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COMPATIBLE MAPS AND COMMON FIXED POINTS IN MENGER PROBABILISTIC METRIC SPACES

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ABSTRACT. In the present work, we introduce two types of compatible maps and prove a common fixed point theorem for such maps in Menger probabilistic metric spaces. Our result generalizes and extends many known results in metric spaces and fuzzy metric spaces.

1. Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space introduced in 1942 by Menger [10] who used distribution functions instead of nonnegative real numbers as values of the metric. Schweizer and Sklar [13] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [15]. Sessa [16] introduced weakly commuting maps in metric spaces. Jungck [7] enlarged this concept to compatible maps. The notion of compatible maps in Menger spaces has been introduced by Mishra [11]. Cho [1] et al. and Sharma [17] gave fuzzy version of compatible maps and proved common fixed point theorems for compatible maps in fuzzy metric spaces.

In this paper, we introduce the concept of compatible maps of type (P-1) and type (P-2), show that they are equivalent to compatible maps under certain conditions and prove a common fixed point theorem for such maps in Menger spaces illustrating with an example which generalize, extend and fuzzify several well known fixed point theorems for contractive type maps on metric spaces, Menger spaces, uniform spaces and fuzzy metric spaces.

2. Preliminaries

In this section, we recall some definitions and known results in Menger space. For more details we refer the readers to [2, 4-6, 8-13, 15, 18].

Definition 1. A triangular norm $*$ (shorty t-norm) is a binary operation on the unit interval $[0,1]$ such that for all $a,b,c,d \in [0,1]$ the following conditions are satisfied:

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(a) $a * 1 = a$;
(b) $a * b = b * a$;
(c) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
(d) $a * (b * c) = (a * b) * c$.

Examples of t-norms are $a * b = \max\{a + b - 1, 0\}$ and $a * b = \min\{a, b\}$.

Definition 2. A distribution function is a function $F : [-\infty, \infty] \to [0, 1]$ which is left continuous on $\mathbb{R}$, non-decreasing and $F(-\infty) = 0$, $F(\infty) = 1$.

We will denote by Δ the family of all distribution functions on $[-\infty, \infty]$. H is a special element of Δ defined by

$$H(t) = \begin{cases} 0, \; t \leq 0 \\ 1, \; t > 0. \end{cases}$$

If X is a nonempty set, $F : X \times X \to \Delta$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by F_{xy}.

Definition 3 ([13]). The ordered pair (X, F) is called a probabilistic semimetric space (shortly PSM-space) if X is a nonempty set and F is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,
(PM-1) $F_{xy}(t) = H(t) \Longleftrightarrow x = y$;
(PM-2) $F_{xy} = F_{yx}$.
If, in addition, the following inequality takes place:
(PM-3) $F_{xz}(t) = 1$, $F_{zy}(s) = 1 \Rightarrow F_{xy}(t + s) = 1$,
then (X, F) is called a probabilistic metric space (shortly PM-space).
The ordered triple $(X, F, *)$ is called Menger probabilistic metric space (shortly Menger space) if (X, F) is a PM-space, $*$ is a t-norm and the following condition is also satisfies: for all $x, y, z \in X$ and $t, s > 0$,
(PM-4) $F_{xy}(t + s) \geq F_{xz}(t) * F_{zy}(s)$.

For every PSM-space (X, F), we can consider the sets of the form

$$U_{\varepsilon,\lambda} = \{(x, y) \in X \times X : F_{xy}(\varepsilon) > 1 - \lambda\}.$$

The family $\{U_{\varepsilon,\lambda}\}_{\varepsilon>0,\lambda\in(0,1)}$ generates a semi-uniformity denoted by U_F and a topology τ_F called the F-topology or the strong topology. Namely,

$$A \in \tau_F \text{ if and only if } \forall x \in A \; \exists \varepsilon > 0 \text{ and } \lambda \in (0, 1) \text{ such that } U_{\varepsilon,\lambda}(x) \subset A.$$

U_F is also generated by the family $\{V_\delta\}_{\delta>0}$ where $V_\delta := U_{\delta,\delta}$ ([9]).

In [14], it is proved that if $\sup_{t<1}(t * t) = 1$, then U_F is a uniformity, called F-uniformity, which is metrizable.

