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EXTENDED CESÀRO OPERATORS FROM $F(p,q,s)$ SPACES TO BLOCH-TYPE SPACES IN THE UNIT BALL

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ABSTRACT. In this paper, we characterize the boundedness and compactness of the extended Cesàro operators from general function spaces $F(p,q,s)$ to Bloch-type spaces $\mathcal{B}_\mu$, where μ is normal function on $[0,1)$.

1. Introduction

Let $\mathbf{B}$ be the open unit ball of $\mathbf{C}^n$, and let $\partial\mathbf{B}$ be its boundary. $H(\mathbf{B})$ denotes the family of all holomorphic functions on $\mathbf{B}$. For $a \in \mathbf{B}$, let $h(z,a) = \log \frac{1}{|\varphi_a(z)|}$ be the Green's function for $\mathbf{B}$ with logarithmic singularity at a, where φ_a is the Möbius transformation of $\mathbf{B}$, satisfying $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a = \varphi_a^{-1}$. For $0 < p, s < \infty$, $-n-1 < q < \infty$, we say $f \in F(p,q,s)$ provided that $f \in H(\mathbf{B})$ and

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{a\in\mathbf{B}} \int_{\mathbf{B}} |\Re f(z)|^p (1-|z|^2)^q h^s(z,a) dv(z) < \infty. \tag{1.1}$$

In one variable, the spaces $F(p,q,s)$ were first introduced by Zhao [12]. We call $F(p,q,s)$ general function space because we can get many function spaces, such as Hardy space, Bergman space, Q_p space, BMOA space, Besove space and α-Bloch space, if we take some special parameters of p, q and s, see [7]. Notice that $F(p,q,s)$ is the space of constant functions if $q + s \le -1$.

A positive continuous function μ on $[0,1)$ is called normal if there are three constants $0 \le \delta < 1$ and $0 < a < b$ such that

$$(P_1) \qquad \frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\mu(r)}{(1-r)^a} = 0;$$

$$(P_2) \qquad \frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

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We extend it to $\mathbf{B}$ by $\mu(z) = \mu(|z|)$. A function $f \in H(\mathbf{B})$ is said to belong to the Bloch-type space $\mathcal{B}_\mu$ if

$$\|f\|_\mu = \sup_{z\in\mathbf{B}} \mu(z)|\nabla f(z)| < \infty,$$

and it is said to belong to the little Bloch-type space $\mathcal{B}_{\mu,0}$ if

$$\lim_{|z|\to 1} \mu(z)|\nabla f(z)| = 0.$$

Here $\nabla f(z) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ is the complex gradient of f. It is clear that both $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$ are Banach spaces with the norm $\|f\|_{\mathcal{B}_\mu} = |f(0)| + \|f\|_\mu$, and $\mathcal{B}_{\mu,0}$ is a closed subspace of $\mathcal{B}_\mu$. When $\mu(r) = 1 - r^2$, the induced space $\mathcal{B}_\mu$ is the classic Bloch space.

In the unit ball, given $g \in H(\mathbf{B})$, we define the extended Cesàro operator to be

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \quad z \in \mathbf{B},$$

where $\Re f(z)$ is the radial derivative of f. Hu got the characterization on g for which the operator T_g is bounded or compact on the Bergman space in [2]. Stević [8] considered the boundedness of T_g on α-Bloch space. Xiao [10] obtained the property on g such that T_g is bounded or compact on α-Bloch space and little α-Bloch space. Recently, Li discussed the boundedness of T_g from $F(p, q, s)$ to α-Bloch spaces for some restricted p, q, s and α in [3]. The purpose of this work is to obtain the boundedness and compactness of T_g from $F(p, q, s)$ to $\mathcal{B}_\mu$ (or $\mathcal{B}_{\mu,0}$) for all $0 < p, s < \infty$, $-n - 1 < q < \infty$. Our work will generalize [3] and [8].

In what follows we always suppose $0 < p, s < \infty$, $-n-1 < q < \infty$, $q+s > -1$. C will stand for positive constants whose value may change from line to line but not depend on the functions in $H(\mathbf{B})$. The expression $A \simeq B$ means $C^{-1}A \le B \le CA$.

