0$, *sub-Markovian* if $0 \leq S_t(A) \leq \mathbf{1}$ for all $0 \leq A \leq \mathbf{1}$ and for all $t \geq 0$. A semigroup $\{S_t\}_{t \geq 0}$ is said to be *Markovian* if S_t is positive preserving and $S_t(\mathbf{1}) = \mathbf{1}$ for all $t \geq 0$.

Next, we consider a complex valued, closed, positive sesquilinear form on some linear manifold of $\mathcal{H}$: $\mathcal{E}(\cdot, \cdot) : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$ satisfying $\mathcal{E}(\xi, \xi) \geq 0$ for all $\xi \in D(\mathcal{E})$, and also the associated quadratic form: $\mathcal{E}[\cdot] : D(\mathcal{E}) \to \mathbb{C}$, $\mathcal{E}[\xi] := \mathcal{E}(\xi, \xi)$. A quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is said to be *J-real* if $JD(\mathcal{E}) \subset D(\mathcal{E})$ and $\mathcal{E}[J\xi] = \mathcal{E}[\xi]$ for any $\xi \in D(\mathcal{E})$, equivalently, $\mathcal{E}(J\xi, J\eta) = \mathcal{E}(\eta, \xi)$ for all $\xi, \eta \in D(\mathcal{E})$.

A J-real, real-valued, densely defined quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is called *Markovian* (with respect to ξ_0) if

$$\xi \in D(\mathcal{E}) \cap \mathcal{H}^J \Rightarrow \xi_I \in D(\mathcal{E}), \quad \mathcal{E}[\xi_I] \leq \mathcal{E}[\xi].$$

A closed Markovian form is called a *Dirichlet form.*

For a positive closed form $(\mathcal{E}, D(\mathcal{E}))$, there exists a positive self-adjoint operator $(H, D(H))$ such that

$$\mathcal{E}(\xi, \eta) = \langle \xi, H\eta \rangle, \ \xi \in D(\mathcal{E}), \ \eta \in D(H),$$

and a strongly continuous, symmetric semigroup $\{T_t\}_{t\geq 0}, T_t = e^{-tH}$. Moreover, when $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form, H is called a Dirichlet operator. For the details, see Section 3.1 of [1].

Let the function $f_0 : \mathbb{R} \to \mathbb{R}$ be given by

$$f_0(t) = 2(e^{2\pi t} + e^{-2\pi t})^{-1}. \tag{2.2}$$

The function f_0 has an analytic extension, denoted by f_0 again, to the interior of $I_{1/4}$. It is not an admissible function, but plays important roles in [10].

Using a fixed element $x \in \mathcal{M}_{1/2}$ and the function f_0, the following (bounded) Dirichlet form $(\mathcal{E}, \mathcal{H})$ was constructed in [10]:

$$\begin{aligned} \mathcal{E}(\eta, \xi) \ = \ & \int \big\langle \big[\sigma_{t-i/4}(x) - j\,(\sigma_{t-i/4}(x^*))\big]\eta, \\ & \qquad \big[\sigma_{t-i/4}(x) - j\,(\sigma_{t-i/4}(x^*))\big]\xi \big\rangle f_0(t)dt \\ & + \int \big\langle \big[\sigma_{t-i/4}(x^*) - j\,(\sigma_{t-i/4}(x))\big]\eta, \\ & \qquad \big[\sigma_{t-i/4}(x^*) - j\,(\sigma_{t-i/4}(x))\big]\xi \big\rangle f_0(t)dt \end{aligned} \tag{2.3}$$

for any $\xi, \eta \in \mathcal{H}$.