The F-topology is generated by the F-uniformity and is determined by the F-convergence:

$$x_n \to x \Leftrightarrow F_{x_n x}(t) \to 1, \; \forall t > 0.$$

Proposition 1 ([15]). *Let (X, d) be a metric space. Then the metric d induces a distribution function F defined by $F_{xy}(t) = H(t - d(x, y))$ for all $x, y \in X$ and $t > 0$. If t-norm $*$ is $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$, then $(X, F, *)$*

is a Menger space. Further, $(X, F, *)$ is a complete Menger space if (X, d) is complete.

Definition 4 ([11]). Let $(X, F, *)$ be a Menger space and $*$ be a continuous t-norm.

(a) A sequence $\{x_n\}$ in X is said to be *converge* to a point x in X (written $x_n \to x$) if and only if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.

(b) A sequence $\{x_n\}$ in X is said to be *Cauchy* if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n x_{n+p}}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$.

(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 1. If $*$ is a continuous t-norm, it follows from (FM-4) that the limit of sequence in Menger space is uniquely determined.

Definition 5 ([11]). Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible if $F_{ABx_n BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.

Following lemma can be selected from the proof of Theorem 3 of Sehgal and Bharucha-Reid [15].

Lemma 1. *Let $\{x_n\}$ be a sequence in a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$. If there exists a constant $k \in (0,1)$ such that*

$$F_{x_n x_{n+1}}(kt) \geq F_{x_{n-1} x_n}(t)$$

for all $t > 0$ and $n = 1, 2, \ldots$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2 ([18]). *Let $(X, F, *)$ be a Menger space. If there exists $k \in (0,1)$ such that*

$$F_{xy}(kt) \geq F_{xy}(t)$$

for all $x, y \in X$ and $t > 0$, then $x = y$.

3. Compatible maps of type (P-1) and type (P-2)

In this section, we introduce the concept of compatible mappings of type (P-1) and type (P-2) in Menger spaces and show that they are equivalent to compatible mappings under certain conditions.

Definition 6. Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible of type (P) if $F_{ABx_n BBx_n}(t) \to 1$ and $F_{BAx_n AAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.

Definition 7. Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible of type (P-1) if $F_{ABx_n BBx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.

Definition 8. Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible of type (P-2) if $F_{BAx_n AAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.

Remark 2. Clearly, if a pair of mappings (A, B) is compatible of type (P-1), then the pair (B, A) is compatible of type (P-2). Further, if A and B compatible mappings of type (P), then the pair (A, B) is compatible of type (P-1) as well as type (P-2).

The following is an example of pair of self maps in a Menger space which are compatible of type (P-1) and type (P-2) but not compatible.

Example 1. Let (X, d) be a metric space with the usual metric d where $X = [0, 2]$ and $(X, F, *)$ be the induced Menger space with $F_{xy}(t) = H(t - d(x, y))$, $\forall x, y \in X$ and $\forall t > 0$. Define self maps A and B as follows:

$$Ax = \begin{cases} 2 - x, & \text{if } 0 \leq x < 1, \\ 2, & \text{if } 1 \leq x \leq 2, \end{cases} \quad \text{and} \quad Bx = \begin{cases} x, & \text{if } 0 \leq x < 1, \\ 2, & \text{if } 1 \leq x \leq 2. \end{cases}$$

Take $x_n = 1 - 1/n$. Then $F_{Ax_n 1}(t) = H(t - (1/n))$ and $\lim_{n\to\infty} F_{Ax_n 1}(t) = H(t) = 1$. Hence $Ax_n \to 1$ as $n \to \infty$. Similarly, $Bx_n \to 1$ as $n \to \infty$. Also $F_{ABx_n BAx_n}(t) = H(t - (1 - 1/n))$ and $\lim_{n\to\infty} F_{ABx_n BAx_n}(t) = H(t - 1) \neq 1$, $\forall t > 0$. Hence the pair (A, B) is not compatible. But $F_{ABx_n BBx_n}(t) = H(t - (2/n))$ and $\lim_{n\to\infty} F_{ABx_n BBx_n}(t) = H(t) = 1, \forall t > 0$. Hence the pair (A, B) is compatible of type (P-1). Similarly, the pair (A, B) is compatible of type (P-2).