2. Some preliminary results

Lemma 2.1 ([11]). *Suppose $f \in F(p, q, s)$. Then $f \in \mathcal{B}_{(1-r^2)^{\frac{n+1+q}{p}}}$ and*

$$\|f\|_{\mathcal{B}_{(1-r^2)^{\frac{n+1+q}{p}}}} \le C\|f\|_{F(p,q,s)}.$$

Lemma 2.2 ([9]). *Let μ be normal and $f \in H(\mathbf{B})$. Then*

(i) *$f \in \mathcal{B}_\mu$ if and only if $\sup_{z\in\mathbf{B}} \mu(z)|\Re f(z)| < \infty$. Moreover,*

$$\|f\|_{\mathcal{B}_\mu} \simeq |f(0)| + \sup_{z\in\mathbf{B}} \mu(z)|\Re f(z)|.$$

(ii) *$f \in \mathcal{B}_{\mu,0}$ if and only if $\lim_{|z|\to 1} \mu(z)|\Re f(z)| = 0$.*

Lemma 2.3 ([8]). *For $0<\alpha<\infty$, if $f\in\mathcal{B}_{(1-r^2)^\alpha}$, then for any $z\in\mathbf{B}$,*

$$|f(z)|\le\begin{cases} C\|f\|_{\mathcal{B}_{(1-r^2)^\alpha}}, & 0<\alpha<1;\\ C\|f\|_{\mathcal{B}_{(1-r^2)^\alpha}}\log\frac{2}{1-|z|^2}, & \alpha=1;\\ C(1-|z|^2)^{1-\alpha}\|f\|_{\mathcal{B}_{(1-r^2)^\alpha}}, & \alpha>1.\end{cases}$$

Lemma 2.4 ([5]). *For $s>-1$, $r,t\ge 0$ and $r+t-s>n+1$, then*

$$\int_{\mathbf{B}}\frac{(1-|z|^2)^s}{|1-<a,z>|^r|1-<w,z>|^t}dv(z)$$
$$\le\begin{cases}\frac{C}{|1-<w,a>|^{r+t-s-n-1}}, & \text{if } r-s,t-s<n+1;\\ \frac{C}{(1-|a|^2)^{r-s-n-1}|1-<w,a>|^t}, & \text{if } t-s<n+1<r-s.\end{cases}$$

Lemma 2.5. *Let $p=n+1+q$. Suppose that for each $w\in\mathbf{B}$, z-variable functions g_w satisfy $|g_w(z)|\le\frac{C}{|1-<z,w>|}$, then*

$$\int_{\mathbf{B}}|g_w(z)|^p(1-|z|^2)^q h^s(z,a)dv(z)\le C.$$

Proof. If $0<s<n+1+q$, Lemma 2.4 implies

$$\int_{\mathbf{B}}\frac{(1-|a|^2)^s(1-|z|^2)^{q+s}}{|1-<z,w>|^{n+1+q}|1-<a,z>|^{2s}}dv(z)\le\frac{C(1-|a|^2)^s}{|1-<w,a>|^s}\le C. \tag{2.1}$$

If $s>n+1+q$, by Lemma 2.4 we have

$$\begin{aligned}&\int_{\mathbf{B}}\frac{(1-|a|^2)^s(1-|z|^2)^{q+s}}{|1-<z,w>|^{n+1+q}|1-<a,z>|^{2s}}dv(z)\\ &\le\frac{C(1-|a|^2)^s}{(1-|a|^2)^{s-n-1-q}|1-<w,a>|^{n+1+q}}\le C.\end{aligned} \tag{2.2}$$