From now on, for a fixed family $\{x_k\}_{k=1}^n \subset \mathcal{M}_{1/2}$ and f_0 in (2.2), we consider the form $(\mathcal{E}, \mathcal{H})$ given by

$$\mathcal{E}(\eta, \xi) = \sum_{k=1}^{n} \mathcal{E}_k(\eta, \xi) \tag{2.4}$$

for any $\xi, \eta \in \mathcal{H}$, where

$$\begin{aligned} \mathcal{E}_k(\eta, \xi) \ = \ & \int \big\langle \big[\sigma_{t-i/4}(x_k) - j\,(\sigma_{t-i/4}(x_k^*))\big]\eta, \\ & \qquad \big[\sigma_{t-i/4}(x_k) - j\,(\sigma_{t-i/4}(x_k^*))\big]\xi \big\rangle f_0(t)dt \\ & + \int \big\langle \big[\sigma_{t-i/4}(x_k^*) - j\,(\sigma_{t-i/4}(x_k))\big]\eta, \\ & \qquad \big[\sigma_{t-i/4}(x_k^*) - j\,(\sigma_{t-i/4}(x_k))\big]\xi \big\rangle f_0(t)dt. \end{aligned} \tag{2.5}$$

The above bounded Dirichlet form $(\mathcal{E}_k, \mathcal{H})$ is obtained from (2.3) changing x by x_k. The above form $(\mathcal{E}, \mathcal{H})$ in (2.4) is a Dirichlet form (Theorem 5.2 of [2]).

Denote by H a Dirichlet operator associated to the Dirichlet form $(\mathcal{E}, \mathcal{H})$ in (2.4) satisfying

$$\mathcal{E}(\eta, \xi) = \langle \eta, H\xi \rangle \tag{2.6}$$

and the Markovian semigroup $\{T_t\}_{t\geq 0}$, $T_t = e^{-tH}$. Using the symmetric embedding map i_0, we can define the operator G on $\mathcal{M}$ and the semigroup

$\{S_t\}_{t\geq 0}$, $S_t = e^{-tG}$ on $\mathcal{M}$ given by

$$\Delta^{1/4}G(A)\xi_0 = H\Delta^{1/4}A\xi_0,\ A \in \mathcal{M}.$$

Concretely, the bounded generator G is expressed as

$$\begin{aligned}(2.7) \qquad G(A) = \sum_{k=1}^{n}\Big(\int &[\sigma_{t+i/2}(x_k^*)\sigma_t(x_k)A + A\sigma_t(x_k)\sigma_{t-i/2}(x_k^*) \\ &- \sigma_{t+i/2}(x_k^*)A\sigma_t(x_k) - \sigma_t(x_k)A\sigma_{t-i/2}(x_k^*)]f_0(t)dt \\ + \int &[\sigma_{t+i/2}(x_k)\sigma_t(x_k^*)A + A\sigma_t(x_k^*)\sigma_{t-i/2}(x_k) \\ &- \sigma_{t+i/2}(x_k)A\sigma_t(x_k^*) - \sigma_t(x_k^*)A\sigma_{t-i/2}(x_k)]f_0(t)dt\Big)\end{aligned}$$

for any $A \in \mathcal{M}$. Then the semigroup $\{S_t\}_{t\geq 0}$ is a KMS-symmetric Markovian semigroup on $\mathcal{M}$ (Theorem of 2.1 of [10]).

Now we state the main results and will give their proofs in Section 3.

Proposition 2.1. *The generator G is the Lindblad type operator: for any $A \in \mathcal{M}$*

$$\begin{aligned}(2.8) \qquad G(A) \;=\; \sum_{k=1}^{n}\Big(&\frac{1}{2}(y_k^*y_kA - 2y_k^*Ay_k + Ay_k^*y_k) - i[Q_k, A] \\ &+\frac{1}{2}(z_kz_k^*A - 2z_kAz_k^* + Az_kz_k^*) - i[R_k, A]\Big),\end{aligned}$$

where $y_k = \sigma_{-i/4}(x_k)$, $z_k = \sigma_{i/4}(x_k)$ and $Q_k, R_k \in \mathcal{M}$ are some self-adjoint elements.