Next, we cite the following propositions which gives the condition under which the Definitions 5, 7 and 8 becomes equivalent.

Proposition 2. *Let A and B be self maps of a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0, 1]$.*

(i) *If B is continuous then the pair (A, B) is compatible of type* (P-1) *if and only if A and B are compatible.*
(ii) *If A is continuous then the pair (A, B) is compatible of type* (P-2) *if and only if A and B are compatible.*

Proof. (i) Let $\{x_n\}$ be a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$ and let the pair (A, B) be compatible of type (P-1). Since B is continuous, we have $BAx_n \to Bz$ and $BBx_n \to Bz$. Therefore, by (PM-4), we have

$$F_{ABx_n BAx_n}(t) \geq F_{ABx_n BBx_n}(t/2) * F_{BBx_n BAx_n}(t/2) \to 1 * 1 \geq 1$$

as $n \to \infty$. Hence the mappings A and B are compatible.

Now, let A and B be compatible. Therefore, using the continuity of B, we have

$$F_{ABx_n BBx_n}(t) \geq F_{ABx_n BAx_n}(t/2) * F_{BAx_n BBx_n}(t/2) \to 1 * 1 \geq 1$$

as $n \to \infty$. Hence the mappings A and B are compatible of type (P-1).

(ii) It is similar to the proof of (i). □

Next, we give some properties of compatible mappings of type (P-1) and type (P-2) which will be used in our main theorem.

Proposition 3. *Let A and B be self maps of a Menger space $(X, F, *)$. If the pair (A, B) is compatible of type* (P-1) *and $Az = Bz$ for some z in X, then $ABz = BBz$.*

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n = z$ for $n \in \mathbb{N}$ and let $Az = Bz$. Then we have $Ax_n \to Az$ and $Bx_n \to Bz$. Since the pair (A, B) is compatible of type (P-1), we have $F_{ABzBBz}(t) = F_{ABx_nBBx_n}(t) \to 1$ as $n \to \infty$. Hence $ABz = BBz$. □

Proposition 4. *Let A and B be self maps of a Menger space $(X, F, *)$. If the pair (A, B) is compatible of type* (P-2) *and $Az = Bz$ for some z in X, then $BAz = AAz$.*

Proof. It is similar to the proof of Proposition 3. □

Proposition 5. *Let A and B be self maps of a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0, 1]$. If the pair (A, B) is compatible of type* (P-1) *and $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$, then $BBx_n \to Az$ if A is continuous at z.*

Proof. Since A is continuous at z and the pair (A, B) is compatible of type (P-1), we have $ABx_n \to Az$ and $F_{ABx_nBBx_n}(t) \to 1$ as $n \to \infty$. Therefore

$$F_{AzBBx_n}(t) \geq F_{AzABx_n}(t/2) * F_{ABx_nBBx_n}(t/2) \to 1 * 1 \geq 1$$

as $n \to \infty$. Hence $BBx_n \to Az$ as $n \to \infty$. □

Proposition 6. *Let A and B be self maps of a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0, 1]$. If the pair (A, B) is compatible of type* (P-2) *and $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$, then $AAx_n \to Bz$ if B is continuous at z.*

Proof. It is similar to the proof of Proposition 5. □

4. Main results

Theorem 1. *Let A, B, P, Q, S and T be self maps on a complete Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$, for all $t \in [0, 1]$, satisfying:*

(a) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$,

(b) *there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned} F_{PxQy}(kt) \;\geq\; & F_{ABxSTy}(t) * F_{PxABx}(t) * F_{QySTy}(t) \\ & * F_{PxSTy}(\alpha t) * F_{QyABx}((2-\alpha)t) \end{aligned}$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$,

(c) $AB = BA, ST = TS, PB = BP, QT = TQ$,

(d) *either P or AB is continuous,*

(e) *the pairs* (P, AB) *and* (Q, ST) *are compatible of type* (P-1) *or type* (P-2).