If $s=n+1+q$, choose $s_1=\frac{n}{2}$, $s_2=2s$, $x=\frac{s_2-s_1}{s_2-s}$. By the fact $q+s>-1$, we know

$$0<s_1<n+1+q<s_2,\ q+s_2>q+s_1>-1 \text{ and } x>1.$$

Take $t_1=\frac{q+s_1}{x}$, $t_2=\frac{s_1}{x}$, $\frac{1}{x}+\frac{1}{x'}=1$. By (2.1), (2.2) and Hölder inequality,

$$\begin{aligned}&\int_{\mathbf{B}}\frac{(1-|a|^2)^s(1-|z|^2)^{q+s}}{|1-<z,w>|^{n+1+q}|1-<a,z>|^{2s}}dv(z)\\ &=\int_{\mathbf{B}}\frac{(1-|a|^2)^{t_2+(s-t_2)}(1-|z|^2)^{t_1+(q+s-t_1)}}{|1-<z,w>|^{\frac{n+1+q}{x}+\frac{n+1+q}{x'}}|1-<a,z>|^{2t_2+2(s-t_2)}}dv(z)\\ &\le\left\{\int_{\mathbf{B}}\frac{(1-|a|^2)^{s_1}(1-|z|^2)^{q+s_1}}{|1-<z,w>|^{n+1+q}|1-<a,z>|^{2s_1}}dv(z)\right\}^{\frac{1}{x}}\\ &\times\left\{\int_{\mathbf{B}}\frac{(1-|a|^2)^{s_2}(1-|z|^2)^{q+s_2}}{|1-<z,w>|^{n+1+q}|1-<a,z>|^{2s_2}}dv(z)\right\}^{\frac{1}{x'}}\\ &\le C\frac{(1-|a|^2)^{s_1}}{|1-<w,a>|^{s_1}}\cdot\frac{(1-|a|^2)^{s_2}}{(1-|a|^2)^{s_2-n-1-q}|1-<w,a>|^{n+1+q}}\le C.\end{aligned} \tag{2.3}$$

Given any $a \in \mathbf{B}$, let $x = 1 - |\varphi_a(z)|^2$, we have

$$h(z,a) = -\frac{1}{2}\log(1-x) \leq \frac{x}{2}\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right] = 2x \quad \text{for } \frac{1}{2} < |\varphi_a(z)| < 1.$$

Notice that

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1 - \langle a, z \rangle|^2}.$$

Hence, (2.1), (2.2), and (2.3) yield, for $p = n + 1 + q$,

$$\begin{aligned}
&\int_{\frac{1}{2}<|\varphi_a(z)|<1} |g_w(z)|^p (1-|z|^2)^q h^s(z,a)\,dv(z) \\
(2.4)\qquad &\leq C \int_{\frac{1}{2}<|\varphi_a(z)|<1} \frac{(1-|a|^2)^s (1-|z|^2)^{q+s}}{|1-\langle z,w\rangle|^{n+1+q} |1-\langle a,z\rangle|^{2s}}\,dv(z) \\
&\leq C \int_{\mathbf{B}} \frac{(1-|a|^2)^s (1-|z|^2)^{q+s}}{|1-\langle z,w\rangle|^{n+1+q} |1-\langle a,z\rangle|^{2s}}\,dv(z) \leq C.
\end{aligned}$$

At the same time,

$$\begin{aligned}
&\int_{|\varphi_a(z)|\leq\frac{1}{2}} |g_w(z)|^p (1-|z|^2)^q h^s(z,a)\,dv(z) \\
&\leq C \int_{|\varphi_a(z)|\leq\frac{1}{2}} \frac{(1-|z|^2)^q}{|1-\langle z,w\rangle|^p} h^s(z,a)\,dv(z) \\
&= C \int_{|u|\leq\frac{1}{2}} \frac{(1-|\varphi_a(u)|^2)^q}{|1-\langle \varphi_a(u),w\rangle|^{n+1+q}} \cdot \frac{(1-|a|^2)^{n+1}}{|1-\langle u,a\rangle|^{2n+2}} \cdot \log^s \frac{1}{|u|}\,dv(u) \\
&\leq C \int_{|u|\leq\frac{1}{2}} \frac{(1-|a|^2)^{n+1+q}(1-|u|^2)^q}{(1-|\varphi_a(u)|^2)^{n+1+q} |1-\langle u,a\rangle|^{2n+2+2q}} \log^s \frac{1}{|u|}\,dv(u) \\
&= C \int_{|u|\leq\frac{1}{2}} \frac{1}{(1-|u|^2)^{n+1}} \log^s \frac{1}{|u|}\,dv(u) \\
&\leq C \int_{\mathbf{B}} \log^s \frac{1}{|u|}\,dv(u) = C \int_0^1 2nr^{2n-1} \log^s \frac{1}{r}\,dr \int_{\partial\mathbf{B}} d\sigma(\xi) \leq C,
\end{aligned}$$