We introduce two subspaces of $\mathcal{M}$:

$\mathcal{F} = \{A \in \mathcal{M} \mid S_t(A) = A\ \forall t \geq 0\}$,

$\mathcal{N} = \{A \in \mathcal{M} \mid S_t(A^*A) = S_t(A^*)S_t(A),\ S_t(AA^*) = S_t(A)S_t(A^*),\ \forall t \geq 0\}$.

Notice that the Schwarz inequality $S_t(A^*A) \geq S_t(A^*)S_t(A)$ for all $t \geq 0$, $A \in \mathcal{M}$ is equivalent to $G(A^*A) \leq G(A^*)A + A^*G(A)$ for all $A \in \mathcal{M}$. See (3.36) of [8]. The semigroup $\{S_t\}_{t\geq 0}$ with bounded generator G in (2.8) satisfies Schwarz inequality (see (3.18)). In fact, by (2.8) and Corollary 30.13 of [12], the semigroup $\{S_t\}_{t\geq 0}$ satisfies the completely positivity. Since ω associated to ξ_0 is a faithful normal stationary state for the semigroup $\{S_t\}_{t\geq 0}$ satisfying Schwarz inequality, $\mathcal{F} \subset \mathcal{N}$, and $\mathcal{F}$ and $\mathcal{N}$ are von Neumann subalgebras of $\mathcal{M}$ [4, 7].

Theorem 2.2. (a) $\mathcal{F} = \{A \in \mathcal{M} | A\sigma_t(x_k) = \sigma_t(x_k)A,\ A\sigma_t(x_k^*) = \sigma_t(x_k^*)A \forall t \in \mathbb{R},\ k = 1, \ldots, n\}$.

(b) *$\mathcal{N}$ is the commutant of $\{\sigma_{\pm i/4}(x_k), \sigma_{\pm i/4}(x_k^*) | k = 1, \ldots, n\}$.*

Corollary 2.3. *Assume that $\{x_k\}_{k=1}^n \subset \mathcal{M}_{1/2}$ generates $\mathcal{M}$ and $\mathcal{M}$ is a factor, i.e., $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}\mathbf{1}$. Then $\mathcal{F} = \mathcal{N} = \mathbb{C}\mathbf{1}$. Moreover, the semigroup $\{S_t\}_{t\geq 0}$ is* *ergodic in the sense that* $\mathcal{F} = \mathbb{C}\mathbf{1}$, *and mixing in the sense that for any normal state* ρ *on* $\mathcal{M}$, $\rho(S_t(A))$ *converges to* $\omega(A)$ *as* $t \to \infty$ [4].

Proof. Notice that $\mathcal{M}_0$ is dense in $\mathcal{M}$. Let $A \in \mathcal{N} \cap \mathcal{M}_0$. A satisfies $y_k A = Ay_k$, $z_k A = Az_k$, $y_k^* A = Ay_k^*$ and $z_k^* A = Az_k^*$ for all k. Thus $\sigma_{\pm i/4}(A)$ belongs to the commutant of $\{x_k, x_k^* \,|\, k = 1, 2, \ldots, n\}$, and A is a multiplier of the identity. This means $\mathcal{N} = \mathbb{C}\mathbf{1}$ and $\mathcal{F} = \mathcal{N} = \mathbb{C}\mathbf{1}$.