Then A, B, P, Q, S *and* T *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of X. By (a), there exists $x_1, x_2 \in X$ such that $Px_0 = STx_1 = y_0$ and $Qx_1 = ABx_1 = y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Px_{2n} = STx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \ldots$.

Step 1. By taking $x = x_{2n}$, $y = x_{2n+1}$ for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$ in (b), we have

$$\begin{aligned} F_{Px_{2n}Qx_{2n+1}}(kt) &= F_{y_{2n}y_{2n+1}}(kt) \\ &\geq F_{y_{2n-1}y_{2n}}(t) * F_{y_{2n}y_{2n-1}}(t) * F_{y_{2n+1}y_{2n}}(t) * F_{y_{2n}y_{2n}}((1-q)t) \\ &\quad * F_{y_{2n+1}y_{2n-1}}((1+q)t) \\ &\geq F_{y_{2n-1}y_{2n}}(t) * F_{y_{2n-1}y_{2n}}(t) * F_{y_{2n}y_{2n+1}}(t) * 1 * F_{y_{2n-1}y_{2n}}(t) \\ &\quad * F_{y_{2n}y_{2n+1}}(qt) \\ &\geq F_{y_{2n-1}y_{2n}}(t) * F_{y_{2n}y_{2n+1}}(t) * F_{y_{2n}y_{2n+1}}(qt). \end{aligned}$$

Since t-norm is continuous, letting $q \to 1$, we have

$$F_{y_{2n}y_{2n+1}}(kt) \geq F_{y_{2n-1}y_{2n}}(t) * F_{y_{2n}y_{2n+1}}(t).$$

Similarly, we also have

$$F_{y_{2n+1}y_{2n+2}}(kt) \geq F_{y_{2n}y_{2n+1}}(t) * F_{y_{2n+1}y_{2n+2}}(t).$$

In general, for all n even or odd, we have

$$F_{y_ny_{n+1}}(kt) \geq F_{y_{n-1}y_n}(t) * F_{y_ny_{n+1}}(t).$$

Consequently, for $p = 1, 2, \ldots$, it follows that,

$$F_{y_ny_{n+1}}(kt) \geq F_{y_{n-1}y_n}(t) * F_{y_ny_{n+1}}(\frac{t}{k^p}).$$

By noting that $F_{y_ny_{n+1}}(\frac{t}{k^p}) \to 1$ as $p \to \infty$, we have

$$F_{y_ny_{n+1}}(kt) \geq F_{y_{n-1}y_n}(t)$$

for $k \in (0, 1)$, all $n \in \mathbb{N}$ and $t > 0$. Hence, by Lemma 1, $\{y_n\}$ is a Cauchy sequence in X. Since $(X, F, *)$ is complete, it converges to a point z in X. Also its subsequences converge as follows: $\{Px_{2n}\} \to z$, $\{ABx_{2n}\} \to z$, $\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$.

Case I. AB is continuous, and (P, AB) and (Q, ST) are compatible of type (P-1).

Since AB is continuous, $AB(AB)x_{2n} \to ABz$ and $(AB)Px_{2n} \to ABz$. Since (P, AB) is compatible of type (P-1), $PPx_{2n} \to ABz$.

Step 2. By taking $x = Px_{2n}$, $y = x_{2n+1}$ with $\alpha = 1$ in (b), we have

$$F_{PPx_{2n}Qx_{2n+1}}(kt) \geq F_{ABPx_{2n}STx_{2n+1}}(t) * F_{PPx_{2n}ABPx_{2n}}(t) * F_{Qx_{2n+1}STx_{2n+1}}(t) * F_{PPx_{2n}STx_{2n+1}}(t) * F_{Qx_{2n+1}ABPx_{2n}}(t).$$

This implies that, as $n \to \infty$

$$\begin{aligned} F_{zABz}(kt) &\geq F_{zABz}(t) * F_{ABzABz}(t) * F_{zz}(t) * F_{zABz}(t) * F_{zABz}(t) \\ &= F_{zABz}(t) * 1 * 1 * F_{zABz}(t) * F_{zABz}(t) \\ &\geq F_{zABz}(t). \end{aligned}$$

Thus, by Lemma 2, it follows that $z = ABz$.