where $u = \varphi_a(z)$. This, together with (2.4), means

$$\begin{aligned}
&\int_{\mathbf{B}} |g_w(z)|^p (1-|z|^2)^q h^s(z,a)\,dv(z) \\
&= \left(\int_{\frac{1}{2}<|\varphi_a(z)|<1} + \int_{|\varphi_a(z)|\leq\frac{1}{2}}\right) |g_w(z)|^p (1-|z|^2)^q h^s(z,a)\,dv(z) \leq C.
\end{aligned}$$

The proof is completed. □

Lemma 2.6. *Let μ be normal and $g \in H(\mathbf{B})$. Suppose $T_g : F(p,q,s) \to \mathcal{B}_\mu$ is bounded. Then $T_g : F(p,q,s) \to \mathcal{B}_\mu$ is compact if and only if for any bounded* *sequence* $\{f_j\} \subseteq F(p,q,s)$ *which converges to 0 uniformly on any compact subset of* $\mathbf{B}$, *we have* $\lim_{j\to\infty} \|T_g f_j\|_{\mathcal{B}_\mu} = 0$.

Proof. It can be proved by Lemma 2.1, Lemma 2.3 and the Montel Theorem. The details are omitted here. □

To characterize the compactness of T_g from $F(p,q,s)$ to $\mathcal{B}_{\mu,0}$, we give the following lemma, whose proof is similar to that of Lemma 1 in [4].

Lemma 2.7. *Let* μ *be a normal function. A closed subset* E *in* $\mathcal{B}_{\mu,0}$ *is compact if and only if it is bounded and satisfying*

$$\lim_{|z|\to 1} \sup_{f\in E} \mu(z)|\Re f(z)| = 0.$$

3. Main results

Theorem 3.1. *Let* μ *be normal,* $g \in H(\mathbf{B})$, $n+1+q \geq p$. *Then* $T_g : F(p,q,s) \to \mathcal{B}_\mu$ *is bounded if and only if*

(i) *for* $n+1+q > p$,

$$\sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} < \infty. \tag{3.1}$$

In this case,

$$\|T_g\| \simeq \sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}.$$

(ii) *for* $n+1+q = p$,

$$\sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)| \log\frac{2}{1-|z|^2} < \infty. \tag{3.2}$$

In this case,

$$\|T_g\| \simeq \sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)| \log\frac{2}{1-|z|^2}.$$

Proof. (i) First, for $f, g \in H(\mathbf{B})$, direct calculation shows

$$\Re(T_g f)(z) = f(z)\Re g(z).$$

Suppose $n+1+q > p$, $f \in F(p,q,s)$, by Lemmas 2.1, 2.2 and 2.3, we obtain

$$\begin{aligned} \|T_g f\|_{\mathcal{B}_\mu} &\simeq |T_g f(0)| + \sup_{z\in\mathbf{B}} \mu(z)|f(z)||\Re g(z)| \\ &\leq C\|f\|_{\mathcal{B}_{(1-r^2)^{\frac{n+1+q}{p}}}} \sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} \\ &\leq C\|f\|_{F(p,q,s)} \sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}. \end{aligned} \tag{3.3}$$

Hence, (3.1) implies that $T_g : F(p,q,s) \to \mathcal{B}_\mu$ is bounded.