Since $\mathcal{F} = \mathbb{C}\mathbf{1}$, the semigroup is ergodic. Notice that ω is a faithful normal stationary state for the semigroup $\{S_t\}_{t\geq 0}$. By Theorem 4.3 of [4] and the fact that $\mathcal{N} = \mathcal{F} = \mathbb{C}\mathbf{1}$, for any normal state ρ on $\mathcal{M}$, $\rho(S_t(A))$ converges to $\omega(A)$ as $t \to \infty$. □

Remark 2.4. (a) We can extend the form $(\mathcal{E}, \mathcal{H})$ in (2.4). One may choose $\{x_k\}_{k=1}^{\infty} \in \mathcal{M}_{1/2}$. Let $(\mathcal{E}, D(\mathcal{E}))$ be the sesquilinear form defined by

$$D(\mathcal{E}) = \{\xi \in \mathcal{H} \mid \sum_{k=1}^{\infty} \mathcal{E}_k(\xi, \xi) < \infty\},$$

$$\mathcal{E}(\eta, \xi) = \sum_{k=1}^{\infty} \mathcal{E}_k(\eta, \xi), \quad \eta, \xi \in D(\mathcal{E}),$$

where $\mathcal{E}_k$ is defined as in (2.5). If $D(\mathcal{E})$ is densely defined in $\mathcal{H}$, then $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form.

(b) In fact, if $\{x_k\}_{k=1}^{n} \subset \mathcal{M}_{1/2}$ generates $\mathcal{M}$ and $\mathcal{M}$ is a factor, then $\{\xi \in \mathcal{H} \mid T_t\xi = \xi \; \forall t \geq 0\} = \mathbb{C}\xi_0$ and the semigroup $\{T_t\}_{t\geq 0}$ is ergodic in the sense that zero is a simple eigenvalue of the generator H of $\{T_t\}_{t\geq 0}$ with eigenvector ξ_0 (Corollary 2.1 of [11]).

3. Proof of Theorem 2.2

Let us introduce linear maps on $\mathcal{L}(\mathcal{H})$ the space of the bounded operators on $\mathcal{H}$. For any $\lambda > 0$, denote by $\mathcal{L}_\lambda(\mathcal{H})$ the dense subset of $\mathcal{L}(\mathcal{H})$ consisting of every σ_t-analytic elements of $\mathcal{L}(\mathcal{H})$ on the strip $I_\lambda = \{z \mid |\mathrm{Im} z| \leq \lambda\}$. Let $D_{1/4}$ and $D_{-1/4}$ be the linear maps on $\mathcal{L}(\mathcal{H})$ defined by

$$\begin{aligned} D(D_{1/4}) &= \mathcal{L}_{1/4}(\mathcal{H}), \\ D_{1/4}(A) &= \sigma_{-i/4}(A), \quad A \in \mathcal{L}_{1/4}(\mathcal{H}), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} D(D_{-1/4}) &= \mathcal{L}_{1/4}(\mathcal{H}), \\ D_{-1/4}(A) &= \sigma_{i/4}(A), \quad A \in \mathcal{L}_{1/4}(\mathcal{H}). \end{aligned} \tag{3.2}$$

Define two maps T and S on $\mathcal{L}_{1/4}(\mathcal{H})$ by

$$\begin{aligned} T &= D_{1/4} + D_{-1/4}, \\ S &= D_{1/4} - D_{-1/4}. \end{aligned} \tag{3.3}$$

Let I_0 be the linear map defined by

$$D(I_0) = \mathcal{L}(\mathcal{H}), \qquad (3.4)$$
$$I_0(A) = \int \sigma_t(A) f_0(t)dt, \ A \in \mathcal{L}(\mathcal{H}),$$

where f_0 is the function given in (2.2). The relations

$$T(I_0(A)) = I_0(T(A)) = A \qquad (3.5)$$

holds for any $A \in \mathcal{L}_{1/4}$. See Lemma 3.1 of [10].

In [10], Park gave the sufficient condition so that the Lindblad type generator [12] can be expressed as the type of the Dirichlet operator G in (2.7). Using the converse course of Proposition 2.1 and Theorem 2.2 in [10], we show that the generator G is written as the Lindblad type.