Step 3. By taking $x = z$, $y = x_{2n+1}$ with $\alpha = 1$ in (b), we have

$$\begin{aligned} F_{PzQx_{2n+1}}(kt) &\geq F_{ABzSTx_{2n+1}}(t) * F_{PzABz}(t) * F_{Qx_{2n+1}STx_{2n+1}}(t) \\ &\quad * F_{PzSTx_{2n+1}}(t) * F_{Qx_{2n+1}ABz}(t). \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} F_{zPz}(kt) &\geq F_{zz}(t) * F_{zPz}(t) * F_{zz}(t) * F_{zPz}(t) * F_{zz}(t) \\ &= 1 * F_{zPz}(t) * 1 * F_{zPz}(t) * 1 \\ &\geq F_{zPz}(t). \end{aligned}$$

Thus, by Lemma 2, it follows that $z = Pz$. Therefore, $z = ABz = Pz$.

Step 4. By taking $x = Bz$, $y = x_{2n+1}$ with $\alpha = 1$ in (b) and using (c), we have

$$F_{P(Bz)Qx_{2n+1}}(kt) \geq F_{AB(Bz)STx_{2n+1}}(t) * F_{P(Bz)AB(Bz)}(t) * F_{Qx_{2n+1}STx_{2n+1}}(t) * F_{P(Bz)STx_{2n+1}}(t) * F_{Qx_{2n+1}AB(Bz)}(t).$$

This implies that, as $n \to \infty$

$$\begin{aligned} F_{zBz}(kt) &\geq F_{zBz}(t) * F_{BzBz}(t) * F_{zz}(t) * F_{zBz}(t) * F_{zBz}(t) \\ &= F_{zBz}(t) * 1 * 1 * F_{zBz}(t) * F_{zBz}(t) \\ &\geq F_{zBz}(t). \end{aligned}$$

Thus, by Lemma 2, it follows that $z = Bz$. Since $z = ABz$, we have $z = Az$. Therefore, $z = Az = Bz = Pz$.

Step 5. Since $P(X) \subseteq ST(X)$, there exists $w \in X$ such that $z = Pz = STw$. By taking $x = x_{2n}$, $y = w$ with $\alpha = 1$ in (b), we have

$$\begin{aligned} F_{Px_{2n}Qw}(kt) &\geq F_{ABx_{2n}STw}(t) * F_{Px_{2n}ABx_{2n}}(t) * F_{QwSTw}(t) \\ &\quad * F_{Px_{2n}STw}(t) * F_{QwABx_{2n}}(t) \end{aligned}$$

which implies that, as $n \to \infty$

$$\begin{aligned} F_{zQw}(kt) &\geq F_{zz}(t) * F_{zz}(t) * F_{zQw}(t) * F_{zz}(t) * F_{zQw}(t) \\ &= 1 * 1 * F_{zQw}(t) * 1 * F_{zQw}(t) \\ &\geq F_{zQw}(t). \end{aligned}$$

Thus, by Lemma 2, we have $z = Qw$. Hence, $STw = z = Qw$. Since (Q, ST) is compatible of type (P-1), we have $Q(ST)w = ST(ST)w$. Thus, $STz = Qz$.

Step 6. By taking $x = x_{2n}$, $y = z$ with $\alpha = 1$ in (b) and using Step 5, we have

$$\begin{aligned} F_{Px_{2n}Qz}(kt) &\geq F_{ABx_{2n}STz}(t) * F_{Px_{2n}ABx_{2n}}(t) * F_{QzSTz}(t) \\ &\quad * F_{Px_{2n}STz}(t) * F_{QzABx_{2n}}(t) \end{aligned}$$

which implies that, as $n \to \infty$

$$\begin{aligned} F_{zQz}(kt) &\geq F_{zQz}(t) * F_{zz}(t) * F_{QzQz}(t) * F_{zQz}(t) * F_{zQz}(t) \\ &= F_{zQz}(t) * 1 * 1 * F_{zQz}(t) * 1 * F_{zQz}(t) \\ &\geq F_{zQz}(t). \end{aligned}$$

Thus, by Lemma 2, we have $z = Qz$. Since $STz = Qz$, we have $z = STz$. Therefore, $z = Az = Bz = Pz = Qz = STz$.