Conversely, suppose $T_g : F(p,q,s) \to \mathcal{B}_\mu$ is bounded. For any $w \in \mathbf{B}$, set

$$f_w(z) = \frac{1-|w|^2}{(1-\langle z,w\rangle)^{\frac{n+1+q}{p}}}, \quad z\in\mathbf{B}.$$

Then $\|f_w\|_{F(p,q,s)} \leq C$ by [11]. Hence,

$$\mu(w)|\Re g(w)|(1-|w|^2)^{1-\frac{n+1+q}{p}} = \mu(w)|\Re g(w)||f_w(w)| \leq C\|T_g f_w\|_{\mathcal{B}_\mu} \leq C\|T_g\|.$$

Therefore,

$$\sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} \leq C\|T_g\| < \infty. \tag{3.4}$$

Moreover, (3.3) and (3.4) yield

$$\|T_g\| \simeq \sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}.$$

(ii) If $n+1+q=p$, by Lemma 2.1, $F(p,q,s) \subseteq \mathcal{B}_{1-r^2}$. For $f \in F(p,q,s)$, combining Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned}\|T_g f\|_{\mathcal{B}_\mu} &\simeq |T_g f(0)| + \sup_{z\in\mathbf{B}} \mu(z)|f(z)||\Re g(z)| \\ &\leq C\|f\|_{\mathcal{B}_{1-r^2}} \sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)| \log\frac{2}{1-|z|^2} \\ &\leq C\|f\|_{F(p,q,s)} \sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)| \log\frac{2}{1-|z|^2}.\end{aligned} \tag{3.5}$$

Thus, (3.2) yields that $T_g : F(p,q,s) \to \mathcal{B}_\mu$ is bounded.

Conversely, suppose $T_g : F(p,q,s) \to \mathcal{B}_\mu$ is bounded. Given any $w \in \mathbf{B}$, set

$$f_w(z) = \log\frac{2}{1-<z,w>}, \quad z \in \mathbf{B}.$$

Then $|\Re f_w(z)| \leq \frac{C}{|1-<z,w>|}$, by Lemma 2.5

$$\|f_w\|_{F(p,q,s)} \leq C.$$

By the boundedness of T_g, we have

$$\mu(w)|\Re g(w)| \log\frac{2}{1-|w|^2} = \mu(w)|\Re g(w)||f_w(w)| \leq C\|T_g f_w\|_{\mathcal{B}_\mu} \leq C\|T_g\|.$$

This means

$$\sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)| \log\frac{2}{1-|z|^2} \leq C\|T_g\| < \infty. \tag{3.6}$$

Furthermore, (3.5) and (3.6) imply

$$\|T_g\| \simeq \sup_{z\in\mathbf{B}} \mu(z)|\Re g(z)| \log\frac{2}{1-|z|^2}.$$

The proof is completed. □

Remark. Set $\mu(z) = (1-|z|^2)^\alpha$, when $n+1+q \leq p\alpha$ in (i) and $\alpha \geq 1$, $s > n$ in (ii), respectively, Theorem 3.1 is just the main results in [3], which are Theorem 2.4 and Theorem 2.10.

Theorem 3.2. *Let μ be normal, $g \in H(\mathbf{B})$, $n+1+q \geq p$. Then the following statements are equivalent:*

(A) T_g*:* $F(p,q,s) \to \mathcal{B}_\mu$ *is compact;*

(B) T_g*:* $F(p,q,s) \to \mathcal{B}_{\mu,0}$ *is compact;*

(C) (i) *for* $n+1+q > p$,

$$\lim_{|z|\to 1} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} = 0; \tag{3.7}$$

(ii) *for* $n+1+q = p$,

$$\lim_{|z|\to 1} \mu(z)|\Re g(z)| \log \frac{2}{1-|z|^2} = 0. \tag{3.8}$$

Proof. The implication (B)⇒(A) is trivial.

(C)⇒(B) Suppose (3.7) holds for the case of $n+1+q > p$. For $f \in F(p,q,s)$, by Lemmas 2.1 and 2.3, we obtain

$$\begin{aligned} \mu(z)|f(z)||\Re g(z)| &\leq C\|f\|_{\mathcal{B}_{(1-r^2)^{\frac{n+1+q}{p}}}} \mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}} \\ &\leq C\|f\|_{F(p,q,s)}\mu(z)|\Re g(z)|(1-|z|^2)^{1-\frac{n+1+q}{p}}. \end{aligned}$$

Thus, (3.7) shows

$$\lim_{|z|\to 1} \sup_{\|f\|_{F(p,q,s)}\leq 1} \mu(z)|\Re(T_g f)(z)| = 0.$$

Similarly, we can obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{F(p,q,s)}\leq 1} \mu(z)|\Re(T_g f)(z)| = 0$$

for the case of $n+1+q = p$ by (3.8). Therefore, $T_g : F(p,q,s) \to \mathcal{B}_{\mu,0}$ is compact by Lemma 2.7.