Proof of Proposition 2.1'. Put $y_k = \sigma_{-i/4}(x_k)$ and $z_k = \sigma_{i/2}(y_k) = \sigma_{i/4}(x_k)$. Using the map I_0, $J^2 = \mathbf{1}$ and $j(B) \in \mathcal{M}'$ for any $B \in \mathcal{M}$, the Dirichlet operator H in (2.6) can be expressed as

$$H = \sum_{k=1}^{n} H_k, \qquad (3.6)$$

where

$$\begin{aligned} H_k &= I_0([y_k - j(z_k^*)]^*[y_k - j(z_k^*)]) \\ &\quad + I_0([z_k^* - j(y_k)]^*[z_k^* - j(y_k)]) \\ &= I_0(y_k^* y_k + j(z_k z_k^*) - y_k^* j(z_k^*) - y_k j(z_k)) \\ &\quad + I_0(z_k z_k^* + j(y_k^* y_k) - z_k j(y_k) - z_k^* j(y_k^*)). \end{aligned} \qquad (3.7)$$

Using the symmetric embedding map, we have

$$\begin{aligned} G(A)\xi_0 &= D_{-1/4}(H)A\xi_0 \\ &= \sum_{k=1}^{n} D_{-1/4}(H_k)A\xi_0 \\ &= \sum_{k=1}^{n} \Big(I_0(D_{-1/4}[y_k^* y_k + j(z_k z_k^*) - y_k^* j(z_k^*) - y_k j(z_k)]) \\ &\quad + I_0(D_{-1/4}[z_k z_k^* + j(y_k^* y_k) - z_k j(y_k) - z_k^* j(y_k^*)]) \Big) A\xi_0. \end{aligned} \qquad (3.8)$$

It follows from $z_k = \sigma_{i/2}(y_k)$ and $\sigma_{i/2}(j(z_k)) = j(\sigma_{-i/2}(z_k)) = j(y_k)$ that

$$\begin{aligned} D_{-1/4}(y_k j(z_k)) &= D_{1/4}(\sigma_{i/2}(y_k)\sigma_{i/2}(j(z_k))) \\ &= D_{1/4}(z_k j(y_k)), \\ D_{-1/4}(z_k^* j(y_k^*)) &= D_{1/4}(y_k^* j(z_k^*)). \end{aligned} \qquad (3.9)$$

By (3.9) and (3.5), we have

$$I_0(D_{-1/4}[y_k^* j(z_k^*) + y_k j(z_k) + z_k j(y_k) + z_k^* j(y_k^*)]) \qquad (3.10)$$

$$
\begin{aligned}
&= I_0(D_{-1/4}(y_k^* j(z_k^*) + D_{1/4}(z_k j(y_k)) \\
&\quad + D_{-1/4}(z_k j(y_k)) + D_{1/4}(y_k^* j(z_k^*))) \\
&= I_0(T[y_k^* j(z_k^*) + z_k j(y_k)]) \\
&= y_k^* j(z_k^*) + z_k j(y_k).
\end{aligned}
$$

Since $j(B) = JBJ \in \mathcal{M}'$ and $JB\xi_0 = \Delta^{1/2} B^* \xi_0$ for any $B \in \mathcal{M}$, we have

$$
\begin{aligned}
y_k^* j(z_k^*) A\xi_0 &= y_k^* A y_k \xi_0, \\
z_k j(y_k) A\xi_0 &= z_k A z_k^* \xi_0.
\end{aligned} \tag{3.11}
$$

Combining (3.10) and (3.11), we have

$$
\begin{aligned}
&I_0(D_{-1/4}[y_k^* j(z_k^*) + y_k j(z_k) + z_k j(y_k) + z_k^* j(y_k^*)]) A\xi_0 \\
&= (y_k^* A y_k + z_k A z_k^*)\xi_0
\end{aligned} \tag{3.12}
$$

for any $A \in \mathcal{M}$.