Step 7. By taking $x = x_{2n}$, $y = Tz$ with $\alpha = 1$ in (b) and using (c), we have

$$\begin{aligned} F_{Px_{2n}Q(Tz)}(kt) &\geq F_{ABx_{2n}ST(Tz)}(t) * F_{Px_{2n}ABx_{2n}}(t) * F_{Q(Tz)ST(Tz)}(t) \\ &\quad * F_{Px_{2n}ST(Tz)}(t) * F_{Q(Tz)ABx_{2n}}(t) \end{aligned}$$

which implies that, as $n \to \infty$

$$\begin{aligned} F_{zTz}(kt) &\geq F_{zTz}(t) * F_{zz}(t) * F_{TzTz}(t) * F_{zTz}(t) * F_{zTz}(t) \\ &= F_{zTz}(t) * 1 * 1 * F_{zTz}(t) * 1 * F_{zTz}(t) \\ &\geq F_{zTz}(t). \end{aligned}$$

Thus, by Lemma 2, we have $z = Tz$. Since $z = STz$, we have $z = Sz$. Therefore, $z = Az = Bz = Pz = Qz = Sz = Tz$, that is, z is the common fixed point of A, B, P, Q, S and T.

Similarly, it is clear that z is also the common fixed point of A, B, P, Q, S and T in the case AB is continuous, and (P, AB) and (Q, ST) are compatible of type (P-2).

Case II. P is continuous, and (P, AB) and (Q, ST) are compatible of type (P-1).

Since P is continuous, $PPx_{2n} \to Pz$ and $P(AB)x_{2n} \to Pz$. Since (P, AB) is compatible of type (P-1), $AB(AB)x_{2n} \to Pz$.

Step 8. By taking $x = ABx_{2n}$, $y = x_{2n+1}$ with $\alpha = 1$ in (b), we have

$$\begin{aligned} F_{P(AB)x_{2n}Qx_{2n+1}}(kt) &\geq F_{AB(AB)x_{2n}STx_{2n+1}}(t) * F_{P(AB)x_{2n}AB(AB)x_{2n}}(t) \\ &\quad * F_{Qx_{2n+1}STx_{2n+1}}(t) * F_{P(AB)x_{2n}STx_{2n+1}}(t) \\ &\quad * F_{Qx_{2n+1}AB(AB)x_{2n}}(t). \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} F_{zPz}(kt) &\geq F_{zPz}(t) * F_{PzPz}(t) * F_{zz}(t) * F_{zPz}(t) * F_{zPz}(t) \\ &= F_{zPz}(t) * 1 * 1 * F_{zPz}(t) * F_{zPz}(t) \\ &\geq F_{zPz}(t). \end{aligned}$$

Thus, by Lemma 2, it follows that $z = Pz$. Now using Step 5-7, we have $z = Qz = STz = Sz = Tz$.

Step 9. Since $Q(X) \subseteq AB(X)$, there exists $w \in X$ such that $z = Qz = ABw$. By taking $x = w$, $y = x_{2n+1}$ with $\alpha = 1$ in (b), we have

$$\begin{aligned} F_{PwQx_{2n+1}}(kt) &\geq F_{ABwSTx_{2n+1}}(t) * F_{PwABw}(t) * F_{Qx_{2n+1}STx_{2n+1}}(t) \\ &\quad * F_{PwSTx_{2n+1}}(t) * F_{Qx_{2n+1}ABw}(t) \end{aligned}$$

which implies that, as $n \to \infty$

$$\begin{aligned} F_{zPw}(kt) &\geq F_{zz}(t) * F_{zPw}(t) * F_{zz}(t) * F_{zPw}(t) * F_{zz}(t) \\ &= 1 * F_{zPw}(t) * 1 * F_{zPw}(t) * 1 \\ &\geq F_{zPw}(t). \end{aligned}$$

Thus, by Lemma 2, we have $z = Pw$. Since $z = Qz = ABw$, $Pw = ABw$. Since (P, AB) is compatible of type (P-1), we have $Pz = ABz$. Also $z = Bz$ follows from Step 4. Thus, $z = Az = Bz = Pz$. Hence, z is the common fixed point of the six maps in this case also.