(A)⇒(C) First, we deal with the case of $n+1+q > p$. Suppose (3.7) did not hold. Then there would be some $\varepsilon_0 > 0$ and some sequence $\{z^j\} \subseteq \mathbf{B}$ satisfying $\lim_{j\to\infty} |z^j| = 1$, but for each j,

$$\mu(z^j)|\Re g(z^j)|(1-|z^j|^2)^{1-\frac{n+1+q}{p}} \geq \varepsilon_0. \tag{3.9}$$

Set

$$f_j(z) = \frac{1-|z^j|^2}{(1-<z,z^j>)^{\frac{n+1+q}{p}}}, \quad z \in \mathbf{B}. \tag{3.10}$$

Then $\|f_j\|_{F(p,q,s)} \leq C$, and $\{f_j\}$ converges to 0 uniformly on any compact subset of $\mathbf{B}$. By Lemma 2.6 and (A),

$$\|T_g f_j\|_{\mathcal{B}_\mu} \to 0 \quad (j \to \infty). \tag{3.11}$$

On the other hand, (3.9) implies

$$\begin{aligned}\|T_g f_j\|_{\mathcal{B}_\mu} &\simeq |T_g f_j(0)| + \sup_{z\in\mathbf{B}} \mu(z)|f_j(z)\Re g(z)| \\ &\geq \mu(z^j)|f_j(z^j)\Re g(z^j)| \\ &= \mu(z^j)|\Re g(z^j)|(1-|z^j|^2)^{1-\frac{n+1+q}{p}} \geq \varepsilon_0.\end{aligned}$$

This is a contradiction to (3.11). If $n+1+q=p$, suppose (3.8) did not hold. Then there would be some $\varepsilon_0 > 0$ and some sequence $\{z^j\} \subseteq \mathbf{B}$ satisfying $\lim_{j\to\infty} |z^j| = 1$, but for each j,

$$\mu(z^j)|\Re g(z^j)| \log \frac{2}{1-|z^j|^2} \geq \varepsilon_0. \tag{3.12}$$

Take the test function

$$f_j(z) = \frac{\left(\log \frac{2}{1-<z,z^j>}\right)^2}{\log \frac{2}{1-|z^j|^2}}, \quad z \in \mathbf{B}.$$

Then

$$\begin{aligned}|\Re f_j(z)| &= \left|\frac{2<z,z^j>\log\frac{2}{1-<z,z^j>}}{(1-<z,z^j>)\log\frac{2}{1-|z^j|^2}}\right| \leq 2\left|\frac{\log\frac{2}{1-<z,z^j>}}{\log\frac{2}{1-|z^j|^2}}\right| \frac{1}{|1-<z,z^j>|} \\ &\leq 2\frac{2\pi+\log\frac{2}{1-|z^j|}}{\log\frac{2}{1-|z^j|^2}}\cdot\frac{1}{|1-<z,z^j>|} \leq \frac{C}{|1-<z,z^j>|}.\end{aligned}$$

Then $\|f_j\|_{F(p,q,s)} \leq C$ by Lemma 2.5, and $\{f_j\}$ converges to 0 uniformly on any compact subset of $\mathbf{B}$. By Lemma 2.6 and (A), we have

$$\|T_g f_j\|_{\mathcal{B}_\mu} \to 0 \quad \text{as } j \to \infty. \tag{3.13}$$

However, (3.12) yields

$$\begin{aligned}\|T_g f_j\|_{\mathcal{B}_\mu} &\simeq |T_g f_j(0)| + \sup_{z\in\mathbf{B}} \mu(z)|f_j(z)\Re g(z)| \\ &\geq \mu(z^j)|f_j(z^j)\Re g(z^j)| \\ &= \mu(z^j)|\Re g(z^j)|\log\frac{2}{1-|z^j|^2} \geq \varepsilon_0.\end{aligned}$$

This is a contradiction to (3.13). The proof is completed. □

Theorem 3.3. *Let μ be normal, $g \in H(\mathbf{B})$, $n+1+q<p$. Then the following statements are equivalent:*

(A) T_g*:* $F(p,q,s) \to \mathcal{B}_\mu$ *is bounded;*

(B) T_g*:* $F(p,q,s) \to \mathcal{B}_\mu$ *is compact;*

(C) $g \in \mathcal{B}_\mu$.