Notice that

$$
\begin{aligned}
I_0(D_{-1/4}(y_k^* y_k)) &= \frac{1}{2}[I_0(T(y_k^* y_k) - S(y_k^* y_k))] \\
&= \frac{1}{2}[y_k^* y_k - I_0(S(y_k^* y_k))], \\
I_0(D_{-1/4}(j(y_k^* y_k))) &= I_0(D_{1/4}(j(\sigma_{-i/2}(y_k^* y_k)))) \\
&= \frac{1}{2}[I_0(T(j(\sigma_{-i/2}(y_k^* y_k))) + S(j(\sigma_{-i/2}(y_k^* y_k)))] \\
&= \frac{1}{2}[j(\sigma_{-i/2}(y_k^* y_k)) + I_0(S(j(\sigma_{-i/2}(y_k^* y_k)))]
\end{aligned} \tag{3.13}
$$

and for any $A \in \mathcal{M}$

$$
\begin{aligned}
S(j(\sigma_{-i/2}(y_k^* y_k)))A\xi_0 &= AS(j(\sigma_{-i/2}(y_k^* y_k)))\xi_0 \\
&= A[\Delta^{1/4} j(\sigma_{-i/2}(y_k^* y_k))\xi_0 - \Delta^{-1/4} j(\sigma_{-i/2}(y_k^* y_k))\xi_0] \\
&= A[\Delta^{1/4} y_k^* y_k \xi_0 - \Delta^{-1/4} y_k^* y_k \xi_0] \\
&= AS(y_k^* y_k)\xi_0.
\end{aligned} \tag{3.14}
$$

Hence we have

$$
I_0(D_{-1/4}(y_k^* y_k + j(y_k^* y_k)))A\xi_0 = \frac{1}{2}(y_k^* y_k A + A y_k^* y_k - [I_0(S(y_k^* y_k)), A])\xi_0. \tag{3.15}
$$

Replacing y_k^* by z_k in the above, we get

$$
I_0(D_{-1/4}(z_k z_k^* + j(z_k z_k^*)))A\xi_0 = \frac{1}{2}(z_k z_k^* A + A z_k z_k^* - [I_0(S(z_k z_k^*)), A])\xi_0. \tag{3.16}
$$

Substituting (3.12), (3.15) and (3.16) into (3.8), we have

$$
G(A)\xi_0 = \sum_{k=1}^{n} \Big(\frac{1}{2}(y_k^* y_k A - 2 y_k^* A y_k + A y_k^* y_k) - i[Q_k, A] \tag{3.17}
$$

$$+\frac{1}{2}(z_k z_k^* A - 2z_k A z_k^* + A z_k z_k^*) - i[R_k, A]\Big)\xi_0,$$

where $Q_k = -i/2I_0(S(y_k^* y_k))$ and $R_k = -i/2I_0(S(z_k z_k^*))$. Note that $I_0(S(A))^* = -I_0(S(A^*))$ for any $A \in \mathcal{M}_{1/4}$. Thus $Q_k^* = Q_k$, $R_k^* = R_k$ for all k. Since ξ_0 is a cyclic vector for $\mathcal{M}$, the operator G holds (2.8). □

Proof of Theorem 2.2. (a) It follows from the symmetric embedding mapping that

$$\begin{aligned}\mathcal{F} &= \{A \in \mathcal{M} | G(A) = 0\}\\ &= \{A \in \mathcal{M} | H\Delta^{1/4} A\xi_0 = 0\}\\ &= \{A \in \mathcal{M} | \mathcal{E}(\Delta^{1/4} A\xi_0, \Delta^{1/4} A\xi_0) = 0\}.\end{aligned}$$

Notice that

$$\begin{aligned}&[\sigma_{t-i/4}(\tilde{x_k}) - j(\sigma_{t-i/4}(\tilde{x_k}^*))]\Delta^{1/4} A\xi_0\\ =\ & \Delta^{1/4}\sigma_t(\tilde{x_k})A\xi_0 - \sigma_{-i/4}(A) j(\sigma_{t-i/4}(\tilde{x_k}^*))\xi_0\\ =\ & \Delta^{1/4}[\sigma_t(\tilde{x_k}), A]\xi_0,\end{aligned}$$

where $\tilde{x_k}$ is either x_k or x_k^*.