Similarly, it is clear that z is also the common fixed point of A, B, P, Q, S and T in the case P is continuous, and (P, AB) and (Q, ST) are compatible of type (P-2).

Step 10. For uniqueness, let v $(v \neq z)$ be another common fixed point of A, B, P, Q, S and T. Taking $x = z$, $y = v$ with $\alpha = 1$ in (b), we have

$$\begin{aligned} F_{PzQv}(kt) &\geq F_{ABzSTv}(t) * F_{PzABz}(t) * F_{QvSTv}(t) \\ &\quad * F_{PzSTv}(\alpha t) * F_{QvABz}((2-\alpha)t) \end{aligned}$$

which implies that

$$\begin{aligned} F_{zv}(kt) &\geq F_{zv}(t) * F_{zz}(t) * F_{vv}(t) * F_{zv}(t) * F_{vz}(t) \\ &= F_{zv}(t) * 1 * 1 * F_{zv}(t) * F_{zv}(t) \\ &\geq F_{zv}(t). \end{aligned}$$

Thus, by Lemma 2, we have $z = v$. This completes the proof of the theorem. □

If we take $A = B = S = T = I_X$ (the identity map on X) in Theorem 1, we have the following:

Corollary 1. *Let P and Q be self maps on a complete Menger space $(X, F, *)$ with continuous t-norm $*$ and $t*t \geq t$ for all $t \in [0, 1]$. If there exists a constant*

$k \in (0,1)$ such that

$$F_{PxQy}(kt) \geq F_{xy}(t) * F_{xPx}(t) * F_{yQy}(t) * F_{yPx}(\alpha t) * F_{xQy}((2-\alpha)t)$$

for all $x, y \in X, \alpha \in (0,2)$ and $t > 0$, then P and Q have a unique common fixed point.

In [17], Sehgal and Bharucha-Reid presented the probabilistic version of the Banach contraction theorem as follows:

Corollary 2. *Let P be a self map on a complete Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0,1]$. If there exists a constant $k \in (0,1)$ such that*

$$F_{PxPy}(kt) \geq F_{xy}(t)$$

for all $x, y \in X$ and $t > 0$, then P has a unique fixed point.

Proof. The proof follows from Corollary 1 since $P = Q$ and $F_{xy}(t) = \min\{F_{xy}(t), F_{xPx}(t), F_{yQy}(t), F_{yPx}(t), F_{xQy}(t)\}$. □

Remark 3. In Theorem 1, Corollaries 1 and 2, the condition "the t-norm $*$ is continuous and $t * t \geq t$ for all $t \in [0,1]$" can be replaced by the condition "$s * t = \min\{s, t\}$ for all $s, t \in [0,1]$."

Example 2. Let (X, d) be a metric space with the usual metric d where $X = [0,1]$ and $(X, F, *)$ be the induced Menger space with $F_{xy}(t) = H(t - d(x,y))$ for all $x, y \in X, t > 0$. Clearly $(X, F, *)$ is a complete Menger space where t-norm $*$ is defined by $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$. Let A, B, P, Q, S and T be maps from X into itself defined as $Ax = x/5, Bx = x/3, Px = x/6, Qx = 0, Sx = x, Tx = x/2$ for all $x \in X$. Then $P(X) = [0, \frac{1}{6}] \subset [0, \frac{1}{2}] = ST(X)$ and$Q(X) = \{0\} \subset [0, \frac{1}{15}] = AB(X)$. Clearly, conditions (b), (c) and (d) of the main Theorem are satisfied. Moreover, the pairs (P, AB) and (Q, ST) are compatible of type (P-1). In fact, if $\lim_{n\to\infty} x_n = 0$, where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = 0$ and $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} STx_n = 0$ for some $0 \in X$, then $\lim_{n\to\infty} F_{P(AB)x_n AB(AB)x_n}(t) = H(t) = 1$ and $\lim_{n\to\infty} F_{(AB)Px_n PPx_n}(t) = H(t) = 1$. Similarly, the pairs (P, AB) and (Q, ST) are also compatible of type (P-2). Thus, all conditions of the main Theorem are satisfied and 0 is the unique common fixed point of A, B, P, Q, S and T.

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