In this case,

$$\|T_g\| \simeq \|g - g(0)\|_{\mathcal{B}_\mu}.$$

Proof. The implication (B)⇒(A) is trivial.

(A)⇒(C) Suppose T_g: $F(p,q,s) \to \mathcal{B}_\mu$ is bounded. By the fact that $g(z) = g(0) + T_g(1)(z)$, we know $g \in \mathcal{B}_\mu$. Moreover,

$$\|g - g(0)\|_{\mathcal{B}_\mu} = \|T_g(1)\|_{\mathcal{B}_\mu} \le C\|T_g\| < \infty. \tag{3.14}$$

(C)⇒(B) Suppose $\{f_j\} \subseteq F(p,q,s)$ is any bounded sequence converging to 0 uniformly on any compact subset of $\mathbf{B}$. By Lemma 2.1 and [9, Lemma 4.2],

$$\lim_{j\to\infty} \sup_{z\in\mathbf{B}} |f_j(z)| = 0.$$

Hence,

$$\begin{aligned}\|T_g f_j\|_{\mathcal{B}_\mu} &\simeq |T_g f_j(0)| + \sup_{z\in\mathbf{B}} \mu(z)|f_j(z)\Re g(z)| \\ &\le C\|g\|_{\mathcal{B}_\mu} \sup_{z\in\mathbf{B}} |f_j(z)| \to 0 \quad (j\to\infty).\end{aligned}$$

This means T_g: $F(p,q,s) \to \mathcal{B}_\mu$ is compact.

Furthermore, for any $f \in F(p,q,s)$, Lemmas 2.1 and 2.3 yield

$$\begin{aligned}\|T_g f\|_{\mathcal{B}_\mu} &\simeq |T_g f(0)| + \sup_{z\in\mathbf{B}} \mu(z)|f(z)\Re g(z)| \\ &\le C\|g - g(0)\|_{\mathcal{B}_\mu}\|f\|_{\mathcal{B}_{(1-r^2)^{\frac{n+1+q}{p}}}} \\ &\le C\|g - g(0)\|_{\mathcal{B}_\mu}\|f\|_{F(p,q,s)}.\end{aligned}$$

This, combining with (3.14), shows

$$\|T_g\| \simeq \|g - g(0)\|_{\mathcal{B}_\mu}.$$

The proof is completed. □

Theorem 3.4. *Let μ be normal, $g \in H(\mathbf{B})$, $n+1+q < p$. Then the following statements are equivalent:*

(A) T_g*: $F(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded;*

(B) T_g*: $F(p,q,s) \to \mathcal{B}_{\mu,0}$ is compact;*

(C) $g \in \mathcal{B}_{\mu,0}$.

Proof. The implication (B)⇒(A) is trivial.

(A)⇒(C) It is trivial from the fact that $g(z) = g(0) + T_g(1)(z)$.

(C)⇒(B) By Theorem 3.3, the condition (C) implies that T_g is compact from the $F(p,q,s)$ space to Bloch-type space $\mathcal{B}_\mu$. We claim that $T_g(F(p,q,s)) \subseteq \mathcal{B}_{\mu,0}$. In fact, for any $f \in F(p,q,s) \subseteq \mathcal{B}_{(1-r^2)^{\frac{n+1+q}{p}}}$, Lemmas 2.2 and 2.3 imply

$$0 \le \mu(z)|\Re g(z)||f(z)| \le C\|f\|_{\mathcal{B}_{(1-r^2)^{\frac{n+1+q}{p}}}} \mu(z)|\Re g(z)| \to 0 \quad \text{as } |z| \to 1.$$

The proof is completed. □

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