By (2.4), (2.5) and the above, $A \in \mathcal{F}$ equals to $\Delta^{1/4}[\sigma_t(\tilde{x_k}), A]\xi_0 = 0$. Since $\sigma_t(\mathcal{M}) = \mathcal{M}$ and ξ_0 is a separating vector for $\mathcal{M}$, $A \in \mathcal{F}$ if and only if $[\sigma_t(\tilde{x_k}), A] = 0$ for all k and all $t \in \mathbb{R}$. The part (a) is proved.

(b) Define

$$\Gamma(A, B) = G(AB) - G(A)B - AG(B)$$

for all $A, B \in \mathcal{M}$. Notice that $\Gamma(A^*, A) \le 0$ for all $A \in \mathcal{M}$, and $A \in \mathcal{N}$ if and only if $\Gamma(A^*, A) = 0 = \Gamma(A, A^*)$. See also the proof of Proposition 6.1 of [6].

In fact, a straightforward computation shows that

$$\begin{aligned}\text{(3.18)}\quad \Gamma(A^*, A) &= G(A^*A) - G(A^*)A - A^*G(A)\\ &= \sum_{k=1}^{n}\Big(\frac{1}{2}(y_k^* y_k A^* A - 2y_k^* A^* A y_k + A^* A y_k^* y_k) - i[Q_k, A^* A]\\ &\quad + \frac{1}{2}(z_k z_k^* A^* A - 2z_k A^* A z_k^* + A^* A z_k z_k^*) - i[R_k, A^* A]\Big)\\ &\quad - \sum_{k=1}^{n}\Big(\frac{1}{2}(y_k^* y_k A^* - 2y_k^* A^* y_k + A^* y_k^* y_k) - i[Q_k, A^*]\\ &\quad + \frac{1}{2}(z_k z_k^* A^* - 2z_k A^* z_k^* + A^* z_k z_k^*) - i[R_k, A^*]\Big)A\\ &\quad - A^* \sum_{k=1}^{n}\Big(\frac{1}{2}(y_k^* y_k A - 2y_k^* A y_k + A y_k^* y_k) - i[Q_k, A]\\ &\quad + \frac{1}{2}(z_k z_k^* A - 2z_k A z_k^* + A z_k z_k^*) - i[R_k, A]\Big)\end{aligned}$$

$$= -\sum_{k=1}^{n} \big(y_k^* A^* A y_k - y_k^* A^* y_k A + A^* y_k^* y_k A - A^* y_k^* A y_k + z_k A^* A z_k^* - z_k A^* z_k^* A + A^* z_k z_k^* A - A^* z_k A z_k^*\big)$$

$$= -\sum_{k=1}^{n} \big([A, y_k]^*[A, y_k] + [A, z_k^*]^*[A, z_k^*]\big)$$

for $A \in \mathcal{M}$. Here we have used $[A, BC] = [A, B]C + B[A, C]$ for $A, B, C \in \mathcal{M}$.

Similarly,

$$\Gamma(A, A^*) = -\sum_{k=1}^{n} \big([y_k^*, A][y_k^*, A]^* + [z_k, A][z_k, A]^*\big).$$

$\Gamma(A^*, A) = 0 = \Gamma(A, A^*)$ if and only if for each $[A, y_k]^*[A, y_k] + [A, z_k^*]^*[A, z_k^*] = 0$ and $[y_k^*, A][y_k^*, A]^* + [z_k, A][z_k, A]^* = 0$, which means that $A \in \mathcal{N}$ if and only if $Ay_k = y_k A$, $Az_k^* = z_k^* A$, $Ay_k^* = y_k^* A$, $Az_k = z_k A$. The part (b) is proved. □

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